The Category of X-Nets

What are Networks ???

Networks = Graphs ?

What are Networks ???

Networks = Metric Spaces !

Why ?

The *lengths* of edges

- **is not only highly informative indicating the presumed relative time spans of evolutionary phases under investigation !**

- **more importantly: it is essential for many algorithms that are designed to searching for “network structures” whose edge lengths are “optimally” adapted to given data !**

More specifically:

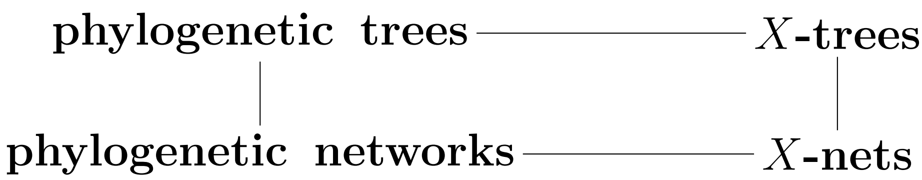
Our topic here is the structure of

X**-nets:**

A natural framework for taxonomic analysis in terms of *phylogenetic networks*

quite in analogy to X***-trees***

supporting taxonomic analysis in terms of (the much more familiar) *phylogenetic trees*.



Outline

- **Picture Show**
- **Basic Terminology**
- **Basic Results**
- X**-nets and** $\mathbb{R}$**-valued split systems**

Basic Terminology

(i) $\mathbf{M} = (M, D) : M \times M \rightarrow \mathbb{R} :$

$$(u, v) \mapsto uv$$

a (finite) metric space,

(ii) **Intervals:** $u, v \in M$**:**

$$[u, v] = [u, v]_D :=$$
$$\{w \in M : uv = uw + wv\}$$

(iii) **Parallel pairs:** $(u, u'), (v, v') \in M^2$:

$$uu' \| vv' \iff$$
$$u', v \in [u, v'] \text{ and } u, v' \in [u', v]$$

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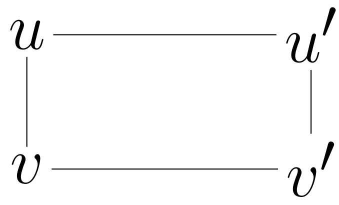

Basic Terminology

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(iv) M an (abstract) L_1-*space*

$$\iff$$

"$\|$" is an equivalence relation on M^2

Basic Terminology

(v) An **edge** in **M** (or an **M-edge**) is a two-subset $\{u, v\} \subseteq M$ with

$$\#[u, v] = 2.$$

$\Rightarrow$ Definition of **M-paths**:

A sequence $a_0, a_1, a_2, \ldots, a_k$ of point in **M** is an **M-path** if all consecutive pairs $\{a_{i-1}, a_i\}$ are **M**-edges

A sequence $a_0, a_1, a_2, \ldots, a_k$ of point in **M** is a **geodesic M-path** if it is an **M**-path and one has

$$a_0 a_k = \sum_{i=1}^{k} a_{i-1} a_i$$

Basic Terminology

(v) An **edge** in **M** (or an **M-edge**) is a two-subset $\{u, v\} \subseteq M$ with

$$\#[u, v] = 2.$$

$\Rightarrow$ Fundamental for relating graphs and metric spaces!

Basic Terminology

(v) An **edge** in **M** (or an **M-edge**) is a two-subset $\{u,v\} \subseteq M$ with

$$\#[u,v] = 2.$$

(vi) **M** is **bipartite** $\iff$

every edge $\{u,v\}$ is a **gated** subset of **M**,

i.e., $\forall\, x \in M \quad \exists\, y = y(x) \in \{u,v\}:$
$\forall\ w \in [u,v]:\ xw = xy + yw$

A Basic Result

A **net** is a bipartite L_1-space.

Theorem 1

A metric space $\mathbf{N}$ is a net $\iff$ it can be embedded isometrically into a **hypercuboid**, i.e., a product

$$\mathbf{M} := \mathbf{M}_1 \times \mathbf{M}_2 \times \cdots \times \mathbf{M}_k$$

of a metric spaces of cardinality 2 so that any two points in $\mathbf{N}$ can be connected by a geodesic path $a_0, a_1, a_2, \ldots, a_k$ in $\mathbf{M}$ all of whose points are points in $\mathbf{N}$.

Basic Terminology

A an X**- net** is a net $\mathbf{N} = (N, D)$ together with a (labelling) map

$$\psi : X \to N$$

with the property that, for every point $v \in N$ and any two edges e, e' in $\mathbf{N}$, there exists some $x \in X$ such that v and $\psi(x)$ can be connected by an $\mathbf{N}$-path not involving any edge that is parallel to either e or e'.

A **morphism**

$$\varphi : \mathbf{N} = (N, D; \psi) \to \mathbf{N}' = (N', D'; \psi')$$

from an X-net $\mathbf{N} = (N, D; \psi : X \to N)$ into an X-net $\mathbf{N}' = (N', D'; \psi' : X \to N')$ is a map φ from N into N' such that

$$D'(\varphi(u), \varphi(v)) \leq D(u, v)$$

holds for all $u, v \in N$,

$$D'(\varphi(u), \varphi(v)) + D'(\varphi(v), \varphi(w)) = D'(\varphi(u), \varphi(w))$$

holds for all $u, v, w \in V$ with $uv + vw = uw$, $\{\varphi(u), \varphi(v)\}$ is an $\mathbf{N}'$-edge for every $\mathbf{N}$-edge $\{u, v\}$ with $\varphi(u) \neq \varphi(v)$, and $\psi' = \varphi \circ \psi$ holds.

Basic Terminology

That is, morphisms are maps that are

non-expansive **(**$D'(\varphi(u), \varphi(v)) \leq D(u, v)$**),**

additive or **interval-preserving** **(**$v \in [u, w] \Rightarrow \varphi(v) \in [(\varphi(u), \varphi(w))]$**),**

edge-preserving ($\{u, v\}$ an $\mathbf{N}$-edge and $\varphi(u) \neq \varphi(v) \Rightarrow \{\varphi(u), \varphi(v)\}$ is an $\mathbf{N}'$-edge).

and labelling-compatible $(\psi'(v) = \varphi(\psi(v)))$.

More basic results

Theorem 2

(i) There exists at most one morphism from one X-net into another one.

(ii) A morphism from one X-net into another one is an isometry if and only both nets induce the same metric an X.

(iii) For every X-net $(\mathbf{N}, \psi)$, there exists an X-net $(\mathbf{N}, \psi)^*$, also called the **injective hull** of $(\mathbf{N}, \psi)$, together with an isometric embedding $\varphi_{(\mathbf{N},\psi)}$ from $(\mathbf{N}, \psi)$ into $(\mathbf{N}, \psi)^*$ such that, for every isometric embedding φ of $(\mathbf{N}, \psi)$ into another X-net $(\mathbf{N}', \psi')$, there exists a (necessarily unique) isometric embedding φ' from $(\mathbf{N}', \psi)$ into $(\mathbf{N}, \psi)^*$ with $\varphi_{(\mathbf{N},\psi)} = \varphi' \circ \varphi$

(iv) The underlying metric space $\mathbf{N}$ of X-net $(\mathbf{N}, \psi)$ is a median metric space if and only if $(\mathbf{N}, \psi)$ is its own injective hull.

Main Theorem

(i) Associating, to any edge $e = \{u, v\}$ in an X-net $(\mathbf{N}, \psi)$, the X**-split**

$$S_e := \{A_{u<v}, A_{v<u)}\}$$

whose two split halves $A_{u<v}$ and $A_{v<u}$ are those two subsets of X for which, for any x in $A_{u<v}$ or $A_{v<u}$, respectively, the point $\psi(x)$ is closer to u than to v or closer to v than to u, respectively, induces a **well-defined** and **injective** map from the set of equivalence classes of $\mathbf{N}$-edges of $\mathbf{N}$ into the set $\mathcal{S}(X)$ of **splits** of X.

(ii) Associating, to any X-net $(\mathbf{N}, \psi)$, the map $\mu = \mu_{(\mathbf{N},\psi)}$ from $\mathcal{S}(X)$ into $\mathbb{R}$ that maps any split of the form $S = S_e$ for some edge $e = \{u, v\}$ in $(\mathbf{N}, \psi)$ onto its **length** uv, and every other split of X onto 0, defines a **well-defined** map from the set of isomorphism classes of X-nets **onto** the set of all such maps.

(iii) And one has $\mu = \mu'$ for the maps μ, μ' associated with two X-nets $(\mathbf{N}, \psi)$and $(\mathbf{N}', \psi')$ if and only if the injective hulls of these two nets are isomorphic objects in the category of X-nets.