4 Short time existence and curvature estimates

We give a very rough description of how DeTurck's proof (a.k.a. **DeTurck's trick**) of short time existence of the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ on closed manifolds works. Recall that given a variation $\frac{\partial}{\partial s}g_{ij} = v_{ij}$ of a metric, the associated variation of the Ricci tensor is (see equation (10) in lecture 2)

$$-2\frac{\partial}{\partial s}R_{ij} = \Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i\left(\operatorname{div} v\right)_j - \nabla_j\left(\operatorname{div} v\right)_i$$

We search for a flow equivalent to the Ricci flow for which the linearization of the RHS is given by the laplacian operator. Note that the extra terms on the RHS may be written as a Lie derivative of the metric

$$\nabla_i\nabla_j V - \nabla_i\left(\operatorname{div} v\right)_j - \nabla_j\left(\operatorname{div} v\right)_i = \left(\mathcal{L}_X g\right)_{ij}$$

where

$$X = \frac{1}{2}\nabla V - \operatorname{div} v.$$

Motivated by the above discussion, given a fixed background Levi-Civita connection $\tilde{\Gamma}$ of a metric $\tilde{g}$, we define the **Ricci–DeTurck flow** by

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i\,, \\ g\left(0\right) &= g_0, \end{aligned} \tag{1}$$

where the time-dependent 1-form $W = W\left(g\right)$ is defined by

$$W_j \doteqdot g_{jk}g^{pq}\left(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k\right). \tag{2}$$

Note that if $g\left(s\right)$ is a one-parameter family of metrics with

$$\left.\frac{\partial}{\partial s}\right|_{s=0} g_{ij} = v_{ij}$$

and if $\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols of $g\left(0\right)$, then $W\left(g\left(0\right)\right) = 0$ and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} W\left(g\left(s\right)\right)_j = -X_j.$$

where $X = \frac{1}{2}\nabla V - \operatorname{div} v$ as above. We compute

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-2R_{ij} + \nabla_i W_j + \nabla_j W_i\right) = \Delta_L v_{ij}. \tag{3}$$

Exercise 1 *Verify formula (3).*

From (3) it follows that the Ricci-DeTurck flow is strictly parabolic and that given any smooth initial metric g_0 on a closed manifold, there exists a unique solution $g(t)$ to the Ricci-DeTurck flow with $g(0) = g_0$.

Now given a solution of the Ricci-DeTurck flow, we can solve the following ODE at each point in M :

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_t &= -W^* \\ \varphi_0 &= \mathrm{id}, \end{aligned} \tag{4}$$

where $W^*(t)$ is the vector field dual to $W(t)$ with respect to $g(t)$. Pulling back $g(t)$ by the diffeomorphisms φ_t, we obtain a solution

$$\bar{g}(t) \doteqdot (\varphi_t)^* g(t) \tag{5}$$

to the Ricci flow with $\bar{g}(0) = g_0$ (see p.81 of [1] for instance). One can also show that this solution is unique (p. 90 of [1]).

Theorem 2 (Hamilton, DeTurck - Short time existence) *If M^n is a closed Riemannian manifold and if g_0 is a C^∞ Riemannian metric, then there exists a unique smooth solution $\bar{g}(t)$ to the Ricci flow define on some time interval $[0,\delta)$, $\delta > 0$, with $\bar{g}(0) = g_0$.*

Now recall from (16) in Lecture 1 the evolution of the volume form is

$$\frac{\partial}{\partial t} d\mu = -R d\mu \tag{6}$$

and hence $\mathrm{Vol}(g) \doteqdot \int_{M^n} d\mu$ satisfies

$$\frac{d}{dt}\mathrm{Vol}(g(t)) = -\int_M R d\mu. \tag{7}$$

We normalize the Ricci flow and consider:

$$\frac{\partial}{\partial t}\hat{g}_{ij} = -2\hat{R}_{ij} + \frac{2}{n}\hat{r}\hat{g}_{ij} \tag{8}$$

where $\hat{r} = \mathrm{Vol}(\hat{g})^{-1} \cdot \int_{M^n} \hat{R}\, d\hat{\mu}$ is the average scalar curvature. Then

$$\mathrm{Vol}(\hat{g}(t)) \equiv \text{constant}. \tag{9}$$

Exercise 3 *Given a solution $g(t)$, $t \in [0,T)$, of the Ricci flow, show that the metrics $\hat{g}\left(\hat{t}\right) \doteqdot c(t)\, g(t)$, where*

$$c(t) \doteqdot \exp\left(\frac{2}{n}\int_0^t r(\tau)\, d\tau\right), \qquad \hat{t}(t) \doteqdot \int_0^t c(\tau)\, d\tau$$

form a solution of the normalized Ricci flow with $\hat{g}(0) = g(0)$.

The main result which started Ricci flow is the following classification of closed 3-manifolds with positive Ricci curvature by Hamilton in 1982.

Theorem 4 *If* (M^3, g_0) *is a closed Riemannian* 3*-manifold with* $\operatorname{Rc}(g_0) > 0$*, then there exists a unique solution* $g(t)$ *of the normalized Ricci flow with* $g(0) = g_0$ *for all* $t \geq 0$*. As* $t \to \infty$*, the metrics* $g(t)$ *converge exponentially fast in every* C^m*-norm to a* C^∞ *metric* g_∞ *with constant positive sectional curvature.*

Knowing that we have short time existence, one would like to understand the long time behavior of the solution. A fundamental tool in this study are pointwise estimates for the curvature tensor. Before we go to this, we recall that the Riemann curvature tensor may be considered as bundle map (operator)

$$\operatorname{Rm} : \wedge^2 M^n \to \wedge^2 M^n$$

defined by

$$\operatorname{Rm}(\alpha)_{ij} \doteqdot R_{ijk\ell}\alpha_{\ell k}. \tag{10}$$

We call Rm the curvature operator. We say that (M^n, g) has positive (nonnegative) curvature operator if the eigenvalues of Rm are positive (nonnegative); we denote this by $\operatorname{Rm} > 0$ ($\operatorname{Rm} \geq 0$).

In (11) of Lecture 2 we gave the formula for the evolution of Rm. We can simplify the appearance of this evolution equation as follows. Define the square of Rm by $\operatorname{Rm}^2 = \operatorname{Rm} \circ \operatorname{Rm} : \wedge^2 M^n \to \wedge^2 M^n$. There is another quadratic related to the Lie algebra structure on $\wedge^2 M^n \cong \mathfrak{so}(n)$ defined by

$$[U, V]_{ij} \doteqdot g^{k\ell}\left(U_{ik}V_{\ell j} - V_{ik}U_{\ell j}\right)$$

for $U, V \in \wedge^2 M^n$. Let $\{\varphi^\alpha\}$ be a basis of $\wedge^2 M^n$ and $C_\gamma^{\alpha\beta}$ be the structure constants defined by

$$\left[\varphi^\alpha, \varphi^\beta\right] \doteqdot \sum_\gamma C_\gamma^{\alpha\beta}\varphi^\gamma.$$

The Lie algebra square

$$\operatorname{Rm}^\# : \wedge^2 M^n \to \wedge^2 M^n$$

is defined by

$$(\operatorname{Rm}^\#)_{\alpha\beta} \doteqdot C_\alpha^{\gamma\delta} C_\beta^{\varepsilon\zeta} \operatorname{Rm}_{\gamma\varepsilon} \operatorname{Rm}_{\delta\zeta}.$$

Since Rm is self-adjoint, we can choose $\{\varphi^\alpha\}$ so that Rm is diagonal. Then for any 2-form η, we have $(\operatorname{Rm}^\#)_{\alpha\beta}\eta^\alpha\eta^\beta = \left(C_\alpha^{\gamma\delta}\eta^\alpha\right)^2 \operatorname{Rm}_{\gamma\gamma} \operatorname{Rm}_{\delta\delta}$. Hence, if $\operatorname{Rm} \geq 0$, then $\operatorname{Rm}^\# \geq 0$ and hence $\operatorname{Rm}^2 + \operatorname{Rm}^\# \geq 0$.

The evolution equation for Rm may be rewritten as:

Lemma 5

$$\boxed{\frac{\partial}{\partial t}\operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^\#.} \tag{11}$$

Rm is a section of the bundle $\pi : \wedge^2 M^n \otimes_S \wedge^2 M^n \to M$, which has a natural bundle metric and connection induced by the Riemannian metric and connection on TM. Let $\pi^{-1}(x)$ denote the fiber over x. For each $x \in M$, consider the system of ODE on $\pi^{-1}(x)$ corresponding to the PDE (11) obtained by dropping the laplacian term:

$$\boxed{\frac{d}{dt}\mathbf{M} = \mathbf{M}^2 + \mathbf{M}^{\#}} \tag{12}$$

where $\mathbf{M} \in \pi^{-1}(x)$ is a symmetric $N \times N$ matrix, where $N = \frac{n(n-1)}{2} = \dim \mathfrak{so}(n)$. The maximum principle for systems says the following. A set K in a vector space is said to be convex if for any $X, Y \in K$, we have $sX + (1-s)Y \in K$ for all $s \in [0,1]$. A subset K of the vector bundle E is said to be invariant under parallel translation if for every path $\gamma : [a,b] \to M$ and vector $X \in K \cap E_{\gamma(a)}$, the unique parallel section $X(s) \in E_{\gamma(s)}$, $s \in [a,b]$, along $\gamma(s)$ with $X(a) = X$ is contained in K.

Theorem 6 *Let $g(t)$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold M^n. Let $K \subset E$ be a subset which is invariant under parallel translation and whose intersection $K_x \doteqdot K \cap E_x$ with each fiber is closed and convex. Suppose the* ODE *(12) has the property that for any $\mathbf{M}(0) \in K$, we have $\mathbf{M}(t) \in K$ for all $t \in [0,T)$. If $\mathrm{Rm}(0) \in K$, then $\mathrm{Rm}(t) \in K$ for all $t \in [0,T)$.*

By (11), we have the following.

Corollary 7 (Positive (nonnegative) curvature operator is preserved) *If $(M^n, g(t))$, $t \in [0,T)$, is a solution to the Ricci flow on a closed manifold with $\mathrm{Rm}(g(0)) \geq 0$ (or $\mathrm{Rm}(g(0)) > 0$), then $\mathrm{Rm}(g(t)) \geq 0$ (or $\mathrm{Rm}(g(t)) > 0$) for all $t \in [0,T)$.*

In dimension 3, if $\mathbf{M}(0)$ is diagonal, then $\mathbf{M}(t)$ stays diagonal under the ODE. Let $\lambda_1(\mathbf{M}) \leq \lambda_2(\mathbf{M}) \leq \lambda_3(\mathbf{M})$ be the eigenvalues of $\mathbf{M}$. Under the ODE the ordering of the eigenvalues is preserved and we have

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_2\lambda_3 \\ \frac{d\lambda_2}{dt} &= \lambda_2^2 + \lambda_1\lambda_3 \\ \frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1\lambda_2. \end{aligned} \tag{13}$$

References

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