Student's Guide to Ricci Flow I, II, III lectures

Note: actual lectures may differ from the notes.

Ricci Flow I - Bennett Chow

Hamilton's 3-manifolds with positive Ricci curvature theorem: background and basic techniques used in its proof - linearization of the Ricci tensor, short time existence, basic evolution equations, maximum principles, curvature pinching estimates, convergence criteria.

1. Connections, curvatures, and variation formulas

We review basic formulas in Riemannian geometry with emphasis on local coordinate calculations and variation formulas. We also discuss the Lie derivative of the metric and the Einstein-Hilbert (total scalar curvature) functional.

2. The Ricci flow equation and associated equations

We introduce the Ricci flow equation and compute some basic evolution equations (for the Christoffel symbols and curvatures) under the Ricci flow. We find that the curvatures satisfy heat type equations. We also compute the variation of the Ricci tensor and introduce the Lichnerowicz laplacian operator. Some exercises on computing the commutation of covariant derivatives are included.

3. Heat equations and maximum principles

Since the study of the Ricci flow involves studying heat type equations for geometric quantities, the maximum principle is an important tool. We first discuss maximum principles for scalar heat type equations and then maximum principles for heat type equations for symmetric 2-tensors (such as the Ricci tensor). In the tensor case, we present a sufficient condition for positivity of a tensor to be preserved and applying this to the Ricci tensor in dimension 3.

4. Short time existence and curvature estimates

We discuss DeTurck's trick for proving short time existence of the Ricci flow on closed manifolds. We introduce the volume normalized Ricci flow and state Hamilton's 1982 theorem on the classification of closed 3-manifolds with positive Ricci curvature. We consider the evolution equation for the Riemann curvature tensor and relate this to the Lie algebra structure on the fibers of the bundle of 2-forms. In dimension 3 we consider the ode corresponding to the pde for the curvature obtained by dropping the laplacian term.

5. Convergence of Ricci flow for closed 3-manifolds with positive Ricci curvature

We present some aspects of the Hamilton's proof of the classification of closed 3-manifolds with positive Ricci curvature. Our focus is on pointwise curvature estimates and the application of the maximum principle for tensors (systems). This reduces the study of the pde for the curvature to a suitable analysis of the corresponding ode. Long time existence, exponential convergence, and derivative estimates are stated but not proved.

Supplement A. Divergence theorem and integration by parts

We discuss the basic technique of applying the divergence theorem and performing integration by parts. These techniques are used in Perelman's proof of his entropy formula and will be discussed in the lectures of Kleiner/Lott/Tian.

Ricci Flow II - Bennett Chow

Special solutions: Ricci solitons and homogeneous solutions - gradient Ricci solitons and basic associated formulas, examples: cigar soliton, expanding soliton on $\mathbb{R}^2$, Bryant soliton, Rosenau solution, homogeneous solutions in dimension 3.

6. Gradient Ricci solitons, related monotonicity on surfaces and the Kazdan-Warner identity

We discuss self-similar solutions of the Ricci flow, also called Ricci solitons. In low dimensions (2 and 3) the only examples on closed manifolds are Einstein solutions. We discuss the Kazdan-Warner identity (used in one of the proofs of nonexistence of shrinking 2-dimensional solitons) and the entropy formula for Hamilton's surface entropy. The latter is related to gradient shrinking solitons.

7. The cigar soliton, the Rosenau solution, and moving frames calculations

We discuss the fundamental cigar steady soliton. For this solution, although the metric evolves, all of the metrics are isometric to each other. Before Perelman's work, the possibility of a singularity of this type was one of the stumbling blocks to Hamilton's program on 3-manifolds. We also discuss another ancient (existing from time negative infinity) solution on the 2-sphere, called the Rosenau solution and related to the cigar solution. To perform the calculations in this lecture we use the method of moving frames.

8. Expanding soliton on $\mathbb{R}^2$, the 3-dimensional Bryant soliton, and no closed 3-dimensional shrinkers

In this lecture we discuss an expanding rotationally symmetric Ricci soliton on a topological $\mathbb{R}^2$. This example is essentially explicit, has positive curvature and is cone-like at infinity. We also discuss the higher dimensional Bryant steady soliton which is qualitatively different than its 2-dimensional cousin the cigar. In dimensions 3 and above, the Bryant soliton is like a paraboloid and the curvature decays linear in the distance to the origin.

9. Basic formulas associated to the gradient Ricci soliton equation

We present a result which puts gradient Ricci solitons in canonical form. We derive associated formulas for gradient Ricci solitons by differentiating (in different ways) the Ricci soliton equation. This section is partly in preparation for the monotonicity formulas presented in other lectures for this summer school.

10. Homogeneous solutions in dimension

We consider the special class of solutions where the metrics are homogeneous. In this case the Ricci flow pde reduces to a system of odes which are more easy to analyze. First we recall some basic connection and curvature formulas for left-invariant metrics on Lie groups. Then we analyze the 3-dimensional examples of $SU(2)$ and the Heisenberg group.

Ricci Flow III - Bennett Chow

Analytic and geometric techniques: more maximum principle and monotonicity — Li-Yau Harnack estimate for the heat equation, Hamilton's Harnack estimates for the Ricci flow, consequences for eternal solutions, Shi's local and global derivative estimates, Hamilton-Ivey estimate and its consequences.

11. The Li-Yau differential Harnack estimate for the heat equation

In this lecture our discussion turns more analytic in nature. We use the heat equation as a model for some types of estimates for the Ricci flow. Of fundamental importance is the Li-Yau differential Harnack estimate for the heat equation on Riemannian manifolds. This estimate takes the form of a gradient estimate. The precise form of the estimate in the case of nonnegative Ricci curvature is easy to motivate using the explicit euclidean heat kernel as an example. Integrating the gradient estimate (using the fundamental theorem of calculus) yields a pointwise comparison of the solution at different points in space-time. Such estimates are more generally known as Harnack inequalities. In the appendix we carry out some of the calculations in more detail.

12. Hamilton's trace Harnack estimate for the Ricci flow on surfaces and its consequences

What is somewhat miraculous is that the Li-Yau gradient estimate has analogues for the Ricci flow. Since the general estimate of Hamilton at first appears rather complicated we first consider the case of surfaces where the link between Hamilton's and Li-Yau's estimates is more evident. We present Hamilton's scalar Harnack estimate. We also show that there is a matrix estimate. This so-called matrix estimate should be thought of as a hessian estimate whereas the scalar estimate is like a laplacian estimate.

13. More gradient Ricci solitons, Hamilton's matrix Harnack estimate for the Ricci flow and eternal solutions

We consider Hamilton's general dimensional matrix Harnack estimate which holds for solutions with bounded positive curvature operator. To motivate the form of this estimate we consider the associated equations for gradient expanding solitons. We then give a corollary of the matrix inequality which is a more digestible scalar inequality very similar to the 2-dimensional scalar Harnack estimate.

14. Shi's local and global derivative estimates

Bounds on the curvature gives bounds on the derivatives of curvature (as long as we wait a little time). There are both global and local versions of this result. They reflect the smoothing property of the Ricci flow. For the global version we sketch the proof of the first derivative estimate. We then give applications to the long time existence theorem which says that a solution to the Ricci flow exists as long as the pointwise norm of the Riemann curvature tensor is bounded.

15. The Hamilton-Ivey 3-dimensional curvature pinching estimate and some consequences

In dimension 3 the Ricci flow makes curvatures tend to positive in a sense. One has to be careful in what one means by this. The precise statement is an estimate of Hamilton and Ivey which is a nice application of the maximum principle for systems and the analysis of the ode system corresponding to the pde system satisfied by the curvature tensor. A consequence of this result is roughly if a sectional curvature tends to negative infinity at points and times, then at these same points and times another sectional curvature tends to positive infinity much faster. This simple result has important applications to the study of singularities in dimension 3.