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Manjul Bhargava
Counting field extensions of the rational numbers

“Number fields” – i.e., finite extensions of the rational numbers – are perhaps the most fundamental objects in algebraic number theory. Yet, amazingly, we have almost no idea how to systematically enumerate them, or even describe their distribution asymptotically with respect to basic invariants. The only general guess regarding their distribution is an old and simple (but very elusive!) conjecture of Linnik.

In the first talk, we will begin by stating Linnik’s conjecture, and present some old and new results that now allow for a proof in the case of number fields of degree up to five.

In the second talk, we will discuss some new heuristics for understanding the distribution of number fields. These heuristics, in particular, imply Linnik’s conjecture, but also yield very refined predictions regarding precise constants in asymptotics, prime splitting and ramification, and more. These heuristics too can now be proven up to degree 5. For general degrees, these heuristics turn out to have a beautiful probabilistic interpretation, and in the local case, are closely related to Serre’s mass formulae for totally ramified local extensions. We will explain these interesting connections.

Dennis Gaitsgory

Two faces of geometric Eisenstein series

Lecture 1

Let G be a reductive group, and X a complete algebraic curve. Let Bun_G be the moduli space of G -bundles on X . I'll first remind the objectives of the geometric Langlands program, i.e., that starting with a local system E with respect to the Langlands dual group \check{G} , one aims to construct an ℓ -adic perverse sheaf, S_E , called the automorphic sheaf, which satisfies the Hecke property with respect to E .

In this talks we will be concerned with the simplest case, when E is split, i.e., factors through the maximal torus \check{T} . In this case one can attempt to construct S_E by the Eisenstein series procedure. I will review the motivation for this through the connection with the classical notion of Eisenstein series in number theory.

However, there are two types of Eisenstein series: usual (non-normalized) and compactified (normalized); the definition of the former is rather straightforward, and of the latter requires some geometry in the form of intersection cohomology sheaves. It turns out that the automorphic sheaf S_E , corresponding to a split local system is exactly the compactified Eisenstein series, denoted \overline{Eis}_E .

A natural question now is: what is the role of the usual Eisenstein series, Eis_E . The answer to this question was proposed by Drinfeld.

Lecture 2

Let us view E as a homomorphism $\pi_1(X) \rightarrow \check{T}$, and let us consider the following space of deformations of E . Namely, we consider homomorphisms $\pi_1(X) \rightarrow \check{B}$ (here $\check{B} \subset \check{G}$ is the Borel subgroup) with the property that the composition $\pi_1(X) \rightarrow \check{B} \rightarrow \check{T}$ is the initial local system E . If E is generic, the space of deformations exists, and will be denoted by $Def(E)$, and is in fact isomorphic to an affine space. Let E_{univ} be the universal local system $\pi_1(X) \rightarrow \check{B}$ over $Def(E)$.

Drinfeld's conjecture states that the usual Eisenstein series Eis_E is the automorphic sheaf $S_{E_{univ}}$, corresponding to the above universal family.

This conjecture amounts to certain additional structures possessed by the objects Eis_E and \overline{Eis}_E . In the case of Eis_E this is the action of the algebra of functions on $Def(E)$, and in the case of \overline{Eis}_E this is the action of a certain DG algebra attached to \check{B} and E .

In the talk we will first analyze in detail the case of GL_2 and then indicate the structure in the general case.

Maryam Mirzakhani

Counting simple closed geodesics on hyperbolic surfaces and intersection theory on the moduli spaces of curves

Let X be a complete hyperbolic Riemann surface of genus g with n punctures. In this talk we address the problem of estimating $s_X(L)$, the number of primitive *simple* closed geodesics of hyperbolic length less than L on X . By using some results on the symplectic geometry of moduli spaces of curves which will be discussed in the second talk, we show that $s_X(L)$ has the asymptotic behavior

$$s_X(L) \sim n_X L^{6g-6+2n}$$

as $L \rightarrow \infty$. we will also discuss the problem of the frequencies of different types of simple closed geodesics on hyperbolic surfaces and show that relative frequencies are universal rational numbers independent of X .

In the second talk, we discuss a recursive formula for the Weil-Petersson volume of the moduli space of hyperbolic Riemann surfaces with geodesic boundary components. We then use this formula to obtain a geometric proof of the Virasoro constraints for the intersection numbers on the moduli space of curves with marked points. We also relate these results to the frequencies of different types of simple closed geodesics on hyperbolic surfaces.