10 Homogeneous solutions in dimension 3

A **Lie group** is a C^∞ manifold G with a group structure such that $\mu : G\times G \to G$, defined by $\mu(\sigma,\tau) = \sigma\cdot\tau^{-1}$ is C^∞. Given $\sigma \in G$, we define left multiplication $\sigma_L : G \to G$ by $\sigma_L(\tau) = \sigma\cdot\tau$ and right multiplication σ, $\sigma_R : G \to G$ by $\sigma_R(\tau) = \tau\cdot\sigma$.

Definition 1 *A Riemannian metric g on G is **left-invariant** if for any $\sigma \in G$, σ_L is an isometry of (G,g): $(\sigma_L)^* g = g$.*

The connection and curvature of a left-invariant metric may be computed algebraically as follows.

Lemma 2 *If g is a left-invariant metric on G and if X, Y, Z, W are left-invariant vector fields, then*

1.
$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle\right), \tag{1}$$

2.
$$\langle \mathrm{Rm}(X,Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \left\langle \nabla_{[X,Y]} Z, W\right\rangle. \tag{2}$$

We leave the proofs as exercises. For example the formula for the Levi-Civita connection follows from the formula for the connection given in lecture 1 and the fact that $X\langle Y,Z\rangle = 0$, etc.

Left-invariant metrics on Lie groups (and more generally, homogeneous metrics) provide nice examples to study the behavior of the Ricci flow.

A Riemannian manifold (M^n, g) is (globally) **homogeneous** if for every $x, y \in M^n$ there exists an isometry $\iota : M^n \to M^n$ with $\iota(x) = y$. A homogeneous manifold looks the same at every point (but not necessarily in every direction). A Lie group with a left-invariant metric is homogeneous. Suppose G is a 3-dimensional **unimodular** (i.e., its volume form is bi-invariant) Lie group with a left invariant metric g. Then there exists a left-invariant frame field $\{f_i\}_{i=1}^3$ with dual coframe field $\left\{\eta^i\right\}_{i=1}^3$ such that there are positive constants A, B, C where the metric is diagonal:

$$g = A\eta^1\otimes\eta^1 + B\eta^2\otimes\eta^2 + C\eta^3\otimes\eta^3$$

and the Lie brackets are of the form:

$$[f_i, f_j] = c_{ij}^k f_k$$

where $c_{ij}^k \in \{1, 0, -1\}$ and $c_{ij}^k = 0$ unless i, j, k are distinct. (See [7].) Let $\lambda \doteqdot c_{23}^1$, $\mu \doteqdot c_{31}^2$, $\nu \doteqdot c_{12}^3$. The frame field $\{e_i\}_{i=1}^3$ defined by $e_1 \doteqdot A^{-1/2} f_1$,

$e_2 \doteqdot B^{-1/2} f_2$, $e_3 \doteqdot C^{-1/2} f_3$, is orthonormal. The Ricci tensor is diagonal and given by:

$$\begin{aligned}\mathrm{Rc}\left(e_1, e_1\right) &= \frac{\lambda^2 A^2 - \left(\mu B - \nu C\right)^2}{2ABC} \\ \mathrm{Rc}\left(e_2, e_2\right) &= \frac{\mu^2 B^2 - \left(\nu C - \lambda A\right)^2}{2ABC} \\ \mathrm{Rc}\left(e_1, e_1\right) &= \frac{\nu^2 C^2 - \left(\lambda A - \mu B\right)^2}{2ABC}.\end{aligned}$$

Hence the Ricci flow equation is equivalent to the following system:

$$\begin{aligned}\frac{dA}{dt} &= \frac{\left(\mu B - \nu C\right)^2 - \lambda^2 A^2}{BC} \\ \frac{dB}{dt} &= \frac{\left(\nu C - \lambda A\right)^2 - \mu^2 B^2}{AC} \\ \frac{dC}{dt} &= \frac{\left(\lambda A - \mu B\right)^2 - \nu^2 C^2}{AB}.\end{aligned}$$

We first consider the case when the Lie group G is $\mathrm{SU}\left(2\right)$. In this case there exists a frame such that $\lambda = \mu = \nu = -2$.

Exercise 3 *Show that for a left-invariant metric on* $\mathrm{SU}\left(2\right)$

$$R = 2\frac{A^2 + B^2 + C^2 - \left(B - C\right)^2 - \left(A - C\right)^2 - \left(A - B\right)^2}{ABC}.$$

We have the following (see p. 728-9 of [3]).

Theorem 4 (Isenberg-Jackson) *For any left-invariant initial metric g_0 on* $\mathrm{SU}\left(2\right)$ *there exists a solution $g\left(t\right)$ of the normalized Ricci flow defined for all $t \in [0, \infty)$ with $g\left(0\right) = g_0$ such that $g\left(t\right)$ converges to a constant positive sectional curvature metric as $t \to \infty$.*

Under the normalized Ricci flow, the quantity $\left(ABC\right)\left(t\right)$ is proportional to the volume and hence independent of time. Thus we may assume without loss of generality $\left(ABC\right)\left(t\right) \equiv 8/3$. Using this one can show that on $\mathrm{SU}\left(2\right)$ the volume normalized Ricci flow equations become

$$\frac{dA}{dt} = A\left(A\left(B + C - 2A\right) + \left(B - C\right)^2\right) \tag{3}$$

$$\frac{dB}{dt} = B\left(B\left(A + C - 2B\right) + \left(A - C\right)^2\right) \tag{4}$$

$$\frac{dC}{dt} = C\left(C\left(A + B - 2C\right) + \left(A - B\right)^2\right). \tag{5}$$

From this we may compute the evolution equations for the difference:

$$\begin{aligned}\frac{d}{dt}\left(A - C\right) &= 2C^3 - 2A^3 + AB^2 + A^2 B - BC^2 - B^2 C \\ &= \left(-2\left(A^2 + AC + C^2\right) + B^2 + B\left(A + C\right)\right)\left(A - C\right) \qquad (6)\end{aligned}$$

and similarly for $\frac{d}{dt}(A-B)$ and $\frac{d}{dt}(B-C)$. We assume without loss of generality

$$A(0) \geq B(0) \geq C(0).$$

By the above equations, we have

$$A(t) \geq B(t) \geq C(t)$$

for all $t > 0$. Since $A + B - 2C \geq 0$, equation (5) implies $\frac{dC}{dt} \geq 0$ so that $C(t) \geq C(0)$.

Now we may estimate the factor on the RHS of (6)

$$\begin{aligned} & -2\left(A^2 + AC + C^2\right) + B^2 + B(A+C) \\ &= -2C^2 - \left(A^2 - B^2\right) - AC - (A+C)(A-B) \\ &\leq -2C^2 \leq -2C(0)^2. \end{aligned}$$

Thus

$$\frac{d}{dt}(A-C) \leq -2C(0)^2(A-C).$$

We conclude that

$$A(t) - C(t) \leq (A(0) - C(0))\, e^{-2C(0)^2 t}$$

for all $t > 0$. That is, $A - C$ decays exponentially to zero. Since $A \geq B \geq C$ and we have normalized the volume so that $ABC \equiv 8/3$, we conclude that $A(t), B(t), C(t)$ exponentially converge to $A_\infty = B_\infty = C_\infty \doteqdot 2/\sqrt[3]{3}$. That is, $g(t)$ exponentially converges as $t \to \infty$ to a constant sectional curvature metric.

Next we consider the case where G is the Heisenberg group (Nil) of upper-triangular 3×3 real matrices. In this case there is a frame where $\lambda = -2$ and $\mu = \nu = 0$ and the (unnormalized) Ricci flow equations are

$$-\frac{d}{dt}\log A = \frac{d}{dt}\log B = \frac{d}{dt}\log C = 4\frac{A}{BC}. \tag{7}$$

We immediately see that A is decreasing whereas B and C are increasing. In fact B/C, AB and AC are independent of time. We compute

$$\frac{d}{dt}\log\left(\frac{A}{BC}\right) = -12\frac{A}{BC}$$

so that

$$\frac{A}{BC}(t) = \frac{1}{12}\left(\frac{B_0 C_0}{12 A_0} + t\right)^{-1}$$

where $A_0 \doteqdot A(0)$, $B_0 \doteqdot B(0)$, $C_0 \doteqdot C(0)$. Thus one can explicitly solve (7) to get

$$\frac{A_0}{A(t)} = \frac{B(t)}{B_0} = \frac{C(t)}{C_0} = \left(1 + \frac{12 A_0}{B_0 C_0} t\right)^{1/3}.$$

Exercise 5 *Show that the sectional curvatures are*

$$K(e_2 \wedge e_3) = -\frac{3A}{BC} = -3\left(\frac{B_0C_0}{A_0} + 12t\right)^{-1}$$
$$K(e_3 \wedge e_1) = \frac{A}{BC} = \left(\frac{B_0C_0}{A_0} + 12t\right)^{-1}$$
$$K(e_1 \wedge e_2) = \frac{A}{BC} = \left(\frac{B_0C_0}{A_0} + 12t\right)^{-1}$$

and hence satisfy $|\mathrm{sect}(g(t))| \leq \mathrm{const} \cdot t^{-1}$.

Note that the scalar curvature is negative as it must be for the solution to exist for all time. If we consider a compact quotient of the Heisenberg group, such as $G/\mathbb{Z}^3$, the diameters satisfy $\mathrm{diam}(g(t)) \leq \mathrm{const} \cdot (t+1)^{1/6}$. Hence

$$|\mathrm{sect}(g(t))| \,\mathrm{diam}(g(t))^2 \leq \mathrm{const} \cdot t^{-2/3} \to 0 \quad \text{as } t \to \infty. \tag{8}$$

Thus the solution **collapses** as time tends to infinity. In fact, (8) says that the metrics become more and more **almost flat**. The only nonzero bracket is $[e_2, e_3] = -\sqrt{\frac{A}{BC}}e_1 \approx t^{-1/2}e_1$, so that the brackets tend to zero when measured in an orthonormal frame.

See [1], [3], [2], [5] and [6] for further discussion of the Ricci flow of homogeneous 3-manifolds and related **quasi-stability** questions (some in the category of warped products). Besides $\mathrm{SU}(2)$ and Nil, some other 3-dimensional homogeneous geometries correspond to Solv, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\widetilde{\mathrm{Isom}}(\mathbb{R}^2)$ and $\mathbb{E}^3$ (see [8]). In the case of Solv and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, homogeneous solutions exist for all time and $|\mathrm{Rm}| \leq Ct^{-1}$. In the case of $\widetilde{\mathrm{Isom}}(\mathbb{R}^2)$ the solution exists for all time and $|\mathrm{Rm}| \leq Ce^{-ct}$ while for $\mathbb{E}^3$ the solution is flat and hence stationary. These geometries roughly correspond to Thurston's model geometries, but are not in 1-1 correspondence, since the geometries he considers have maximal isotropy groups. See Isenberg-Jackson-Lu [4] for work on the Ricci flow on 4-dimensional homogeneous spaces.

References

[1] Carfora, M.; Isenberg, James; Jackson, Martin. *Convergence of the Ricci flow for metrics with indefinite Ricci curvature.* J. Diff. Geom. **31** (1990) 249-263.

[2] Hamilton, Richard; Isenberg, James. *Quasi-convergence of Ricci flow for a class of metrics.* Comm. Anal. Geom. **1** (1993), no. 3-4, 543–559.

[3] Isenberg, James; Jackson, Martin. *Ricci flow of locally homogeneous geometries on closed manifolds.* J. Differential Geom. **35** (1992), no. 3, 723–741.

[4] Isenberg, James; Jackson, Martin; Lu, Peng. *Ricci flow on locally homogeneous closed 4-manifolds.* To appear in Comm. Anal. Geom. arXiv:math.DG/0502170.

[5] Knopf, Dan. *Quasi-convergence of the Ricci flow.* Comm. Anal. Geom. **8** (2000), no. 2, 375-391.

[6] Knopf, Dan; McLeod, Kevin. *Quasi-convergence of model geometries under the Ricci flow.* Comm. Anal. Geom. **9** (2001), no. 4, 879–919.

[7] Milnor, John. *Curvatures of left invariant metrics on Lie groups.* Advances in Math. **21** (1976), no. 3, 293–329.

[8] Thurston, William P. *Three-dimensional geometry and topology.* Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, **35**. Princeton University Press, Princeton, NJ, 1997.