Contact and Symplectic Geometry of Monge-Ampère Equations: Introduction and Examples

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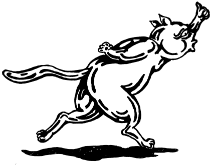
Bibliography:



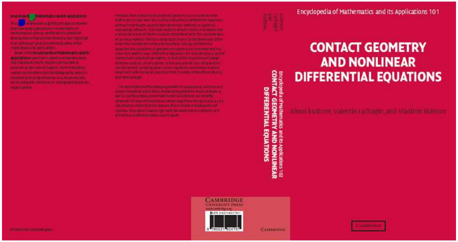


Figure: Cambridge University Press, 2007

Basic object

Figure: Monge and Ampère

$$A\frac{\partial^2 f}{\partial q_1^2} + 2B\frac{\partial^2 f}{\partial q_1 \partial q_2} + C\frac{\partial^2 f}{\partial q_2^2} + D(\frac{\partial^2 f}{\partial q_1^2} \cdot \frac{\partial^2 f}{\partial q_2^2} - (\frac{\partial f}{\partial q_1 \partial q_2})^2) + E = 0$$

Global Solutions: Monge

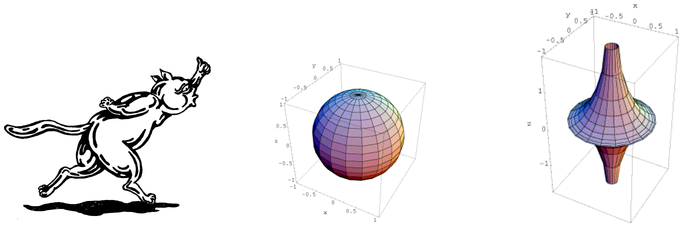


Figure: sphere and pseudosphere

An example: curvature of a surface in $\mathbb{R}^3$

$$\frac{u_{q_1 q_1} \cdot u_{q_2 q_2} - u_{q_1 q_2}^2}{(1 + u_{q_1}^2 + u_{q_2}^2)^2} = \mathcal{K}(u)$$

Main idea

- Let $F : \mathbb{R}^n \to (i)\mathbb{R}^n$ be a vector-function and its graph is a subspace in $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus (i)\mathbb{R}^n$.

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- This graph is a Lagrangian subspace in $T^*(\mathbb{R}^n)$ iff $(dF)_x$ is a symmetric endomorphism . The matrix $\|\ \frac{\partial F_i}{\partial x_j}\ \|$ is symmetric $\forall x$ iff the differential form $\sum_i F_i dx_i \in \Lambda^1(\mathbb{R}^n)$ is closed or, equivalently, exact:

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$$F_i = \frac{\partial f}{\partial x_i} \Longrightarrow F = \nabla f.$$

- The projection of the graph of ∇f on $(\mathbb{R}^n)_x$ is given in coordinates by $\nabla^2(f) = \det || \frac{\partial^2 f_i}{\partial x_j^2} ||$.

Symplectic Linear Algebra Digression

- Let (V, Ω) be a symplectic $2n$-dimensional vector space over $\mathbb{R}$ and $\Lambda^*(V^*)$ the space of exteriors forms on V. Let $\Gamma : V \to V^*$ be the isomorphism determined by Ω and let $X_\Omega \in \Lambda^2(V)$ be the unique bivector such that $\Gamma^*(X_\Omega) = \Omega$.

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- We introduce the operators $\top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*)$, $\omega \mapsto \omega \wedge \Omega$ and $\perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*)$, $\omega \mapsto i_{X_\Omega}(\omega)$. They have the followings properties:

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- $$\begin{cases} [\perp, \top](\omega) = (n-k)\omega \ , \ \forall \omega \in \Lambda^k(V^*); \\ \perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*) \text{ is into for } k \geq n+1; \\ \top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*) \text{ is into for } k \leq n-1. \end{cases}$$

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- A k-form ω is effective if $\bot\omega = 0$ and we will denote by $\Lambda^k_\varepsilon(V^*)$ the vector space of effective k-forms on V. When $k = n$, ω is effective if and only if $\omega \wedge \Omega = 0$.

Hodge-Lepage-Lychagin theorem

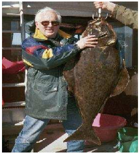

Figure: Hodge and Lychagin

The next theorem plays the fundamental role played by the effective forms in the theory of Monge-Ampère operators :

Theorem (Hodge-Lepage-Lychagin)

- *Every form $\omega \in \Lambda^k(V^*)$ can be uniquely decomposed into the finite sum*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots,$$

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  where all ω_i *are effective forms.*

- *If two effective* k*-forms vanish on the same* k*-dimensional isotropic vector subspaces in* (V, Ω)*, they are proportional.*

MAO definition

- Let M be an n-dimensional smooth manifold. Denote by J^1M the space of 1-jets of smooth functions on M and by $j^1(f) : M \to J^1M$, $x \mapsto [f]^1_x$ the natural section associated with a smooth function f on M.

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- The Monge-Ampère operator

$$\Delta_\omega : C^\infty(M) \to \Omega^n(M)$$

associated with a differential n-form $\omega \in \Omega^n(J^1M)$ is the differential operator

$$\Delta_\omega(f) = j_1(f)^*(\omega).$$

Contact Structure

A contact structures is some analogue of symplectic structure on odd-dimensional manifold. A differential 1-form U on a smooth manifold M is called nondegenerate if the following conditions hold:

- The map $P : a \ni M \mapsto \ker U_a \subset T_a M$ is a 1-codimensional distribution;

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- A restriction of dU to the hyperplane $P(a)$ is a symplectic structure on $P(a)$ for any $a \in M$.
- The last condition means that if a vector $X_a \in P(a)$ and $X_a \rfloor \left(dU|_{P(a)}\right) = 0$, then $X_a = 0$. In other words, a differential 1-form U is nondegenerate if the distribution P has no characteristic symmetries.

Example: Cartan form

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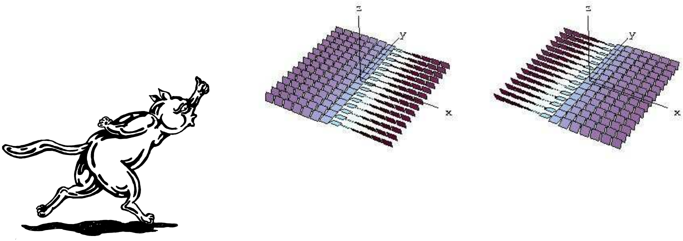
- 

Figure: Contact structure in $\mathbb{R}^3$

Generalized Solutions

- Let U be the contact 1-form on J^1M and X_1 be the Reeb's vector field. Denote by $C(x)$ the kernel of U_x for $x \in J^1M$.

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$$T_x J^1 M = C(x) \oplus \mathbb{R} X_{1x}.$$

- A generalized solution of the equation $\Delta_\omega = 0$ is a legendrian submanifold L^n of (J^1M, U) such that $\omega|_L = 0$. Note that $T_x L$ is a lagrangian subspace of $(C(x), dU_x)$ in each point $x \in L$, and that L is locally the graph of a section $j^1(f)$, where f is a regular solution of the equation $\Delta_\omega(f) = 0$, if and only if the projection $\pi : J^1M \to M$ is a local diffeomorphism on L.

Effective Forms - MAO Correspondence

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$$\Omega^*_\varepsilon(C^*) = \Omega^*(J^1M)/I_C,$$
where I_C is the Cartan ideal generated by U and dU.
- The second part means that two differential n-forms ω and θ on J^1M determine the same Monge-Ampère operator if and only if $\omega - \theta \in I_C$.

Contact Groupe Action

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- $$F(\Delta_\omega) = \Delta_{F^*(\omega)},$$
  and the corresponding infinitesimal action is
  $$X(\Delta_\omega) = \Delta_{L_X(\omega)}.$$

Symplectic MAO-1

- We are interested in particular in a more restrictive class of operators, the class of symplectic operators. These operators satisfy

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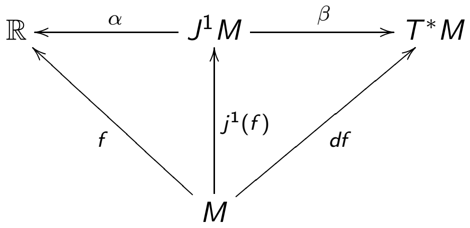
- 

Symplectic MAO-2

- We can naturally identify the space $\{\omega \in \Omega^*_\varepsilon(C^*) : L_{X_1}\omega = 0\}$ with the space of effective forms on (T^*M, Ω) using this projection β. Then, the group acting on these forms is the group of symplectomorphisms of T^*M.

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- Definition

  A Monge-Ampère structure on a 2n-dimensional manifold X is a pair of differential form $(\Omega, \omega) \in \Omega^2(X) \times \Omega^n(X)$ such that Ω is symplectic and ω is Ω-effective i.e. $\Omega \wedge \omega = 0$.

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  - When we locally identify the symplectic manifold (X, Ω) with $(T^*\mathbb{R}^n, \Omega_0)$, we can associate to the pair (Ω, ω) a symplectic Monge-Ampère equation $\Delta_\omega = 0$.
  - Conversely, any symplectic Monge-Ampere equation $\Delta_\omega = 0$ on a manifold M is associated with Monge-Ampère structure (Ω, ω) on T^*M.

Correspondence: Forms -Symplectic MAO

Let M be a smooth $n-$dimensional manifold and ω is a differential n-form on T^*M. A (symplectic) Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ is the differential operator defined by

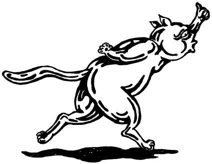
$$\Delta_\omega(f) = (df)^*(\omega),$$

where $df : M \to T^*M$ is the natural section associated to f.

Examples

ω	$\boldsymbol{\Delta_\omega = 0}$
$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	$\Delta f = 0$
$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	$\Box f = 0$
$dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$	$Hess(f) = 1$
$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3$ $+dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$	$\Delta f - Hess(f) = 0$

Generic types of singularities for Generalized solutions of MAE



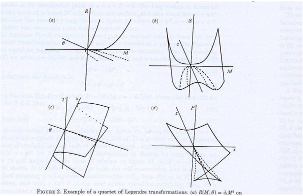


Figure: Lagrangian singularities (Wave fronts, foldings etc.)

Symplectic Monge-Ampère Equations: Solutions

- A generalized solution of a MAE $\Delta_\omega = 0$ is a lagrangian submanifold of (T^*M, Ω) which is an integral manifold for the MA differential form ω:

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- A generalized solution (generically) locally is the graph of an 1-forme df for a regular solution f.

Generalized solution

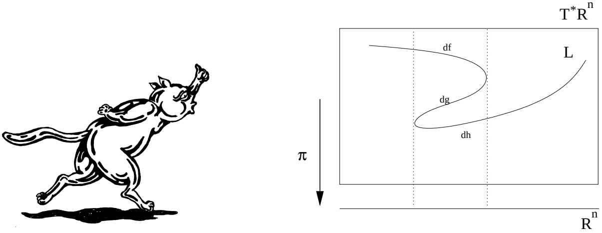

Figure: Generalized solution of a MAE

Symplectic Equivalence-1

- Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are locally equivalent iff there is exist a local symplectomorphism $F : (T^*M, \Omega) \to (T^*M, \Omega)$ such that

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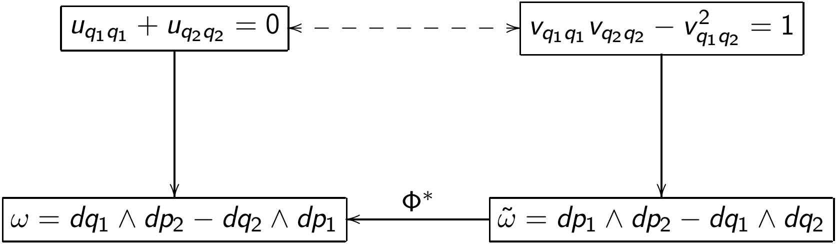
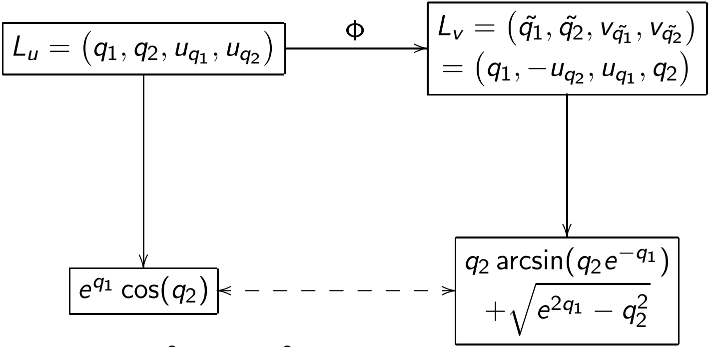
- L is a generalized solution of $\Delta_{F^*\omega_1} = 0$ iff $F(L)$ is a generalized solution of $\Delta_\omega = 0$.

Legendre partial transformation

Figure: Legendre



Legendre partial transformation-2



with $\Phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -p_2, p_1, q_2)$.

Table 1.

$\mathbf{\Delta_\omega = 0}$	ω	$pf(\omega)$
$\Delta f = 0$	$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	1
$\Box f = 0$	$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	-1
$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dq_1 \wedge dp_2$	0

Geometric Structures on $T^*\mathbb{R}^2$.

Let (Ω, ω) be a Monge-Ampère structure on $X = \mathbb{R}^4$. The field of endomorphisms $A_\omega : X \to TX \otimes T^*X$ is defined by

$$\omega(\cdot, \cdot) = \Omega(A_\omega \cdot, \cdot).$$

REMARK The tensor

$$J_\omega = \frac{A_\omega}{\sqrt{|pf(\omega)|}}$$

gives

- an almost-complex structure on X if $pf(\omega) > 0$.

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- an almost-product structure on X if $pf(\omega) < 0$.

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- The tensor J_ω is integrable;
- the normalized form $\frac{\omega}{\sqrt{|pf(\omega)|}}$ is closed.

Courant Bracket

$T-$tangent bundle of M and T^*- cotangent bundle.

$$(X+\xi, Y+\eta) = \frac{1}{2}(\xi(Y)+\eta(X)),$$

-natural indefinite interior product on $T \oplus T^*$.
The Courant bracket on sections of $T \oplus T^*$ is

$$[X+\xi, Y+\eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi).$$

Generalized Complex Geometry

Figure: Hitchin

DEFINITION [Hitchin]: An almost generalized complex structure is a bundle map $\mathbb{J} : T \oplus T^* \to T \oplus T^*$ with

$$\mathbb{J}^2 = -1,$$

and

$$(\mathbb{J}\cdot, \cdot) = -(\cdot, \mathbb{J}\cdot).$$

An almost generalized complex structure is integrable if the spaces of sections of its two eigenspaces are closed under the Courant bracket.

$2D$ SMAE and Generalized Complex Geometry

- **DEFINITION** A Monge-Ampère equation $\Delta_\omega = 0$ has a divergent type if the corresponded form can be chosen closed : $\omega' = \omega + \mu\Omega$.

$2D$ SMAE and Generalized Complex Geometry

- **DEFINITION** A Monge-Ampère equation $\Delta_\omega = 0$ has a divergent type if the corresponded form can be chosen closed : $\omega' = \omega + \mu\Omega$.
- **PROPOSITION** (B.Banos)
  Let $\Delta_\omega = 0$ be a Monge-Ampère divergent type equation on $\mathbb{R}^2$ with closed ω (which might be non-effective). The generalized almost-complex structure defined by

$$\mathbb{J}_\omega = \begin{pmatrix} A_\omega & \Omega^{-1} \\ -\Omega(1 + A_\omega^2 \cdot, \cdot) & -A_\omega^* \end{pmatrix}$$

  is integrable.

Hitchin pairs (after M.Crainic)

A Hitchin pair is a pair of bivectors π and Π, $\Pi-$ non-degenerate, satisfying

$$\begin{cases} [\Pi, \Pi] = [\pi, \pi] \\ [\Pi, \pi] = 0. \end{cases} \tag{1}$$

PROPOSITION There is a 1-1 correspondence between Generalized complex structure

$$\mathbb{J} = \begin{pmatrix} A & \pi_A \\ \sigma & -A^* \end{pmatrix}$$

with σ non degenerate and Hitchin pairs of bivector (π, Π). In this correspondence

$$\begin{cases} \sigma = \Pi^{-1} \\ A = \pi \circ \Pi^{-1} \\ \pi_A = -(1 + A^2)\Pi \end{cases}$$

Hitchin pair of bivectors in $4D$

Π is non-degenerate $\Rightarrow$ two 2-forms ω and Ω, not necessarily closed and $\omega(\cdot,\cdot) = \Omega(A\cdot,\cdot)$.
A generalized lagrangian surface: closed under A, or equivalently, bilagrangian: $\omega|_L = \Omega|_L = 0$.
Locally, L is defined by two functions u and v satisfying a first order system:

Jacobi systems

$$\begin{cases} a + b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} + d\frac{\partial v}{\partial x} + e\frac{\partial v}{\partial y} + f \det J_{u,v} = 0 \\ A + B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y} + D\frac{\partial v}{\partial x} + E\frac{\partial v}{\partial y} + E \det J_{u,v} = 0 \end{cases}$$

$$J_{u,v} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Such a system generalizes both MAE and Cauchy-Riemann systems and is called a Jacobi system.

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- The Hitchin pfaffian defined by

$$pf(\omega) = \frac{1}{6} tr A_\omega^2.$$

	$\Delta_\omega = 0$	ω	$\varepsilon(q_\omega)$	$pf(\omega)$
1	$\text{hess}(f) = 1$	$dq_1 \wedge dq_2 \wedge dq_3 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(3,3)$	ν^4
2	$\Delta f - \text{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - \nu^2 \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(0,6)$	$-\nu^4$
3	$\Box f + \text{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 + \nu^2 \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(4,2)$	$-\nu^4$
4	$\Delta f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(0,3)$	0
5	$\Box f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(2,1)$	0
6	$\Delta_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 - dp_2 \wedge dq_1 \wedge dq_3$	$(0,1)$	0
7	$\Box_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 + dp_2 \wedge dq_1 \wedge dq_3$	$(1,0)$	0
8	$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dp_1 \wedge dq_2 \wedge dq_3$	$(0,0)$	0
9		0	$(0,0)$	0

Table: Classification of effective 3-formes in dimension 6

$3D$ Generalized Calabi-Yau structures

- **A generalized almost Calabi-Yau structure** on a $6D$-manifold X is a 5-uple $(g, \Omega, A, \alpha, \beta)$ where
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    for all tangent vectors U, V,
  - α and β are (eventually complex) decomposable 3-forms whose associated distributions are the distributions of A eigenvectors and such that
    $$\frac{\alpha \wedge \beta}{\Omega^3} \text{ is constant.}$$
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Generalized CY and MA

Each nondegenerate Monge-Ampère structure (Ω, ω_0) defines a generalized almost Calabi-Yau structure $(q_\omega, \Omega, A_\omega, \alpha, \beta)$ with

$$\omega = \frac{\omega_0}{\sqrt[4]{|\lambda(\omega_0)|}}.$$

The generalized Calabi-Yau structure associated with the equation

$$\Delta(f) - \mathrm{hess}(f) = 0$$

is the canonical Calabi-Yau structure of $\mathbb{C}^3$

$$\begin{cases} g = -\sum_{j=1}^{3} dx_j.dx_j + dy_j.dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau associated with the equation

$$\Box(f) + \mathrm{hess}(f) = 0$$

is the pseudo Calabi-Yau structure

$$\begin{cases} g = dx_1.dx_1 - dx_2.dx_2 + dx_3.dx_3 + dy_1.dy_1 - dy_2.dy_2 + dx_3.dx_3 \\ A = \frac{\partial}{\partial x_1} \otimes dy_1 - \frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 \\ +\frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau structure associated with the equation

$$\mathrm{hess}(f) = 1$$

is the "real" Calabi-Yau structure

$$\begin{cases} g = \sum_{j=1}^{3} dx_j . dy_j \\ A = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3 \\ \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

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- iff the correspondingly defined generalized Calabi-Yau structure is integrable and locally flat.