15 The Hamilton-Ivey 3-dimensional curvature pinching estimate and some consequences

We consider the curvature estimate of Hamilton and Ivey. The motivation for the form of this estimate comes from considering the case where $\left(M^3, g(t)\right)$ is rotationally symmetric and forming a neck pinch. In the region of the neck, the two smallest eigenvalues are equal and negative and the largest eigenvalue is positive:

$$\lambda_1(\mathrm{Rm}) = \lambda_2(\mathrm{Rm}) < 0 < \lambda_3(\mathrm{Rm}).$$

These inequalities are preserved under the following ODE system

$$\begin{aligned}\frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1^2 \\ \frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_1\lambda_3,\end{aligned}$$

which follows from (13) in lecture 4:

$$\begin{aligned}\frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_2\lambda_3 \\ \frac{d\lambda_2}{dt} &= \lambda_2^2 + \lambda_1\lambda_3 \\ \frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1\lambda_2,\end{aligned} \tag{1}$$

and where $\lambda_i = \lambda_i(\mathbf{M})$. Solving the homogeneous equation

$$\frac{d\lambda_1}{d\lambda_3} = \frac{\lambda_1^2 + \lambda_1\lambda_3}{\lambda_3^2 + \lambda_1^2},$$

we have

$$\log(-\lambda_1) = \frac{\lambda_3}{-\lambda_1} + 2\log\left(\frac{-\lambda_1}{\lambda_3 - \lambda_1}\right) + C \tag{2}$$

for some constant C. From this one can show that if $-\lambda_1$ is sufficiently large, then

$$\lambda_3 > -\lambda_1 \log(-\lambda_1). \tag{3}$$

Exercise 1 *Derive (2) and deduce (3).*

Before we state the **Hamilton-Ivey estimate**, we recall an analogue of Theorem 6 in lecture 4, which holds for K depending on t (see [3] or [1] for the proof). Let $E = \wedge^2 M^n \otimes_S \wedge^2 M^n$.

Proposition 2 (Maximum principle for time-dependent convex sets) *Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold M^n and let $K(t) \subset E$ be subsets which are invariant under parallel translation and* *whose intersections* $K(t)_x \doteqdot K(t) \cap E_x$ *with each fiber are closed and convex. Suppose*

$$\{(v,t) \in E \times [0,T) : v \in K(t)\}$$

is closed in $E \times [0,T)$ *and suppose the* ODE *(1) has the property that for any* $\mathbf{M}(t_0) \in K(t_0)$, *we have* $\mathbf{M}(t) \in K(t)$ *for all* $t \in [t_0, T)$. *If* $\mathrm{Rm}(0) \in K$, *then* $\mathrm{Rm}(t) \in K$ *for all* $t \in [0,T)$.

The main estimate of this section is the following.

Theorem 3 (Hamilton-Ivey 3-d curvature estimate) *Let* $(M^3, g(t))$ *be a solution of the Ricci flow on a closed* 3*-manifold for* $0 \leq t < T$. *If we normalize the initial metric so that* $\lambda_1(\mathrm{Rm})(x,0) \geq -1$ *for all* $x \in M^3$, *then at any point* $(x,t) \in M^3 \times [0,T)$ *where* $\lambda_1(\mathrm{Rm})(x,t) < 0$, *we have*

$$\boxed{R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| + \log(1+t) - 3\right).} \tag{4}$$

In particular,

$$R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| - 3\right). \tag{5}$$

Remark 4

1. *This implies that in a scaled sense, the sectional curvatures of any solution to the Ricci flow on a closed 3-manifold tends to nonnegative (see the remarks after the proof of the theorem).*

2. *By scaling the solution, we see that if we assume*

$$\lambda_1(\mathrm{Rm})(x,0) \geq -C$$

for all $x \in M^3$, *where* $C > 0$, *then*

$$R \geq |\lambda_1(\mathrm{Rm})| \left(\log|\lambda_1(\mathrm{Rm})| + \log\left(C^{-1} + t\right) - 3\right) \tag{6}$$

at all (x,t) *where* $\lambda_1(\mathrm{Rm})(x,t) < 0$.

Proof. The proof depends on defining a suitable subset of the bundle E which is invariant under parallel translation, fiberwise convex and invariant under the ODE. Define

$$K(t) \doteqdot \left\{ \mathbf{M} : \begin{array}{c} \lambda_1 + \lambda_2 + \lambda_3 \geq -3/(1+t) \\ \text{and if } \lambda_1 \leq -1/(1+t), \text{ then} \\ \lambda_1 + \lambda_2 + \lambda_3 \geq -\lambda_1 \left(\log(-\lambda_1) + \log(1+t) - 3\right) \end{array} \right\} \subset E.$$

For each t, $K(t)$ is invariant under parallel translation and for each (x,t), $K_x(t)$ is closed and convex (see [5] or [2], p. 258-260). By our assumption, it is easy to see that $\mathrm{Rm}[g(0)] \in K(0)$. The theorem will follow from showing $\mathrm{Rm}[g(t)] \in K(t)$ for all $t \in [0,T)$. Indeed, if $-1/(1+t) < \lambda_1(\mathrm{Rm}) < 0$, then (4) follows directly, and when $\lambda_1(\mathrm{Rm}) \leq -1/(1+t)$, then (4) follows from $\mathrm{Rm}[g(t)] \in K(t)$.

From dropping the factor of 2 on the RHS of the sum of (1), we have

$$\frac{d}{dt}\left(\lambda_1+\lambda_2+\lambda_3\right)\geq\frac{1}{3}\left(\lambda_1+\lambda_2+\lambda_3\right)^2.$$

This implies

$$\lambda_1+\lambda_2+\lambda_3\geq -3/\left(1+t\right) \tag{7}$$

is preserved under the ODE (1); that is, if (7) holds at time t_0, then (7) holds for $t\geq t_0$.

Given $\mathbf{M}$ with $\lambda_1<0$, define

$$\phi\left(\mathbf{M}\right)=\frac{\lambda_1+\lambda_2+\lambda_3}{-\lambda_1}-\log\left(-\lambda_1\right).$$

Claim. Under the ODE (1)

$$\frac{d\phi}{dt}\geq-\lambda_1. \tag{8}$$

By (**??**),

$$\lambda_1^2\frac{d\phi}{dt}=-\lambda_1^3-\lambda_1\left(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3\right)+\left(\lambda_2+\lambda_3\right)\lambda_2\lambda_3.$$

On one hand, if $\lambda_2<0$, then

$$\lambda_1^2\frac{d\phi}{dt}=-\lambda_1^3-\lambda_2^3+\left(\lambda_2-\lambda_1\right)\left(\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3\right)\geq-\lambda_1^3.$$

On the other hand, if $\lambda_2\geq 0$, then

$$\lambda_1^2\frac{d\phi}{dt}=-\lambda_1^3-\lambda_1\lambda_2^2+\left(\lambda_2-\lambda_1\right)\left(\lambda_3^2+\lambda_2\lambda_3\right)\geq-\lambda_1^3.$$

By (8), if $\lambda_1\left(\mathbf{M}\right)\leq -1/\left(1+t\right)$, then

$$\frac{d}{dt}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{-\lambda_1}-\log\left(-\lambda_1\right)-\log\left(1+t\right)\right)\geq 0.$$

This is the estimate we need to show that $K\left(t\right)$ is preserved by the ODE (1). By Proposition 2, we conclude $\mathrm{Rm}\left[g\left(t\right)\right]\in K\left(t\right)$ for all $t\in[0,T)$. ■

From (5) we see that if $\lambda_1\left(\mathrm{Rm}\right)\leq -e^{C+3}$ for some constant $C>0$, then

$$\left|\lambda_1\left(\mathrm{Rm}\right)\right|\leq C^{-1}R\leq 3C^{-1}\lambda_3\left(\mathrm{Rm}\right).$$

That is, if we have a large negative sectional curvature, then we have a much larger positive sectional curvature. Note also that if $\lambda_1\left(\mathrm{Rm}\right)\leq -e^6$, then

$$R\geq\frac{1}{2}\left|\lambda_1\left(\mathrm{Rm}\right)\right|\log\left|\lambda_1\left(\mathrm{Rm}\right)\right|.$$

The Hamilton-Ivey estimate implies that limits of dilations about finite time singularities of the Ricci flow on closed 3-manifolds have nonnegative sectional curvature.

References

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