Mathematical Seminar at Göttingen Winter semester 1905/1906 under the direction of Professors Klein, Hilbert, Minkowski

Talks by F. Klein

Notes by Dr. Phil. Otto Toeplitz

First meeting, November 1, 1905

Speaker: Professor Klein

It is the prospect of standing at the edge of developments in the area of *linear differential equations* and *automorphic functions* that motivated me to give you in the coming weeks a coherent overview of my old investigations on this subject, restricting myself to the most elementary case of the so called hypergeometric functions (with only three singular points). These investigations raise a number of questions which are not settled even today.

I will begin today with a **review of the literature**.

a) The modern development of our subject takes its beginning with
Bernhard Riemann (1826-1866), whose collected works were published by H. Weber in 1876, in a 2nd edition in 1892. Of his papers the following are particularly relevant:
"IV. Beiträge zur Theorie der durch Gauss'sche Reihe $F(\alpha, \beta, \gamma; x)$ darstellbaren Funktion" (1857), p. 67-85 in the 2nd edition, and the posthumous paper p. 379-390,
"XXI. Zwei allgemeine Lehrsätze über lineare Differentialgleichungen mit algebraischen Koeffizienten" (1857).
Not only did Riemann develop the foundations on which we descendants all build, he was in command of a variety of results that we later found independently. By a strange coincidence notes of a course on hypergeometric functions from 1859, taken by the physicist von Bezold, surfaced recently. They show that Riemann had progressed even to the uniformization theorem. These remarkable lectures by Riemann, lost for so long, were revised by Noether and Wirtinger and published in 1902, besides some other material, as "Nachträge zu Riemanns gesammelten Werken".
In addition I would like to point to Wirtinger's address at the International Congress of Mathematicians (1904) in Heidelberg (Verhandlungen des III. Internationalen Mathematiker-Congresses..., Leipzig 1904, p. 121-139).

b) Partially following Riemann, partially taking their motivation from Weierstrass, a series of *German investigations* were undertaken. In listing them I would like to place my own quite objectively besides the others. In essence the following have to be named:
Fuchs, who since 1865 developed the theory formally in a certain direction. He trained a number of students at the University of Berlin, where he worked since 1882. His papers are now being published as collected works of which the first volume just appeared: L. Fuchs, Ges. Math. Werke I, Berlin 1904. See p. 111 ff., in particular.
Schwarz, "Über diejenigen Fälle, in welchen die Gauss'sche hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt", J. 75, p. 292-335= ges. Abhandlungen, Berlin 1890, II, p. 211-259.
And then, specifically taking its lead from this paper of Schwarz, and otherwise taking the point of view of the theory of groups,
Klein, with the mention of

α) *published lecture notes*

1. Vorlesungen über das Ikosaeder, Leipzig, 1884,
2. Vorlesungen über die Theorie der elliptischen Modulfunktionen I, II, Leipzig 1890, 1892, edited by Fricke,

3. Fricke-Klein, Vorlesungen über die Theorie der automorphen Funktionen, I, Leipzig 1897, II, 1901; I treats groups of linear substitutions, and only II the functions that belong to them.

β) *autographed lecture notes*
winter 90/91 and summer 91: Ausgewählte Capitel aus der Theorie der linearen Differentialgleichungen 2.O. I, II (not commercially available),
winter 93/94: Über die hypergeometrische Funktion, notes by E. Ritter[1] and summer 94: Über lineare Differentialgleichungen der zweiten Ordnung, notes by E. Ritter; commercially available, but out of stock at present.

The lectures from 1890/91 raise a number of questions that are not formulated very precisely and contain several results without proof. To take these up again will be the task of our seminar. In contrast, the lectures of 1893/94 contain only fully verified, rigorously formulated results.
And then, following Klein, his students

Schilling, Math. Ann. 44,46, 51,
Ritter, Math. Ann. 48 (specifically on hypergeometric functions)
and in addition
Pick, Math. Ann. 25, 26, 27, 34
Papperitz, Math. Ann. 24, 25, 26, 38, 42, Wiener Ber. 1892,
Wälsch, Mitt. d. deutschen Math. Ges. zu Prag, 1892,
Bolze, Math. Ann. 42,
Schellenberg, Dissertation 1892, incorporated into the autographed lecture notes.

c) The work of
Poincaré, beginning in 1881, published in the first five volumes of acta mathematica. I am giving a report just in these weeks to the "mathematische Gesellschaft" on the advances obtained by Poincaré and their relationship to my own investigations.

d) Besides this let me mention the following *textbooks that are relevant for our purpose.*
α) Schlesinger, Handbuch der Theorie der linearen Differentialgleichungen, I, II_1, II_2, Leipzig 1895, 1897, 1897, the sections XIII and XIV in part 2 of volume 2, p. 1-167, deal in particular with triangle functions.
β) Relevant sections in general textbooks on function theory:
Picard, Traité d'analyse III (1896, Paris, G.-V.), p. 291-346,
Darboux, Leçons sur la théorie générale des surfaces, Paris, 4 volumes, passim
Burkhardt, Einführung in die Theorie der ..., Leipzig (Veit), 1897,
Bianchi, Lezioni ... I, II, Pavia 1898, 1899, autographed,
Fricke, Analytisch-funktionentheoretische Vorlesungen, Leipzig, 1900,
Forsyth, Lehrbuch der Differentialgleichungen, Braunschweig 1889, German translation by H. Maser, p. 211-242
and others.

Preliminary remarks

I intend, in the time until Christmas, to discuss triangle functions, or, from the point of view of the theory of differential equations, linear differential equations of the second order with only three singular points. They provide a suitable, simple paradigm to familiarize ourselves with the basic questions of this general theory.

[1] Selfreview in Math. Ann. 45,46

The

necessary prerequisites

I will best characterize by mentioning some elementary facts and ideas from the theory of

functions of a complex variable that will be frequently used in the sequel. Without their fluent mastery a fruitful participation in our circle might not be possible. They are:

1) The habit to interpret, in a function $\zeta = \xi + i\eta$ of the complex argument $z = x + iy$, both the dependent variable ζ and the independent variable z alternatingly as lying now in the plane, now on the sphere, to pass from the plane to the sphere, and in reverse from the sphere to the plane, by the well known stereographic projection, and thus to accord, for certain purposes, a standing to the place ∞ equal with all other places.

2) The interpretation of the linear fractional transformation

$$\zeta = \frac{Az + B}{Cz + D}$$

as a circle relationship between the $z-$ and the $\zeta-$plane (-sphere), and the equal status of lines and circles in the plane. Facts such as the following: the cross-ratio of 4 points in the plane of complex numbers is real if and only if the 4 points lie on a circle, a result that follows easily from the invariance of the cross-ratio, or: if the variables z and ζ are interpreted as lying in one and the same plane (sphere), and if ζ moves on a certain circle $\mathfrak{K}'$, whereas z moves on a circle $\mathfrak{K}$, then the points inside $\mathfrak{K}$ correspond to the points $\zeta = \frac{Az+B}{Cz+D}$ inside or outside $\mathfrak{K}'$ depending on whether z and ζ move on the circles $\mathfrak{K}$ and $\mathfrak{K}'$ in the same or in the opposite sense. Figure 1 illustrates the second case.

figure 1 here

3) The contrast between these circle relationships (projective transformations of the first kind) and projective transformations of the second kind, that is circle relationships with inversion of angles:

$$\overline{\zeta} = \xi - i\eta = \frac{Az + B}{Cz + D},$$

as for instance the reflection in the real axis, or the transformation by reciprocal radii.

Distinguished classes of functions

Among the single valued analytic functions, the class of *rational functions* is characterized by a variety of properties. I would now like to isolate in the totality of all *multiple-valued analytic functions* a similarly distinguished class by requiring that the ramification behavior be a simple one.

a) The simplest type of ramification behaviour that $\zeta(z)$ can exhibit if z moves around a ramification point is that ζ is changed only by an additive constant. Then $\zeta'(z)$ remains unchanged, and if we admit only this type of ramification behaviour on the whole sphere, we identify a distinguished class of functions, the collection of functions ζ whose derivative is a rational function

$$\zeta' = \mathfrak{R}(z), \quad \zeta = \int \mathfrak{R}(z)dz,$$

or that

b) in all ramifications ζ is changed only by a multiplicative constant. Then $\frac{\zeta'}{\zeta}$ remains unchanged, and now the integrals of the differential equation

$$\frac{\zeta'}{\zeta} = \text{rational function of } \ z = \mathfrak{R}(z),$$

that is the functions of the form

$$\zeta = e^{\int \mathfrak{R}(z)dz}$$

form a distinguished class of functions.
One can now replace the two previous types of ramification by the following more general one that is also very simple:

c) ζ is replaced by $\alpha\zeta + \beta$. Then $\frac{\zeta''}{\zeta'}$ is unchanged, and

$$\frac{\zeta''}{\zeta'} = \mathfrak{R}(z) \text{ or } \zeta = \int e^{\int \mathfrak{R}(z)dz} dz$$

gives the class.

d) More generally one can require that on any move around a ramification point ζ undergoes a fractional linear substitution, so is replaced by an expression of the form

$$\frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}.$$

Here a differential expression in ζ that is unchanged under all such ramifications is not quite so immediately given, but can also be constructed without difficulty.
Such a differential expression in the first instance will have to reproduce itself under substitutions of the form $\alpha\zeta + \beta$, as is the case with the very simple expression

$$\frac{\zeta''}{\zeta'} = \frac{d\ lg\ (\zeta')}{dz}.$$

In addition it will have to reproduce itself, for instance, under the substitution $\frac{1}{\zeta} = \eta$. This produces $\eta' = -\frac{\zeta'}{\zeta^2}$, $\ lg\ (\eta') = lg(-1) + lg(\zeta') - 2\ lg\ (\zeta)$, and hence

$$\frac{\eta''}{\eta'} = \frac{d\ lg\ (\eta')}{dz} = \frac{d\ lg\ (\zeta')}{dz} - 2\frac{d\ lg\ (\zeta')}{dz} = \frac{\zeta'}{\zeta''} - 2\frac{d\ lg\ (\zeta)}{dz}.$$

So $\frac{\zeta''}{\zeta'}$ does not reproduce itself under the substitution $\frac{1}{\zeta} = \eta$ and is insufficient for our purpose. If we succeed, however, to form a differential expression in $\frac{\zeta''}{\zeta'}$ that is unchanged as well when ζ is replaced by $\frac{1}{\zeta}$, the simplest non-entire linear substitution, we can conclude that this new differential expression reproduces itself under all linear fractional substitutions $\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$. In fact, one can regard any such substitution as the effect of successive substitutions of the forms $\alpha\zeta + \beta$ and $\frac{1}{\zeta}$, as one convinces oneself easily.[2]

[2]This was presented in a similar way by D. König at the beginning of the second meeting.

With $\frac{\eta'}{\eta''}-\frac{1}{2}(\frac{\eta'}{\eta''})^2$ we we obtain such a differential expression in $\frac{\eta'}{\eta''}$. It remains unchanged if η is replaced by $\frac{1}{\zeta}$. If we carry out the differentiation we obtain in

$$[\zeta]_z=[\zeta]=\frac{\zeta'}{\zeta'''}-\frac{3}{2}(\frac{\zeta''}{\zeta'})^2$$

an expression that reproduces itself under all linear substitutions of the function ζ. One can show that it is in a certain sense the simplest such expression. One finds it already with Lagrange and Kummer, and of course Riemann as well, but it is called, following Cayley, Schwarz' differential parameter.[3]

In fact it was Schwarz who first pointed out the great importance of this expression. The functions for which $[\zeta]$ is a rational function of the independent variable z can be called *"linear-polymorphic functions"*.

The five exhibited distinguished classes of functions, that is the rational functions and the classes described under a), b), c), d), and of which each is contained in the next, seem to have been singled out with a certain arbitrariness. One realizes, though, that they represent all essential classes, and that their sequence can not be refined by intermediate steps, if one takes into account that they correspond to certain subgroups of the group of all projective transformations (linear substitutions), and that these five subgroups are the only ones, as one can show.

The five exhibited classes of functions correspond precisely to the five groups of linear substitutions that one distinguishes in the theory of continuous transformation groups.

A *geometric interpretation of Schwarz' expression* can be obtained as follows. Let $z+e_1, z+e_2, z+e_3, z+e_4$ be four points of the $z-$plane, close to each other within certain bounds, and let $\zeta+\varepsilon_1, \zeta+\varepsilon_2, \zeta+\varepsilon_3, \zeta+\varepsilon_4$ be the four points corresponding to them in the $\zeta-$plane via the function $\zeta(z)$. Let $\mathfrak{Dv}(z)$ be the cross-ratio of the four points $z+e_1, z+e_2, z+e_3, z+e_4$, taken in a definite order, and $\mathfrak{Dv}(\zeta)$ that of the corresponding points $\zeta+\varepsilon_1, \zeta+\varepsilon_2, \zeta+\varepsilon_3, \zeta+\varepsilon_4$, taken in the corresponding order. If now $\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$ is an arbitrary fractional linear function of ζ we have

$$\mathfrak{Dv}(\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta})=\mathfrak{Dv}(\zeta)$$

because of the invariance of the cross-ratio, that is, $\mathfrak{Dv}(\zeta)$ will share with Schwarz' expression its principal property. In contrast to $[\zeta]$, however, $\mathfrak{Dv}(\zeta)$ depends on e, e_2, e_3, e_4 besides z and ζ. If we develop $\mathfrak{Dv}(\zeta)$ in powers of e_1, e_2, e_3, e_4 according to Taylor's theorem, the coefficients of this development will be certain differential expressions in z and ζ, and one can expect a connection between them and $[\zeta]$. One finds in effect[4] the following quite simple relation:

$$\mathfrak{Dv}(\zeta)=\mathfrak{Dv}(z)\{1+\frac{(e_2-e_4)(e_3-e_1)}{3!}[\zeta]+\frac{(e_2-e_4)(e_3-e_1)(e_1+e_2+e_3-3e_4)}{4!}[\zeta]'+\dots\}.$$

So the geometric meaning of Schwarz' expression is characterized as follows: if e_1, e_2, e_3, e_4 are infinitely small quantities, then $\mathfrak{Dv}(\zeta)$ is precisely $\mathfrak{Dv}(z)$; as e_1, e_2, e_3, e_4 grow, $[\zeta]$ gives the change of the two cross-ratios in the second approximation.

[3]For more detailed references see Schwarz, as cited above (Ges. Abh. II), p. 220, and F. Klein, Ikosaeder, p. 74, Ann.

[4]as calculated by Flenburg, Josephson and others

The linear-polymorphic functions appear as integrals of the differential equation

$$[\zeta] = \mathfrak{R}(z).$$

In contrast to the differential equations of the more restricted classes of functions mentioned before it is not integrable by quadratures, as I would like to mention here. It can be said though, regarding its integrals, that conversely they are all linear-polymorphic functions. For if $\zeta(z)$ is an arbitrary integral of $[\zeta] = \mathfrak{R}(z)$, then also $\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$, and this expression contains the three constants of integration that must appear in the integration of a third order differential equation. It is therefore its general solution. Since every branch of ζ, not just the one we thought of initially, satisfies the same differential equation, and since all integrals are composed linearly from one branch, every integral of $[\zeta] = \mathfrak{R}(z)$ is linear-polymorphic, as we maintained.

Definition of the triangle functions

In the weeks till Christmas we will not deal with the most general linear-polymorphic functions, but with those that satisfy certain restrictions, specifically the following:

ζ shall have only three places of ramification $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. At each finite place z_0 either ζ or some integral ζ_0 of the same differential equation $[\zeta] = \mathfrak{R}(z)$ that is satisfied by ζ is

1) $\zeta_0 = (z - z_0)\mathfrak{P}(z - z_0)$.
   Also[5]
2) $\zeta_\infty = \frac{1}{z}\mathfrak{P}(\frac{1}{z})$ at the places $z = \infty$
3) $\zeta_\mathfrak{a} = (z - \mathfrak{a})^\lambda \mathfrak{P}(z - \mathfrak{a})$ at the place $\mathfrak{a}$,
   $\zeta_\mathfrak{b} = (z - \mathfrak{b})^\mu \mathfrak{P}(z - \mathfrak{b})$ at the place $\mathfrak{b}$,
   $\zeta_\mathfrak{c} = (z - \mathfrak{c})^\nu \mathfrak{P}(z - \mathfrak{c})$ at the place $\mathfrak{c}$.

By $\mathfrak{P}$ here we mean a convergent power series in positive powers of the argument and with non-vanishing constant term.

To justify these restrictions let us just point out that Riemann also made them. Less than two ramification points would be of little interest. Functions with just one ramification point do not exist since an orbit on a circle enclosing the ramification point would also be an orbit around the rest of the sphere containing no ramification point. And the assumption of two ramification points would not be compatible[6] with the restrictions we imposed, following Riemann, regarding the character of the developments of ζ at various places.

From these conditions we conclude the following regarding the shape of $[\zeta] = \mathfrak{R}(z)$:

1) $[\zeta] = \mathfrak{P}(z - z_0)$ at all finite places except $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$
2) $[\zeta] = \frac{1}{z^4}\mathfrak{P}(\frac{1}{z})$ at the place ∞,
3) $[\zeta] = \frac{\frac{1-\lambda^2}{2}}{(z-\mathfrak{a})^2}\mathfrak{P}(z - \mathfrak{a})$ at the place $\mathfrak{a}$,
   $[\zeta] = \frac{\frac{1-\mu^2}{2}}{(z-\mathfrak{b})^2}\mathfrak{P}(z - \mathfrak{b})$ at the place $\mathfrak{b}$,
   $[\zeta] = \frac{\frac{1-\nu^2}{2}}{(z-\mathfrak{c})^2}\mathfrak{P}(z - \mathfrak{c})$ at the place $\mathfrak{c}$.

[5]Klein writes ζ_0 in 2), 3) as well. I have adopted the more consistent notation used by Klein on p. 9 (translator's comment).

[6]Worked out explicitly by Laumann

There exists precisely one rational function $\mathfrak{R}(z)$ that satisfies all these conditions, and one finds

$$[\zeta] = \mathfrak{R}(z) = \frac{1}{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})}\{\frac{\frac{1-\lambda^2}{2}(\mathfrak{a}-\mathfrak{b})(\mathfrak{a}-\mathfrak{c})}{z-\mathfrak{a}}+\frac{\frac{1-\mu^2}{2}(\mathfrak{b}-\mathfrak{c})(\mathfrak{b}-\mathfrak{a})}{z-\mathfrak{b}}+\frac{\frac{1-\nu^2}{2}(\mathfrak{c}-\mathfrak{a})(\mathfrak{c}-\mathfrak{b})}{z-\mathfrak{c}}\}.$$

If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are placed at $0, 1, \infty$, as happens frequently, one obtains

$$[\zeta] = \frac{\frac{1-\lambda^2}{2}}{z^2} + \frac{\frac{1-\mu^2}{2}}{(z-1)^2} + \frac{\frac{\lambda^2+\mu^2-\nu^2-1}{2}}{z(z-1)}$$

One can deduce this, for instance, from the general formula by taking limits[7].

The functions ζ that satisfies all conditions of p. 7 is designated by

$$\zeta\begin{pmatrix} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \\ & & & z \\ \lambda & \mu & \nu & \end{pmatrix}.$$

Second meeting, November 8, 1905

Presentation of participants following the alphabetical list

Continuation of the talk of Professor Klein

I will begin today with some abstract considerations concerning the so called *monodromy group* belonging to our differential equation

$$[\zeta] = \mathfrak{R}(z)$$

as defined last time. I will then turn to a more concrete and descriptive question concerning the conformal maps of circular arc triangles that interest us.

1) *The linear substitutions of the function* ζ_0.

We perform suitable cuts in the $z-$plane from $\mathfrak{a}$ to $\mathfrak{b}$ and from $\mathfrak{b}$ to $\mathfrak{c}$ so that the function ζ_0 defined by the conditions of p. 7 is single valued in the cut plane. We now follow a path that encloses $\mathfrak{a}$, disregarding the cuts just made, and thereby move from the branch $S(\zeta)$ that will be composed linearly of ζ_0 in a certain way. This linear substitution we designate by the symbol S. We now follow a second path that encloses $\mathfrak{b}$, and ζ_0 will again be changed by a linear substitution, into $T(\zeta_0)$. In a similar way let U be the linear substitution that belongs to $\mathfrak{c}$.

Orbiting $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *corresponds to certain linear substitutions* S, T, U*, where a fixed branch of* ζ_0 *is used throughout.*

Doing this it matters which branch is used, the S, T, U could change. If ζ_1 is some other branch, possibly of another integral of the differential equation of the third order that we are considering, then ζ_1 is linearly composed of ζ_0, say

$$\zeta_1 = \frac{\alpha\zeta_0 + \beta}{\gamma\zeta_0 + \delta}.$$

If orbiting $\mathfrak{a}$ moves ζ_0 into $S(\zeta_0)$, then

$$\zeta_1 \text{ into } \frac{\alpha\ S(\zeta_0) + \beta}{\gamma\ S(\zeta_0) + \delta}$$

[7] by Förster

since $\alpha, \beta, \gamma, \delta$ are not changed as we move along a path.

Let A be the substitution with coefficients $\alpha, \beta, \gamma, \delta$ and let us denote symbolically the successive application of the two substitutions A, S by writing A and S as factors of a product $A.S$ in the order opposite to that in which the substitutions are applied. Then ζ_1 is moved into a branch $S_1(\zeta_1) = A{\cdot}S(\zeta_0)$. Now $\zeta_1 = A(\zeta_0), S, A\zeta_0) = A\,S(\zeta_0)$, that is $S, A = AS$ and $S_1 = ASA^{-1}$.

I will now subject ζ_0 successively to an orbit around $\mathfrak{a}$, then one around $\mathfrak{b}$ and one around $\mathfrak{c}$. In the first step I obtain $S(\zeta_0)$. The second step I have to take not for ζ_0, but for $S(\zeta_0)$. If I put S for A and T for S in the last argument, I find that the effect of the second step is

$$STS^{-1}(\zeta_0) = ST(\zeta_0),$$

and the effect of all three steps is

$$STU(\zeta_0)$$

in total. We have then followed a path on the $z-$sphere which on one side encloses $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, but on the other side a part of the sphere that is free of ramification points of the functions ζ. It follows that

$$STU = 1.$$

The product STU of the three substitutions corresponding to orbits around $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and based on a fixed branch ζ_0 is 1.

The preceding demonstrates at the same time that all substitutions ζ_0 is subject to under the most varied orbits can be represented symbolically as products of S, T, U with positive or negative integer exponents. They thus form a group with finite basis. It is called the *monodromy group* of the given differential equation of the third order. Besides the differential equation, it depends on the choice of the branch ζ_0, but not in an essential way. With a different choice it is replaced by an isomorphic group.

$\zeta-$functions that belong to the same monodromy group, that is, have the same singular points and are subject there to the same substitutions, are called *"related"*. It will become apparent later why it is of particular interest to introduce such a notion.

I will now discuss

2) *more details regarding the behavior of the function at the singular points.*

As we postulated on p. 7, there are three integrals $\zeta_\mathfrak{c}, \zeta_\mathfrak{b}, \zeta_\mathfrak{c}$ of our differential equation so that

$$\zeta_\mathfrak{a} = (z-\mathfrak{a})^\lambda\ \mathfrak{P}(z-\mathfrak{a}),$$
$$\zeta_\mathfrak{b} = (z-\mathfrak{b})^\mu\ \mathfrak{P}(z-\mathfrak{b}),$$
$$\zeta_\mathfrak{c} = (z-\mathfrak{c})^\nu\ \mathfrak{P}(z-\mathfrak{c}).$$

Nothing essential is changed if I now use $-\lambda, -\mu, -\nu$ instead of λ, μ, ν, in particular since only the squares of these quantities appear in $\mathfrak{R}(z)$.

If I do not insist on the conditions on p. 7, but consider only our differential equation, it is questionable whether $\zeta_\mathfrak{a}, \zeta_\mathfrak{b}, \zeta_\mathfrak{c}$ can always be determined. One can say that this is the the case for general λ, μ, ν, but usually not, and only exceptionally so, if λ, μ, ν are integers. More precisely the situation is as follows.

If one considers the homogeneous linear differential equation of the second order that is equivalent to our third order differential equation[8] one can conclude by well known methods that $\zeta'_{\mathfrak{a}}$ has a development of the following form:

$$\zeta'_{\mathfrak{a}} = \frac{A}{(z-\mathfrak{a})^{\lambda+1}} + \frac{B}{(z-\mathfrak{a})^{\lambda}} + \dots .$$

In general $\zeta_{\mathfrak{a}}$ is found in the desired form through integration. Only if λ is an integer a logarithmic term appears which is not present if quite specially the coefficient of $\frac{1}{z-\mathfrak{a}}$ vanishes.

The question when the logarithmic term is not present in the case λ is an integer has been answered by Schwarz loc. cit. (J. 75), as I only report here: If $\lambda = k$, and if one of the expressions

$$\sigma = \pm\mu \pm \nu$$

appears in the sequence of numbers

$$k-1, k-3, \dots, 0, \dots, -(k-3), -(k-1),$$

then and only then is the logarithmic term not present.

In the sequel we will disregard the appearance of such logarithmic terms, that is, keep to the conditions of p. 7, in order to avoid unduely slowing the exposition. One can also consider the cases with logarithmic terms as limit cases of the general case.

An orbit around $\mathfrak{a}$ moves $\zeta_{\mathfrak{a}}$ into

$$e^{-2i\pi\lambda} \cdot \zeta_{\mathfrak{a}},$$

and $\zeta_{\mathfrak{b}}, \zeta_{\mathfrak{c}}$ will be transformed in a similarly simple fashion when $\mathfrak{b}, \mathfrak{c}$ are orbited. We now put

$$\zeta_{\mathfrak{a}} = S_{\mathfrak{a}}(\zeta_0), \zeta_{\mathfrak{b}} = T_{\mathfrak{b}}(\zeta_0), \zeta_{\mathfrak{c}} = U_{\mathfrak{c}}(\zeta_0)$$

We then obtain

$$S = S_{\mathfrak{a}}^{-1} e^{-2i\pi\lambda} S_{\mathfrak{a}}, T = T_{\mathfrak{a}}^{-1} e^{-2i\pi\mu} T_{\mathfrak{b}}, U = U_{\mathfrak{c}}^{-1} e^{-2i\pi\nu} U_{\mathfrak{c}}$$

and $STU = 1$ now gives

$$S_{\mathfrak{a}}^{-1} e^{-2\pi i\lambda} S_{\mathfrak{a}}\, T_{\mathfrak{b}}^{-1} e^{-2\pi i\mu} T_{\mathfrak{b}}\, U_{\mathfrak{c}}^{-1} e^{-2\pi i\nu} U_{\mathfrak{c}} = 1.$$

I would like to add a few words on how related functions are characterized by λ, μ, ν. Certainly the λ, μ, ν can respectively only differ by integers in order for $e^{-2i\pi\lambda}, \dots$ to agree. A more detailed investigation shows that, apart from some particularly complicated special cases, functions are related if $\pm\lambda, \pm\mu, \pm\nu$ differ by integers whose sum is even.

I turn now from these somewhat abstract considerations to a more geometric-descriptive investigation in the case of real λ, μ, ν. The other cases will be treated later.

1) If λ, μ, ν are real, and if one moves $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ to the real axis by suitable linear substitution, then $\mathfrak{R}(z)$ certainly is real. If one takes a particular solution ζ_0 that together with its first two derivatives is real at a place on the real axis, ζ_0''' is real according to the differential equation of the third order. Through continued differentiation of the differential equation one finds that ζ_0'''', ..., in short all derivatives, are real. Hence the development of ζ_0 at this place is real. So if z moves along the real axis, say from

[8] see p. 23 (translator's comment)

$\mathfrak{a}$ to $\mathfrak{b}$, then also ζ_0 in the ζ_0-plane. If we now move back $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ from the real axis to arbitrary places in the plane, we conclude:

If z traverses the segments $\mathfrak{bc}, \mathfrak{ca}, \mathfrak{ab}$ of the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, then ζ traverses the sides of a triangle composed of circular arcs in the $\zeta-$plane.

One can prove this in very elegant fashion without transformation to the real axis. Let us take four close points on the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. Their cross-ratio is real since they lie on a circle. The cross-ratio of the four corresponding points in the $\zeta-$plane is the same up to a deviation that is in the first approximation given by $[\zeta]$, and precisely by a development that contains only differential expressions in $[\zeta]$. Now $[\zeta]$ is real and hence also the other cross-ratio. So the four corresponding points in the $\zeta-$plane lie on a circle as well.

2) The three circular arcs in the $\zeta-$plane form the angles $\lambda\pi, \mu\pi, \nu\pi$, as one sees easily:

If λ, μ, ν are real and if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ lie on the real axis of the $z-$plane, then the two $z-$half-planes are mapped conformally by the function $\zeta(z)$ onto circular arc triangles with angles $\lambda\pi, \mu\pi, \nu\pi$.

3) If one adds the reflection principle, then it is easy to see, according to the rules of transformation by reciprocal radii, how the conformed map is continued if we alternate between the positive and negative $z-$half-planes, and thereby keep track of the various ramifications of $\zeta(z)$.

This is quite simple in principle. The imagination does not meet with difficulties as long as $\lambda, \mu, \nu < 2$. But here we also have to gain insight into the cases in which λ, μ, ν take larger values and therefore *overlaps* can occur. This forces us into an excursion into elementary geometry.

I have dealt with these questions of elementary geometry in Math. Ann. 37 and will report here on my work.

4) *The shape of circular arc triangles with the angles $\lambda\pi, \mu\pi, \nu\pi$.*
I distinguish two types of circular arc triangles. I order λ, μ, ν by size and label them so that

$$\lambda \geq \mu \geq \nu.$$

I then distinguish the cases $\lambda > \mu + \nu$ and $\lambda \leq \mu + \nu$.

I construct in addition a reduced triangle $(\overline{\lambda}, \overline{\mu}, \overline{\nu})$ for which $\overline{\lambda}, \overline{\mu}, \overline{\nu}$ are the smallest positive remainders of λ, μ, ν modulo 2. This reduced triangle then has the shape one usually expects a circular arc triangle on the sphere to have, in elementary classes for instance.

From such a reduced triangle one can pass to more complicated ones by two operations, which I call *"lateral"* and *"polar" attachment.*

figure 3 here

For the *lateral attachment* I think of completing $\mathfrak{bc}$ to a full circle and regard $\mathfrak{cb}$, instead of $\mathfrak{bc}$, as a side of the enlarged triangle. The circle $\mathfrak{bc}$ cuts out on the sphere - on which I best imagine this process - two calottae, and I add the calotta that meets the reduced triangle laterally and without overlapping with it, to the content of the enlarged triangle. I can then laterally attach to $\overline{\mathfrak{cb}}$ the other, complementary, calotta, replacing again $\overline{\mathfrak{cb}}$ by $\overline{\mathfrak{bc}}$, covering the reduced triangle in this case. In this way I can make unlimited lateral attachments.

figure 4 here. Put an l besides the wiggly line

For the polar attachment I think of the content of the triangle as augmented by just the other spherical calotta which covers the reduced triangle along $\overline{\mathfrak{bc}}$, and attach it along the line l as ramification cut to the reduced triangle in well known fashion. This time only one angle, the one at $\mathfrak{a}$, has been augmented by 2π.

Now I obtain the following theorem:

To pass from the reduced triangle to the general one, two extension processes have to be applied: lateral attachment of circular discs, polar attachment of circular discs. Upon lateral attachment, two angles are augmented by π, upon polar attachment one angle is increased by 2π and to the opposite side a full orbit is added. If one performs an arbitrary number of lateral attachments at all sides of the reduced triangle one obtains the most general triangle of the second kind ($\lambda \leq \mu + \nu$). If one makes lateral attachments on only two sides, and a polar attachment for the enclosed angle, one obtains the most general triangle of the first kind, with $\lambda > \mu + \nu$.

5) Regarding the *fold-overs* that occur during these attachments the following is to be said. If $\lambda > \mu + \nu$, then the side l opposite the angle $\lambda\pi$ is folded over, indeed

$$\mathfrak{E}\left(\frac{\lambda - \mu - \nu + 1}{2}\right)$$

times, where $\mathfrak{E}(x)$ stands for the largest integer $\leq x$.

Extension relation of triangle geometry: If $\lambda \geq \mu \geq \nu$ and $\lambda > \mu + \nu$, then the side opposite the angle $\lambda\pi$ folds over

$$\mathfrak{E}\left(\frac{\lambda - \mu - \nu + 1}{2}\right)$$

times. Here a circular arc triangle is defined as a simply connected domain bounded by three circular arcs. The other two sides do not fold over, and none of the three sides folds over in case $\lambda \leq \mu + \nu$.

As to the proof of these two results, the argument, by elementary geometric means, for the second theorem, the "*extension theorem*", constitutes the principal part of my paper (Ann. 37). An argument by elementary geometric means is missing, though, for the first theorem. In its place I take recourse to the following considerations.

Our differential equation of the third order is uniquely determined by specifying $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda, \mu, \nu$. Each of the ∞^3 integrals of the differential equation maps the interior of the circle through the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ in the $z-$plane to a circular arc triangle with angles $\lambda\pi, \mu\pi, \nu\pi$ in the $\zeta-$plane. All these triangles are produced one from another by a linear transformation. I claim now that every circular arc triangle is one of these ∞^3 triangles. In fact, according to general mapping theorems, given any circular arc triangle with angles $\lambda\pi, \mu\pi, \nu\pi$, the interior of the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ can be mapped conformally to the interior of this circular arc triangle (with known modification at the vertices). By the reflection principle this map, or the function $\zeta(z)$ that gives it, can be extended beyond the interior of the circle $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, and we arrive at a linear-polymorphic function that evidently satisfies our differential equation of the third order with parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda, \mu, \nu$. It follows that every circular arc triangle with angles $\lambda\pi, \mu\pi, \nu\pi$ is contained in the ∞^3 triangles, that is, any two triangles with the angles $\lambda\pi, \mu\pi, \nu\pi$ can be transformed into each other by a linear transformation.

Now there exist reduced triangles with arbitrary angles. By lateral and polar attachments I can, according to the definition of these processes, change the angles at will by multiples of π and so construct triangles with arbitrary angles $\lambda\pi, \mu\pi, \nu\pi$. Since up to projective transformations there exists only <u>one</u> triangle with given $\lambda\pi, \lambda\mu, \lambda\nu$, I in effect obtain all circular arc triangles that exist by the process of lateral and polar attachment.

The above discussions could be misunderstood if we do not add something regarding the definition of a circular arc triangle. Initially (in the first meeting) we obtained a circular arc triangle as conformal image of the positive half plane (or the interior of a circle). We now regarded them as elementary geometric objects and established elementary geometric results concerning them. In one of these proofs, though (the proof above) we assumed that they can be mapped conformally to the positive half-plane according to general mapping theorems.

figures 5a and 5b here

For this it is necessary that the triangle, after the choice of a definite orientation bounds a connected domain, say to the left. In elementary geometry one thinks, in the first instance, of a circular arc triangle as characterized by vertices that are connected by circular arcs, and by an orientation. Figure 5a shows that even if all angles of the triangle are $< \pi$ the triangle will not necessarily bound a connected domain. Admittedly the circular arcs intersect each other, but it is not legitimate to a priori exclude such intersections. Figure 5b shows reduced triangles, specifically triangles with one angle $< 2\pi$ and the others $< \pi$ that overlap and nevertheless bound a connected domain to the left.

Third meeting, November 15, 1905

Continuation of the talk of Professor Klein

To begin with, several applications may illustrate the usefulness of the elementary geometric considerations I presented in the second half of my last talk.

1) The "*extension theorem of triangle geometry*" permits several conclusions that run parallel to the theorems in algebra on the number of real roots (Sturm, Budan-Fourier), and that complement in a certain way the well known results of Sturm-Lionville on the nesting behavior of the zeroes of a second order homogeneous linear differential equation with real coefficients. Let $\lambda\pi$, as always, be the largest of the three angles $\lambda\pi, \mu\pi, \nu\pi$ and denote by l the opposite side. It is the only one that can overlap and is the image of the arc $\mathfrak{bc}$. The behaviour of the function $\zeta(z)$ will gain a particular interest, therefore, if we convert the circular arc $\mathfrak{bc}$ into a piece of the real axis of the $z-$plane (by a suitable linear transformation), and at the same time the side l of the triangle into a piece of the real axis in the $\zeta-$plane. $\zeta(z)$ will then be a single valued real function in the interval $\mathfrak{bc}$. We now think of z and ζ graphed as abscissa and ordinate in a plane and can visualize $\zeta(z)$ as a real curve. As z now traverses the interval $\mathfrak{bc}, \zeta$ will, according to the overlaps of the side l of the circular arc triangle, tend to $+\infty$, only to rise anew beginning at $-\infty$.

figure 6 here

This increase is repeated by ζ as many times as l has overlaps. The number of overlaps, on the other hand, is given by the extension theorem, and we can summarize:

If $\mathfrak{bc}$ is a piece of the real axis and $\zeta(z)$ is real along it, then the extension theorem allows to make explicit the behavior of the real function $\zeta(z)$ in the interval $\mathfrak{bc}$.

2) In analogy with the fundamental theorem of algebra we are able to determine how often a given value ζ is taken by the function $\zeta(z)$ inside the positive $z-$half plane, or inside the $z-$plane, made simply connected by suitable cuts. In the positive half-plane $\zeta(z)$ evidently will take the value ζ_0 as often as there are sheets in the circular arc triangle stacked above the place ζ_0. Now in two lateral attachments performed at the same side of a triangle the $\zeta-$plane is covered precisely once. In a polar attachment at the side l only the interior of the circle of which l is an arc is covered once. It follows that in case of a triangle of the second kind the plane is covered by
$$\frac{(\lambda-\overline{\lambda})+(\mu-\overline{\mu})+(\nu-\overline{\nu})}{2}$$
complete sheets and, in case of a triangle of the first kind, by only
$$\frac{(\mu-\overline{\mu})+(\nu-\overline{\nu})}{2}$$
sheets, whereas the interior of the circle l is covered
$$\frac{(\lambda-\overline{\lambda})+(\mu-\overline{\mu})+(\nu-\overline{\nu})}{2}$$
times as well. In both cases the reduced triangle, which even can overlap itself as we have seen in figure 5b, has to be added as a covering.

Summarizing we can conclude at least the following:

By quite simple means based solely on considerations of elementary geometry it is possible to ascertain how often the value ζ_0 is taken by $\zeta(z)$ in some part of the $z-$plane, for instance the interior of the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ (the positive half-plane if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are real). Moreover, this number can differ for different ζ_0 by at most one or two units.

3) In a similar way one can answer the question, for instance, how far one has to proceed along a given circle to reach one boundary point of the triangle from another, and other such questions.

In the opposite direction the function $z(\zeta)$ assigns to each point in the interior of the circular arc triangle one and only one point in the interior of the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, or more conveniently in the positive $z-$half-plane (if we think of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ as moved to the real axis). So inside the triangle $z(\zeta)$ only takes values with positive imaginary part, but each of these precisely once.

We would now like to enlarge the circular arc triangle in a way to make $z(\zeta)$ take each value z once and only once, similar to the way the function $z = e^{\zeta}$, for instance, takes each complex value precisely once in a strip of width 2π parallel to the real $\zeta-$axis. In short, we want to construct a *fundamental domain* for the function $z(\zeta)$.

By a *fundamental domain* of a function we mean a region in the plane of the argument in which the function takes each complex value precisely once.

This enlargement or our circular arc triangle can be achieved in several ways. We only have to make cuts in the $z-$plane to make it simply-connected. If we then continue $\zeta(z)$ into the negative half-plane it remains single-valued. At the same time $z(\zeta)$ is continued across the boundary of the circular arc triangle according to the reflection principle.
To make things easier to visualize we will think preliminarily of the circular arc triangle as rectilinear, that is as a triangle with straight sides the way one usually thinks of a triangle. We further cut the $z-$plane along the real axis from $-\infty$ to $\mathfrak{b}$ and from $\mathfrak{c}$ to $+\infty$, so that the interval $\mathfrak{bc}$ is the only non-cut past of the real axis. If we continue $\zeta(z)$ between $\mathfrak{b}$ and $\mathfrak{c}$ from the positive half-plane into the negative half-plane, then by the reflection principle $z(\zeta)$ is continued across the side of the triangle into the triangle symmetric to it w.r.t. the axis l.

figure 7 here

So it together with the initial triangle represents a fundamental domain for $z(\zeta)$, a quadrangle. This quadrangle, therefore, is mapped one-to-one and conformally to the entire $z-$plane, except that conformity is violated in the endpoints and single-valuedness along m, n, m', n', and respectively along the cut parts of the real axis in the $z-$plane. If we do not include m', n' in the fundamental domain to begin with, or it we regard their points as identical with the points of m, n whose reflections they are, we can avoid this last exception and can say with increased precision that $z(\zeta)$ takes each complex value precisely once in the fundamental domain thus defined.

If, beginning with the given branch of the function $\zeta(z)$, we make a complete orbit around $\mathfrak{b}$, for instance, ζ is transformed linearly into $T(\zeta)$. We recognize the linear transformation T of the $\zeta-$plane as a simple rotation of the $\zeta-$plane with center at the vertex $\mathfrak{B}$ that belongs to the angle $\mu\pi$. More precisely it is a rotation by the angle $2\mu\pi$ that moves n' into n and the lower left hatched triangle in figure 7 into the given one. There is a similar interpretation for the substitutions S, U that belong to the points $\mathfrak{a}, \mathfrak{c}$. They correspond to orbits around $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ in the positive sense, and all three are interpreted as rotations about resp. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, in the positive sense as well.

The somewhat un-symmetric privileged treatment given to the side l in constructing the fundamental domain can be avoided through a slightly more complicated realization of the fundamental domain as a hexagon.

figure 8 here

We divide the triangle into three subtriangles by the segments $\mathfrak{D}\mathfrak{A}, \mathfrak{D}\mathfrak{B}, \mathfrak{D}\mathfrak{C}$ as shown in figure 8, and reflect each of these in the corresponding side of the triangle. Figure 8 shows as well which cuts in the $z-$plane correspond to this choice of fundamental domain.

The relation $STU = 1$ is put into evidence visually by figure 8. The rotation U transforms $\mathfrak{D}'_2$ into $\mathfrak{D}'_1$, then T moves the point $\mathfrak{D}'_1$ to $\mathfrak{D}'_3$, and finally S transforms $\mathfrak{D}'_3$ back to $\mathfrak{D}'_1$. The succession of three substitutions just carried out we agreed to designate by $S \cdot T \cdot U$. So STU is a congruent transformation of the plane that sends $\mathfrak{D}'_2$ to itself and, as is easily inferred from figure 8, also leaves $\mathfrak{A}$ (and $\mathfrak{C}$) invariant. From this we can conclude that STU is the identical substitution.

We point out in conclusion:
A fundamental domain of the function $z(\zeta)$ can be constructed in the $\zeta-$plane as a hexagon, or more specially as a quadrilateral, whose sides are pairwise put in correspondence with each

other. The substitutions S, T, U are interpreted as rotations of the $\zeta-$plane with centers at resp. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and by the angles $2\lambda\pi, 2\mu\pi, 2\nu\pi$. They provide the correspondence between the paired sides of the hexagon.

Transition to the $\zeta-$sphere. We have up to now only dealt with triangles with straight sides and acute angles instead of circular arc triangles with arbitrary overlaps. This enabled us to gain an easy insight into a situation that would have been more difficult to comprehend in the general case.

It is our task now to make the transition again to the general case. To this end we replace the $\zeta-$plane by the $\zeta-$sphere. In the first instance we give visual expression that way to the equal status the place ∞ has with all other places of the plane of complex numbers. More crucially, by the transition to the sphere we come to the realization that the more general geometry dealing with circular arc instead of rectilinear triangles leads us into non-Euclidian geometry. We thereby make a connection with a well explored area of mathematics.

I will therefore insert here an *excursion into Cayley- or non-Euclidian geometry (projective measurement).*

Let some surface $\mathfrak{F}$ of degree 2 be given in the space of three dimensions. The line connecting two points p, q of space intersects F in two points π, κ. We introduce the quantity

$$\overline{pq} = \frac{i}{2}\ lg(\mathfrak{D}\mathfrak{v}(p, q, \pi, \kappa)).$$

Let further s, t be two planes tangent to $\mathfrak{F}$ and passing through the line of intersection of s and t. We introduce the quantity

$$(s, t) = \frac{i}{2}\ lg(\mathfrak{D}\mathfrak{v}(s, t, \sigma\tau)).$$

If we pick for $\mathfrak{F}$ in particular the degenerate surface of degree 2 consisting of the imaginary sphere-circle, these quantities become what one calls the distance of the points p, q and the angle between the planes s, t. We will therefore quite generally designate these quantities as "distance" and "angle" if we pick $\mathfrak{F}$ arbitrarily, or more specifically when we pick for $\mathfrak{F}$ our $\zeta-$sphere.

All collineations of space leave the cross-ratio invariant. Hence all collineations that send $\mathfrak{F}$ to itself leave unchanged the "distances" and "angles" defined with $\mathfrak{F}$ as measure-determining surface. If $\mathfrak{F}$ is the imaginary sphere-circle ("absolute curve") then these collineations are what one calls ordinary motions, congruent translations and reflections. Even for arbitrary $\mathfrak{F}$ we will continue therefore to call these collineations "*motions*".

It is easy to see that there are ∞^6 collineations of space that transform a surface of degree 2 into itself. There are altogether ∞^{15} collineations of space. In order for a collineation to transform a surface of degree 2 into itself it is sufficient that it transforms 9 points of the surface into some other 9 points of the surface. For there is in general precisely <u>one</u> surface of degree 2 passing through 9 given points. Hence there are 9 conditions for a collineation to leave $\mathfrak{F}$ invariant, and there are $\infty^{15-9} = \infty^6$ collineations that do this.

Let now $\mathfrak{F}$ be our $\zeta-$sphere. Then we find the ∞^6 collineations that move the sphere to itself matched by ∞^6 circle relationships, or linear transformations $\zeta' = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$. For if we normalize δ as 1, then α, β, γ are essential parameters. Since they are complex quantities, each represents a $2-$dimensional space and altogether we have ∞^6 transformations. One now sees by geometric considerations that would lead us too far here, but that are quite simple, that the angle between two circles on the sphere is identical with the non-Euclidian angle of the two circles on the sphere. Here we mean "non-Euclidian" in the sense that measurement is induced by choosing the $\zeta-$sphere as measure-determining surface. We conclude the following from this result: The non-Euclidian motions determined by our $\zeta-$sphere, that is the collineations that assign to a point in the $\zeta-$sphere another such point, induce on the sphere a one-to-one single valued function with the property that corresponding angles (in the Euclidian sense) are equal, hence a conformal map without singularities, that is a linear transformation

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta},$$

resp.

$$\overline{\zeta}' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

in case an inversion of angles occurs.

Skipping the proof of the converse of this result we summarize as follows[9]:

The ∞^6 motions in the sense of the non-Euclidian geometry based on the $\zeta-$sphere as absolute configuration are connected with the ∞^6 transformations of the $\zeta-$sphere given by the linear substitutions of the complex ζ, and the ∞^6 motions with inversion of angle of our non-Euclidian geometry are connected with the ∞^6 transformations of the $\zeta-$sphere in which $\overline{\zeta}'$ is a fractional linear combination of ζ.

In non-Euclidian geometry one distinguishes translations, reflections, screw operations, and shows that every congruent motion is in effect equivalent to a screw operation, and so on. All this can be transferred to our way of measurement. I only give the result: During one of the non-Euclidian motions that we are considering, a point of the $\zeta-$sphere in general moves on a loxodrome that winds around the two fixpoints of the substitution $\zeta' = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$. In special cases the "*pitch*" of this non-Euclidian screw operation can be zero and one has a "rotation". If one makes sure that the two fixpoints are diametrically opposed, then this is an ordinary rotation of the sphere. On the other hand a point can move on the circle that passes through it and the two fixpoints, so doesn't "wind" at all. This corresponds to a translation. As a new phenomenon the *parabolic case* can occur in which the two fixpoints coincide. Circles that pass through the fixpoint in a common direction will then be transformed into each other. The best way to visualize this situation is to put the fixpoint at the north pole of the $\zeta-$sphere and go back to the $\zeta-$plane by stereographic projection. One then arrives at the transformation $\zeta' = \zeta + c$ and the division of the plane into parallel strips.

Overview:

- I) *Loxodromic substitution*, 2 separate fixpoints, $\zeta' = e^{2\pi i\lambda}\zeta$.
  1. First special case, *rotation*: (elliptic substitution, λ reell).
  2. Second special case, *translation*: (hyperbolic substitution, λ purely imaginary).
- II) *Parabolic substitution*, 1 fixpoint, $\zeta' = \zeta +$ constant.

For participants familiar with the concepts of synthetic geometry I would like to add the following.

Let us think of $\mathfrak{F}$ as a hyperboloid for a moment. Then the collineations that leave $\mathfrak{F}$ invariant split into those that transform each of the two pencils of lines into itself and those

[9] See my paper in Math. Ann. 9, p. 183 ff., written 1875, for more details.

that interchange the two pencils. If we return to the $\zeta-$sphere, the two imaginary pencils of minimal lines take the place of the two real pencils of lines. This splitting of the motions amounts to distinguishing between direct and increase congruences,

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \quad \text{and} \quad \overline{\zeta}' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}.$$

So a direct congruence induces separately in each of the two pencils of lines a projective relationship between the lines as elements. In general therefore one finds two fixed lines in each of the pencils of lines. They form 4 edges of a tetrahedron whose other two edges are real: *the principal tetrahedron of the space collineation*. So there exist 2 real lines in space that are transformed into themselves by the collineation and that are conjugate polars w.r.t. the $\zeta-$sphere, the axes of the motion. One of these meets the sphere in two real points, the fixpoints of the linear transformation (the only real vertices of the principal tetrahedron). The other does not meet the sphere at all, and the two have directions that are perpendicular to each other.

According to whether the planes passing through one or the other of these axes are transformed amongst themselves one encounters the special cases of rotation and translation. In the parabolic case the axes are two tangents in the fixpoints perpendicular to each other. They already made their appearance above.

Fourth meeting, November 29, 1905

Continuation of the talk of Professor Klein

We now proceed to a discussion of circular arc triangles within the framework of these concepts. A circular arc triangle determines three planes, those that cut out its three circles on the $\zeta-$sphere. Conversely, however, different circular arc triangles can belong to the same three planes: In the first instance all that are produced from <u>one</u> by lateral or polar attachments, for in all these operations the circles are not changed, only the edges of the triangle as far as their individual arcs and overlaps are concerned. But evidently also among the reduced triangles several can belong to one and the same system of three planes. We obtain here a very simple and easily visualized version of the concept of *related* functions and triangles. Simply put, triangles are related if they belong to the same system of three planes. We will therefore give a special name to this system of three planes and call it the "*core*" of the circular arc triangle. We then can say:

Triangles are related if they have the same core.

There is a distinction to be made now in the study of circular arc triangles according to whether the *center* of the core, the point of intersection of its three planes, lies inside, on or outside the sphere. Let us add the remark that by a suitable motion (a non-Euclidian motion of course with the $\zeta-$sphere as absolute configuration) we can move any interior point of the sphere to any other interior point, for instance the center of the sphere, any exterior point into any other exterior point, for instance a point on the plane at infinity, and any point on the surface of the sphere to any other such point, for instance the center of the stereographic projection we used to relate the $\zeta-$plane and the $\zeta-$sphere. In the first case we have a triangle as they are encountered in spherical trigonometry, that is its sides are great circles. The substitutions S, T, U then transform great circles into great circles, and so are ordinary rotations of the sphere about its center, see figure 9I.

figures 9I, 9II, 9III here

If the center of the core is on the sphere, the triangle can be transformed into a rectilinear one by stereographic projection, see figure 9III. In the third case, finally, one has a triangle of pseudo-spherical trigonometry, see figure 9II, all circles are perpendicular to the equator (circle of orthogonality).

Overview

Triangles of type I, center of the core in the interior, figure 9I.

Triangles of type II, center of the core in the exterior, figure 9II.

Triangles of type III, center of the core on the sphere, figure 9III.

It is our task now to characterize these three cases through the numbers λ, μ, ν. To this end we subject them to additional reductions besides the reduction mod 2. From the geometric point of view we search among the reduced triangles belonging to the same core for a canonical one, the one we are used to consider in genuinely elementary geometry and that is the easiest to visualize. We obtain it arithmetically if we admit the following two types of reductions:
replace λ, μ, ν by

$$2 - \lambda, \mu, \nu$$

or by

$$\lambda, 1 - \mu, 1 - \nu.$$

We then pick among all triples that we obtain from $\overline{\lambda}, \overline{\mu}, \overline{\nu}$ by these processes the triple λ_0, μ_0, ν_0 for which $\lambda_0 + \mu_0 + \nu_0$ is minimal.

We call the corresponding triangle the *minimal triangle*. We can then say that the three types of cores are distinguished by λ_0, μ_0, ν_0 being resp. $> 1, < 1, = 1$, that is by whether the sum of the angles of the triangle is two right angles, or more, or less.

Regarding the totality of linear-polymorphic functions that are our subject we made, among others, the following assumption (p. 7): there is one among the ∞^3 functions that vanishes to order 1 at a given place z_0:

$$\zeta_0 = (z - z_0)\mathfrak{P}(z - z_0),$$

where the constant term in the power series $\mathfrak{P}(z - z_0)$ ist not missing. One could also say that $\frac{d\zeta_0}{dz} \neq 0$, or that the inverse function $z(\zeta_0)$ is not ramified in ζ_0. Points with the property that one of our ∞^3 linear-polymorphic functions, and therefore all, are themselves regular, but the inverse function there exhibits ramification behaviour (of course algebraic), we will call *auxiliary points*. (Poincaré says: points en apparence singuliers, a cumbersome designation I do not find very descriptive.) We now want to investigate the appearance of such *auxiliary points*.

So suppose <u>one</u> auxiliary point $\mathfrak{d}$ exists, more precisely a simple one. (We will call $\mathfrak{d}$ a $(\lambda - 1)-$fold auxiliary point if the power series development is $\zeta_0 = (z - \mathfrak{d})^\lambda\mathfrak{P}$.) So assume

$$\zeta_0 = (z - \mathfrak{d})^2\mathfrak{P}(z - \mathfrak{d}).$$

Obviously the differential equation of the third order is now different, its right hand side is more complicated. A calculation gives:

$$
\begin{aligned}
[\zeta] \;=\; & \frac{1}{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})(z-\mathfrak{d})}\left\{\frac{\frac{1-\lambda^2}{2}}{z-\mathfrak{a}}(\mathfrak{a}-\mathfrak{b})(\mathfrak{a}-\mathfrak{c})(\mathfrak{a}-\mathfrak{d})\right.\\
+\; & \frac{\frac{1-\mu^2}{2}}{z-\mathfrak{b}}(\mathfrak{b}-\mathfrak{a})(\mathfrak{b}-\mathfrak{c})(\mathfrak{b}-\mathfrak{d})\\
+\; & \frac{\frac{1-\nu^2}{2}}{z-\mathfrak{c}}(\mathfrak{c}-\mathfrak{a})(\mathfrak{c}-\mathfrak{b})(\mathfrak{c}-\mathfrak{d})\\
+\; & \left.\frac{-3/2}{z-\mathfrak{d}}(\mathfrak{d}-\mathfrak{a})(\mathfrak{d}-\mathfrak{b})(\mathfrak{d}-\mathfrak{c})+A\right\}.
\end{aligned}
$$

As to the constant A, the situation is as follows: if one gives it an arbitrary value the integrals of the differential equation of the third order will not be free of logarithmic terms. On the other hand one can always, and without great difficulty, choose A in such a way that no logarithmic terms are present. Schilling has carried this out explicitly, Math. Ann. 51, p. 489. We have proved, therefore, the possibility of an auxiliary point appearing in the way we intended.

The appearance of auxiliary points will not alter anything essential regarding the existing monodromy group. The changes that occur are characterized as follows: the cut $z-$plane is nor more mapped in one-to-one fashion onto the one-sheeted fundamental domain, but this fundamental domain, already multiply ramified in $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, is in addition covered by several sheets, each value is taken in it a certain finite number of times, it contains ramification points.

We want to form a more precise picture of this in case $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ lie on a circle, say on the real axis.

figure 10 here

ζ has an extremum at $\mathfrak{d}$. In the $\zeta-$plane we have to consider the triangle with angles $(\lambda+1)\pi, \mu\pi, \nu\pi$. ζ moves to $\mathfrak{D}$ beyond $\mathfrak{A}$ on $\overline{\mathfrak{C}\mathfrak{A}}$, reverts and returns to $\mathfrak{A}$ before moving on. This triangle with angles $(\lambda+1)\pi, \mu\pi, \nu\pi$ is not related to $(\lambda\pi, \mu\pi, \nu\pi)$ since the sums of the angles differ by an odd multiple of π[10], indeed by π.

The appearance of this non-related triangle, caused by the $180-$tear at the auxiliary point, is quite remarkable. For it shows that in the end the monodromy goup is not entirely unaffected by the appearance of auxiliary points, as one might think upon cursory inspection.

To construct the triangle (λ, μ, ν) with one simple auxiliary point, one constructs the triangle $(\lambda+1, \mu, \nu)$ without auxiliary point and performs a cut that turns back on itself, as shown in figure 10.

I add the following general theorem:
If a $\zeta-$function is endowed with n auxiliary points it is related either with the single-sheeted $(\lambda, \mu, \nu)-$triangle, or with $(\lambda+1, \mu, \nu)$, depending on whether n is an even or odd number.

Fifth meeting, December 6, 1905

Continuation of the talk by Professor Klein

[10] Klein writes: by an odd number (translator's comment)

Besides adding some complements to the material I talked about so far I will present to you today several general questions and problems that have barely been touched up to now. As is the aim of this seminar you may, according to your individual capabilities, pick one of the other of these questions and turn to one of us if it is relevant to you.

Let the angle between two concurrent lines, in the sense of non-Euclidian geometry with the $\zeta-$sphere as absolute configuration, be defined as follows. We consider the plane of the two lines and in it their point of intersection and the circle it cuts out on the $\zeta-$sphere. We draw the tangents from the point of intersection to the circle and compute the cross-ratio for the two tangents and the two given lines; its logarithm, multiplied by $\frac{i}{2}$, is called the (non-Euclidian) angle between two lines.

figure 11 here

Just as the non-Euclidian angle of two planes agrees with the Euclidian one in case the line of intersection of the two planes passes through the center of the sphere, the just defined non-Euclidian angle of two concurrent lines is the same as their Euclidian angle if the point of intersection of the two lines is the center of the sphere. This non-Euclidian angle, moreover, is real if the two lines that form it meet inside the sphere, purely imaginary if they meet outside and 0 if on the surface of the sphere.
Note that in Euclidian geometry as well being parallel means that two lines meet in a point of the absolute configuration. The angle of two lines through the center of the sphere is identical with the arc of the great circle between the points in which they meet the sphere. So if one considers a triangle in spherical trigonometry, its so-called "sides" are identical with the angles between the edges of the corresponding core, multiplied by the radius of the sphere. There is complete agreement if we choose the radius as unity. In pseudo-spherical geometry, however, one has to say that the sides of the triangle agree with the non-Euclidian angles between the corresponding edges of the core, multiplied by the $i-$fold radius of the sphere. One has complete agreement here if one thinks of the $\zeta-$sphere as a sphere of radius i. This does not present any difficulties from the point of view of algebraic geometry. In plane geometry finally (or in the geometry on the sphere corresponding to it by stereographic projection) the sides of the triangle agree with the angles at the core if one takes the radius of the sphere infinitely large, just as the plane is a sphere of infinite radius.

The sides of a triangle in spherical, pseudo-spherical, plane trigonometry agree with the non-Euclidian angles between the edges of the corresponding core if one thinks of the radius of the $\zeta-$sphere as resp. $= 1, i, \infty$.

As far as the *formulas of spherical, pseudo-spherical and plane trigonometry* are concerned they now appear under a unified point of view. The formulas of spherical trigonometry immediately become formulas concerning the non-Euclidian angles at the core. They deal with sines and cosines of these angles and since the latter are themselves logarithms of algebraic quantities, indeed of cross-ratios, these formulas are relations between algebraic quantities. They remain unchanged by algebraic transformations, such as collineations, that transform the cores of the various types into each other. If one finally replaces the non-Euclidian angles by the usual ones according to the theorem just formulated, one immediately obtains the formulas of pseudo-spherical and plane trigonometry.

In the case of cores of the second and third kind one again has exactly the formulas of spherical trigonometry since the latter are just algebraic identities between algebraically defined quantities.

After these preliminary remarks I will finally turn to a

discussion of the significance of these considerations to our function-theoretic questions.

We are concerned with a linear-polymorphic function ζ_0 and the totality of the ∞^3 linear fractional combinations of ζ_0. More precisely, we are considering ζ_0 with three ramification points. The other assumptions we made were related solely to the exclusion of certain unpleasant special cases. Our main task now is the following: We think of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \lambda, \mu, \nu$, or put briefly, of the differential equation of order 3, as given. We want to determine the substitutions that ζ_0 is subject to if it is continued along all possible paths in the $z-$plane, that is, the *monodromy group* of ζ_0. Strictly speaking we are asking too much here. For if we take instead of ζ_0 a branch of another one of our ∞^3 functions, $\zeta_1 = S(\zeta_0)$, then the monodromy group of ζ_1, is by no means identical with the one of ζ_0, but it is isomorphic (in fact conjugate[11]) to it:

If along a path ζ_0 is subject to the substitution A, then ζ_1 is subject to the substitution SAS^{-1} along the same path. We now define as our task the precise determination of those properties of the monodromy group of ζ_0 that it has in common with all these other groups.

The difficulties we will encounter undertaking this are the following: We know the substitutions:

$$\zeta'_{\mathfrak{a}} = e^{2\pi i\lambda}\zeta_{\mathfrak{a}}, \zeta'_{\mathfrak{b}} = e^{2\pi i\mu}\zeta_{\mathfrak{b}}, \zeta'_{\mathfrak{c}} = e^{2\pi i\nu}\zeta_{\mathfrak{c}}$$

that certain branches $\zeta_{mathfraka}, \zeta_{\mathfrak{b}}, \zeta_{\mathfrak{c}}$, suitably chosen for each branch point, are subject to, but we do not yet know the connection between these branches, the linear substitutions that transform them into each other.

Of what use for this purpose is the geometric interpretation we prepared? Let me begin by repeating the interpretation:

We visualize the substitutions S, T, U as non-Euclidian rotations of the $\zeta-$sphere about the edges of the core by the angles $2\lambda\pi, 2\mu\pi, 2\nu\pi$. The quantities l, m, n simply tell us how the axes of rotation are located relative to each other. The arbitrariness in the choice of the branch ζ_0 amounts to the freedom of placing the spherical triangle without changing its shape (in the non-Euclidian sense).

So the restrictions we imposed above, that is to study only properties shared by all these isomorphic monodromy groups, become very evident visually: We restrict ourselves to the study of the shape relationships in the triangle. As to the main task, it finds its complete solution through recourse to spherical trigonometry. For it amounts to finding the relationship between the angles λ, μ, ν on the one hand and the sides l, m, n on the other, and these relationships are given by the apparatus of formulas of spherical trigonometry. In this way our geometric interpretation allows us to make use of a well worked over field for considerations that Gauss and Riemann themselves still treated directly with a lot of effort. I summarize:

Spherical trigonometry with its algebraic relations between $\lambda, \mu, \nu; l, m, n$ gives us a complete description of the substitutions S, T, U according to their essential properties. It is not at all alien to our objective, but makes unnecessary an independent investigation into the monodromy group.

Historically these concepts were not found starting from the very natural circle of ideas we developed, but from the theory of a special linear differential of the second order, the so called "*Gaussian differential equation*". Even if this differential equation is no more essential

[11]Klein uses "holoedrisch" (translator's comment)

for our development (which follows Riemann) as is the differential equation used so far, in as much as it is a differential equation of order 3, I will nevertheless give a brief sketch of the

connection with the theory of differential equations of the second order.

Let

$$y'' + py' + qy = 0$$

be this differential equation of order 2, where p, q are rational functions of z. If y_1, y_2 are two particular solutions, one easily calculates for the quotient

$$\zeta = \frac{y_1}{y_2}$$

that

$$[\zeta]_z = 2q - \frac{p^2}{2} - \frac{dp}{dz}.$$

We obtain our differential equation of order 3 *if in the theory of differential equations of order* 2 *we consider the quotient of two particular solutions,* $\zeta = \frac{y_1}{y_2}$*. We find* $[\zeta]_z = 2q - \frac{p^2}{2} - \frac{dp}{dz}$*. So the study of the* $\zeta-$*function serves as a preparation for the study of the linear differential equation of order* 2*.*

In the opposite direction we have

$$\frac{y_1' y_2 - y_2' y_1}{y_2^2} = \zeta'.$$

One finds easily, using the differential equation of order 2, that

$$y_1' y_2 - y_2 y_1' = e^{-\int p dz},$$

that is

$$\zeta' = \frac{e^{-\frac{1}{2}\int p dz}}{y_2^2},$$

and it follows that

$$y_2 = \frac{e^{-\frac{1}{2}\int p dz}}{\sqrt{\zeta'}}$$

and

$$y_1 = \frac{\zeta e^{-\frac{1}{2}\int p dz}}{\sqrt{\zeta'}}$$

If one has studied ζ *as a function of* z*, one can progress from there directly to particular solutions* y_1 *and* y_2 *of the linear differential equation of order* 2*.*

In this way our investigations into triangles provide a treatment of the linear differential equation with 3 *ramification points, also called the differential equation of the hypergeometric function.*

We note that the differential equation of order 2 is not completely determined by specifying the one of order 3, one has leeway, for instance in the choice of the rational function p.

Sixth meeting, December 13, 1905

Continuation of the talk of Professor Klein

I will first give a more thorough exposition on the questions I had just started to talk about at the end of the last meeting. We have assumed so far - and this is the simplest non-trivial assumption - that the differential of the third order has three singular points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. We have also assumed, apart from the very first beginnings, that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; \lambda, \mu, \nu$ are real. This is no essential restriction on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, but it is on λ, μ, ν. Now Schilling has shown how to extend, in a surprisingly simple way, the program we developed for real λ, μ, ν, using the tools of non-Euclidian geometry, to complex λ, μ, ν. In the second instance my talk today will report on these investigations of Schilling. Before that, however, I will talk about the other generalization, namely that there are n ramification points present instead of three. Here only the beginnings of a treatment are available.

Let the n singular points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \ldots$ as well as the exponents λ, μ, ν, ρ initially be real. The extension to complex $\lambda, \mu, \nu, \rho, \ldots$ then can be made in a way similar to the one Schilling uses for the case $n = 3$. The image of the positive $z-$half-plane will then be an $n-$gon composed of circular arcs on the $\zeta-$sphere. Its core consists of n planes. Any two neighbouring planes intersect in an edge of the core. Any two consecutive edges meet.

figure 12 here

The edges form a configuration in space that we may perhaps call a "skew $n-$gon".

Here one has three series of characteristic quantities that determine the shape (in the non-Euclidian sense) of the core: the angles $\lambda\pi, \mu\pi, \nu\pi, \rho\pi, \ldots$ of any two consecutive planes, the angles $l\pi, m\pi, n\pi, \ldots$ between any two consecutive edges, the arcs $L\pi, M\pi, N\pi, \ldots$ between any two consecutive vertices. The task now is to determine the algebraic relations between these $3n$ characteristic quantities that one would designate as the *formulas of spherical polygonometry.*

So our first task would appear as: Establish a spherical polygonometry.

A subsequent task then would be to obtain the analogue of the "extension relation" which allowed us to proceed to overlapping triangles with angles $> 2\pi$. We should expect several such relations. For in addition to the processes of polar and lateral attachment a new phenomenon appears in the form of the transversal attachment of annuli, as figure 13 will illustrate.

A second task following the first is to discuss the necessary extension theorems of such a spherical polygonometry.

figure 13 here

Figure 13 shows the stereographic projection of the lower $\zeta-$half-sphere on which a circular arc $7-$gon is placed. In the transversal attachment the vertically hatched annulus is added on. (Only the image of the part that lies on the lower half-sphere is hatched. In addition the whole upper half-sphere, the whole exterior of the drawn figure, belongs to it.)

Van Vleck has treated the case in which the 4 planes meet in a point that, specially, lies on the surface of the $\zeta-$sphere. One then obtains a rectilinear quadrangle in the plane through stereographic projection.

Schoenflies has dealt with circular arc $4-$gons in general. The contact I had with him at the time has left me with the impression that some things here are still in need of completion.

Fricke attacked the case for circular arc $n-$gons in which all planes of the core pass through a point outside the $\zeta-$sphere. If one puts it on the plane at infinity, then all circles of the polygon are perpendicular to a certain great circle, the so called *principal circle*. A comprehensive exposition by Fricke of these investigations can be found in Fricke-Klein, automorphe Funktionen, I, 2. Abschnitt, 2. Cap. (p. 285-398).

Finally I have myself treated in depth those 4−gons that appear in connection with Lamé's equations, that is those in which all angles are $\equiv \frac{1}{2}\pi \pmod{\pi}$.[12] An exposition can be found in the autographed notes of my lectures of summer 1984 in the chapter on Lamé's equations. (On p. 382/383 one finds a sample sheet of relevant figures.)

The proof of the fact that $S.T.U.\cdots = 1$ can be done analogously a way analogous to the one used above (p. 15, 16) for triangles according to the principle of Hamilton. Again this one relation contains the totality of algebraic formulas in polygonometry. So of these there are always only three. But in addition there are certain unpleasant sign dependencies that can be described algebraically, that is by algebraic equations, if one uses more than three relations all of which follow from three of them, but without being linear combinations.

In second place now I will talk about

complex λ, μ, ν and Schilling's configuration.

Here the situation at first glance appears much more opaque than with real λ, μ, ν. For we do not have available the reality considerations that we used in case of real λ, μ, ν to show that to the circle through $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ there corresponds a circular arc triangle. We further do not have available the reflections in these arcs and cannot carry through the proof of the relation $STU = 1$ in the manner of Hamilton. What remains is the idea of the full fundamental domain, which above may have given the impression of a superfluous complication and only here really takes its proper place.

I may be allowed to first emphasize a few points in the previous developments. We there took some point $\mathfrak{O}$ in the interior of the circular arc triangle $\mathfrak{ABC}$, and its reflections $\mathfrak{O}'_1, \mathfrak{O}'_2, \mathfrak{O}'_3$ w.r.t. the three sides.

figure 14 here

The hexagon $\mathfrak{AO}'_3, \mathfrak{BO}'_1, \mathfrak{CO}'_2$ then was a fundamental domain, and to its 6 sides there corresponded in the $z-$plane the in each case 2 banks of certain three cuts made from the point $\mathfrak{O}$ corresponding to $\mathfrak{O}'$ to $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. These made the $z-$plane simply connected. We could instead make straight cuts from $\mathfrak{O}$ to $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and look for the corresponding curvilinear hexagon-like figure. It too would be a fundamental domain.
The fact that the rotation S of the $\zeta-$plane about A by the angle $2\lambda\pi$ could be seen as the effect of a reflection in m and one in n then put in evidence the relation $STU = 1$.

What of all this will remain for complex λ, μ, ν? We can again make the $\zeta-$plane simply connected, say by straight cuts $\mathfrak{O}'\mathfrak{a}, \mathfrak{O}'\mathfrak{b}, \mathfrak{O}'\mathfrak{c}$. The six banks of the three cuts will correspond to the sides of a curvilinear hexagon on the $\zeta-$sphere, with a correspondence in pairs. This correspondence is no more given by rotations, but by the general loxodromic substitutions S, T, U. So the fundamental domain is present, and one could even achieve that its sides are circular arcs since we do not have to insist on straight cuts in the $z-$plane. We choose $\mathfrak{AO}'_3, \mathfrak{BO}'_1, \mathfrak{CO}'_2$ as circular arcs. This determines three cuts in the $z-$sphere and thereby also the three sides of the hexagon that correspond to their other banks. So one is sure to begin with that these three remaining sides of the hexagon are circular arcs as well since they arise from the first three through the substitutions S, T, U which transform circles into circles.

[12] Klein writes mod 1 (translator's comment)

It now is Schilling's main contribution to have found a replacement for the concept of reflection in the sides of the triangle. With its help the relation $STU = 1$ becomes evident in a way similar to the case of real λ, μ, ν.

Schilling's starting point is the observation in non-Euclidian geometry that two skew lines that meet the $\zeta-$sphere admit two common perpendiculars that are conjugate polars with respect to the $\zeta-$sphere. So one of them meets the sphere, whereas the other passes it on the outside.
Schilling disregards the latter and considers the first as "*the*" common perpendicular. He now considers the following figure (see figure 15).

figure 15 here

He takes the three axes S, T, U of the screw-operations S, T, U, which here need not intersect and provide a certain equivalent to the core, and he adds "*the*" common perpendiculars $S^\sharp, T^\sharp, U^\sharp$ of any two of these three axes. To each side of this skew hexagon he now assigns a complex number. In the case of S, for instance, he first measures the distance $\lambda''\pi$ between the base points on S of the perpendiculars $T^\sharp, U^\sharp$. He then measures the angle $\lambda'\pi$ between the planes spanned by S and $T^\sharp$ and S and $U^\sharp$ and forms the number

$$\lambda\pi = (\lambda' + \lambda'' i)\pi.$$

In this way he assigns complex numbers λ, μ, ν to the axes S, T, U and shows that these are precisely the numbers λ, μ, ν, as they are determined by the substitutions S, T, U. We can, indeed, bring S into the canonical form

$$\zeta' = e^{2i\pi\lambda}\zeta$$

and recognize that the real part of λ determines the rotation and the purely imaginary part determines the translation into which S can be decomposed:

$$\zeta' = e^{2i\pi\lambda'}.e^{-2\pi\lambda''}\zeta.$$

The rotation about S and the translation along S therefore are characterized by the two numbers λ', λ'' that Schilling assigns to the axis S.

I skip the proof of this and prefer to show how one deduces the identity $STU = 1$. Instead of into two reflections, Schilling decomposes each of the substitutions into turns by 180°. He first turns S about $T^\sharp$, and then about $U^\sharp$, each time by 180°. Then S is precisely the composite of these two turn s:

$$S = (T^\sharp)(U^\sharp), T = (U^\sharp)(S^\sharp), U = (S^\sharp)(T^\sharp).$$

In common with reflections these turns by 180 have the property to be involutory, that is $(S^\sharp)^2 = 1, (T^\sharp)^2 = 1, (U^\sharp)^2 = 1$. It follows that $STU = 1$.

Schilling in addition obtains an analogue to the sides l, m, n of the triangle in the complex numbers he assigns to the axes $S^\sharp, T^\sharp, U^\sharp$. He shows that the formulas of spherical trigonometry hold between them and λ, μ, ν. This point, in particular, is discussed in my lectures on the hypergeometric function of 1893/94. Everything can be found, moreover, in Schilling's dissertation, published in Math. Ann. 44. I must be content here to state that l, m, n become the sides of the triangle, and so are connected to λ, μ, ν by the formulas of spherical trigonometry, in case S, T, U are concurrent in a point. For $S^\sharp, T^\sharp, U^\sharp$ become three lines concurrent in the same point, the polars resp. to S, T, U with respect to the $\zeta-$sphere.

For differential equations with more than 3 ramification points I posed our problem above under the assumption that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ are real, and $\lambda, \mu, \nu, \ldots$ as well. The first condition imposes a restriction if $n > 3$ since in general more than 3 points won't lie on a circle, and so cannot be placed on the real line by a linear transformation. No less does the second cause new difficulties: Schilling's idea is not powerful enough to treat complex $\lambda, \mu, \nu, \ldots$ in a similar fashion if $n > 3$. For if we arbitrarily prescribe n axes that meet the $\zeta-$sphere, then there is no more freedom in building the figure besides the $2n$ choices (counted in the complex sense) that determine the n axes. Opposed to these are the $3n-3$ constants of the n substitutions $S, T, U, \ldots$, which, as we know, are subject only to the relation $STU = 1$. For $n = 3$ we have $2n = 3n - 3$, and Schilling has indeed shown that his construction produces all monodromy groups. For $n > 3$, however, we have $2n < 3n-3$ and our count shows that it is a priori impossible for a configuration analogous to Schilling's to produce all monodromy groups.

Tenth meeting, December 20, 1905

Continuation of the talk of Professor Klein

I now leave the most recently discussed questions of elementary geometry and turn to

deeper questions of function theory

as they were approached by Riemann in his famous fragment of Feb. 20, 1857: "Zwei allgemeine Lehrsätze über lineare Differentialgleichungen mit algebraischen Coefficienten", (Werke, Nr. XXI in the first, Nr. XIV in the second edition).

This means I will talk about the general $\zeta-$function with n ramification points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ with n exponents $\lambda, \mu, \nu, \ldots$, with k auxiliary points $\mathfrak{p}, \mathfrak{q}, \ldots$ and the $n-3$ accessory parameters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$:

$$\zeta\begin{pmatrix} \mathfrak{abc}\ldots \\ \lambda\mu\nu\ldots \end{pmatrix}; \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots; \mathfrak{p}, \mathfrak{q}, \ldots; z\Bigg).$$

The only generalization that I will leave aside, and that I have so far always left aside, is that ζ is more generally defined and single valued everywhere on the Riemann surface of an algebraic function instead of just the plane, in other words, that $[\zeta]$ is equated not with a rational function, but with an algebraic function of z.

We called such a function "*linear-polymorphic*", following the nomenclature I developed together with Fricke. I will add on this occasion that the inverse function then has to be called a "*linear-automorphic*" function. For as we continue our function across the ramification cuts in the $z-$plane, on the $\zeta-$sphere one fundamental domain is placed next to the other. One obtains above the $\zeta-$sphere[13] an extraordinarily multiply ramified Riemann surface - one only has to visualize how already in the simplest case the fundamental domain may overlap itself, that new overlaps occur if auxiliary points are added, and that then the fundamental domains placesd next to each other again form an infinite number of layers.

At all places of this sphere-like Riemann surface, however, that are produced from one by the substitutions of the monodromy group, $z(\zeta)$ has one and the same value z. For this reason we want to call $z(\zeta)$ a "*linear-automorphic function of* ζ", that is a function whose value is preserved by certain linear substitutions. It is important, however, that this invariance is properly understood as happening on the Riemann surface we just constructed. It is not

[13]It might be more natural to write $z-$sphere here (translator's comment)

correct, of course, that <u>all</u> branches of $z(\zeta)$ take the same value at <u>all</u> points of the $\zeta-$sphere that are produced from <u>one</u> by $S, T, U, \dots$. Let us note that in special cases $z(\zeta)$ may have finitely many determinations, or may even be single-valued. In the fragment mentioned above, Riemann speaks of differential equations of order n. We will extract only what regards the case $n = 2$.

We so far always thought of the differential equation of order 3, that is the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ and the exponents $\lambda, \mu, \mu \dots$, as given in the first instance, and asked for the corresponding monodromy groups, or for the essential properties they have in common with all isomorphic groups. These properties were made explicit as certain algebraic relations, relations that we moreover succeeded in identifying with the formulas of spherical polygonometry, that is with certain very general facts.

Riemann now takes as given the ramification points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ and the monodromy group, that is the corresponding n substitutions $S, T, U, \dots$, subject only to the unique relation $STU \cdots = 1$. He then asks:

Do there always exist $\zeta-$functions that are ramified precisely according to the substitutions $S, T, U, \dots$? How large is their totality, how are they related to each other?

In contrast to the question in the form we presented it so far, we can call this a *transcendental problem.*

By a constant count we would like to obtain a preliminary orientation regarding this problem and at the same time obtain a more precise formulation. To begin with the monodromy group depends on the $3n - 3$ constants in the substitutions $(S.T.U. \cdots = 1)$. But not everything in them is essential for the essential properties of the group. If it belongs to a branch ζ_0, then each $\frac{\alpha\zeta_0+\beta}{\gamma\zeta_0+\delta}$ defines another, isomorphic one (see p. 9). So we recognize 3 of these $3n - 3$ choices as non-essential and there remain $3n - 6$:

The monodromy group depends on $3n - 6$ essential parameters.

In comparison, how many constants are available to us in the $\zeta-$function? There are n exponents, $n - 3$ accessory parameters, k auxiliary points, so alltogether $n + (n - 3) + k = k + 2n - 3$. If this number is smaller than $3n - 6$ we cannot always expect the existence of a $\zeta-$function belonging to the group. Here the importance of the auxiliary points becomes apparent, and at the same time the fact that the case $n > 3$ offers considerable new interest compared to $n = 3$. In order to obtain

$$k + 2n - 3 \geq 3n - 6$$

we must have

$$k \geq n - 3.$$

So for $n = 3$ the use of auxiliary points is not necessary. But for $n > 3$ we can, given an arbitrary monodromy group, expect the existence of $\zeta-$functions only if we admit $k = n - 3$ auxiliary points:

Given n ramification points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ and corresponding substitutions $S, T, U, \dots$ with product 1, we can expect a positive answer to Riemann's question regarding the existence of $\zeta-$functions belonging to them only if we allow these $\zeta-$functions to have $k = n - 3$ auxiliary points (as a minimum).

For $n = 3$, Riemann's question in essence is moot. In fact, for each ramification point, for instance $\mathfrak{a}$, we can find a branch $\zeta_{\mathfrak{a}}$ which is subject to the "normal" substitution

$$\zeta'_{\mathfrak{a}} = e^{2\pi i\lambda}\zeta_{\mathfrak{a}}$$

by an orbit around $\mathfrak{a}$. So λ, μ, ν are determined by these substitutions up to integers that can be added, and this in essence determines the differential equation of order 3. It is only through the auxiliary points, appearing with $n = 4$, that new complications arise.

Very recently new angles of approach to Riemann's existence problem seem to have been opened: Schlesinger (Klausenburg) treats it through continuity considerations, and Hilbert finds new avenues of attack from the point of view of integral equations. I myself had always hoped to reach the aim by considerations concerning the shape of the fundamental domain. I will now present to you *a particularly simple example* that I have worked out completely.

I make a very special choice of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ and of the monodromy group: $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ are real, and I assume that the group consists entirely of parabolic substitutions with the normal form $\zeta' = \zeta + c$, and even that in a special way. I pass over the precise shape of the four basic substitutions and will give a brief survey only.

To fix the context I will assume right away that a $\zeta-$function with this monodromy group exists. Since orbits about the singular points change it only by an additive constant, I am not forced to go back all the way to the differential equation of order 3, already the first derivative of the function will be an algebraic, even if not a rational, function of z. A more precise calculation gives

$$\zeta = m \int \frac{(z-\mathfrak{p})dz}{\sqrt{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})(z-\mathfrak{d})^3}}.$$

I assume that $\mathfrak{p}$ also is real and that ζ becomes infinite in $\mathfrak{d}$. If we are presented with such a ζ we will first try to form an idea about the image of the positive half-plane, depending on which interval marked off on the real axis by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ contains $\mathfrak{p}$.

figure 16 here

Eighth meeting, January 10, 1906

Continuation of the talk of Professor Klein

These figures (figure 16) show how with fixed $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ (and m) the figures develop in the $\zeta-$plane as $\mathfrak{p}$ wanders along the real axis of the $z-$plane. We see in particular how a full quarter plane is pinched off any time $\mathfrak{p}$ coincides with one of the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$.

With these preparations done we now attempt a proof of Riemann's theorem in our special case. Before we kept m and $\mathfrak{p}$ fixed and only looked for the shape of the figure in the $\zeta-$plane. Now we keep only $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ fixed and on the other side the periods $\Omega, i\Omega$ of our elliptic integral, in other words the points in the $\zeta-$plane corresponding to $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$. We will from now on designate these by Greek letters, that is by $\alpha, \beta, \gamma, \infty$, and ask whether we can choose m and $\mathfrak{p}$ in such a way that our elliptic integral maps the positive $z-$half-plane precisely onto a figure in the $\zeta-$plane with the parameters $\Omega, i\Omega'$. The figure in the $z-$plane characterized by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ we will call $\mathfrak{M}$, the figure in the $\zeta-$plane $\mathfrak{M}'$.
For the figure $\mathfrak{M}$ we have essentially <u>one</u> degree of freedom. For we can place $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ in arbitrary positions in the $\zeta-$plane by a linear transformation, and then only the position of $\mathfrak{d}$ remains variable.

In $\mathfrak{M}'$ we can, once $\Omega, i\Omega'$ are chosen, still dispose of the position of the point π corresponding to $\mathfrak{p}$. Our figures already have made apparent in which way we can choose π. We therefore have a simply extended totality of figures $\mathfrak{M}'$. Riemann's theorem for our case now asserts that for each figure M there is one among all the figures M' that is produced from it by a conformal map.

Let us note first that the converse is quite evident. If only $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are given, and a fixed figure $\mathfrak{M}'$, then according to the principles of conformal mapping the positive $z-$half-plane can be mapped onto the figure $\mathfrak{M}'$ so that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and α, β, γ correspond to each other. Moreover $\mathfrak{d}$ is then uniquely determined as the point corresponding to the boundary point $\delta = \infty$ of the figure $\mathfrak{M}'$. So to each figure $\mathfrak{M}'$ there corresponds precisely one figure $\mathfrak{M}$.

We now think of the figures $\mathfrak{M}'$ as ordered in their totality in a slightly different way (see figure 17).

figure 17 here

While $\mathfrak{M}'$ runs continuously through these series from their initial term to the end term, the $\mathfrak{d}$ of the corresponding $\mathfrak{M}$ moves on the axis of real z from $\mathfrak{a}$ via ∞ to $\mathfrak{c}$, and then back to $\mathfrak{a}$ via $\mathfrak{b}$.

We now make use of an additional fact from the theory of conformal maps, namely that $\mathfrak{d}$ varies *continuously* as $\mathfrak{M}'$ runs continuously from the first term of its continuous series of shapes to the last. This is quite plausible as long as the pinching-off mentioned above does not occur. For the spots in the series of figures $\mathfrak{M}'$, however, at which we encounter the pinching-off of parasitic quarter-planes, the continuity in the change of $\mathfrak{d}$ becomes at least moderately plausible if in visualizing the integral we take recourse to electric currents instead of conformal maps. For shortly before it is pinched off the quarter-plane is connected with the rest of the figure by a very narrow bridge. Hence very little of the current, which essentially moves in the rest of the figure from one of the poles located there to the other, finds its way to the quarter-plane, and this small amount will continuously go to zero as the quarter-plane is severed at the critical limit.

If we observe finally that the limit terms in our series of figures $\mathfrak{M}'$ correspond to definite figures $\mathfrak{M}$ we can now carry out the main argument in our proof by continuity.

We think of the figures $\mathfrak{M}'$ as ordinates in a Cartesian coordinate system and of the point d as abscissa, each moving in a segment, with a beginning point and an endpoint, of the axis of ordinates or resp. abscissas.

figure 18 here

We have shown for each $\mathfrak{M}'$ there is a unique $\mathfrak{d}$ corresponding to it continuously, and that endpoints correspond to endpoints. The intermediate value theorem of differential calculus now allows the conclusion that for each $\mathfrak{d}$ there is an $\mathfrak{M}'$ corresponding to it. This is the proof by continuity.

The fact that the function $\mathfrak{d}(\mathfrak{M}')$ is one-to-one is not superfluous for this argument. The following figure (see figure 19) will illustrate this.

figure 19 here

Here to each $\mathfrak{M}'$ there corresponds an $\mathfrak{M}$, may be even several, and in a continuous way. Nevertheless there are $\mathfrak{M}$ without corresponding $\mathfrak{M}'$.

It is easy - as Professor Minkowski pointed out to me - to give an analytic proof of Riemann's theorem in the special case discussed above. For the elliptic integral under consideration

$$\zeta = m\int \frac{(z-\mathfrak{p})dz}{\sqrt{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})(z-\mathfrak{d})^3}}.$$

can be split into one of the first and one of the second kind if we replace $z-\mathfrak{p}$ by $(z-\mathfrak{d})+(\mathfrak{d}-\mathfrak{p})$. If we put $\mathfrak{d}-\mathfrak{p}=n$ we obtain

$$\zeta = m\ u + n\ Z.$$

If $\Omega, i\Omega'$ are the periods of u and $\eta, i\eta'$ those of Z, then we have

$$m\omega + n\eta = \Omega$$
$$m\omega' + n\eta' = \Omega'.$$

With given $\Omega, i\Omega'$ there are two linear relations in m, n that always have a unique solution[13] and so give m and $\mathfrak{p}$. So here we obtain a proof of existence and uniqueness at the same time. Now the above proof by continuity was supposed only to illustrate, at the hand of this special case, a basic principle which has a wider reach. It would be desirable to test it with further special cases and to explore what the connection is with the general investigations made recently by L. Schlesinger.

I would like to add one more comment regarding the quadrilaterals of figure 17 and their relationship with the above elliptic integral. The hatched quadrilaterals are images only of the positive $z-$half-plane, not yet of the elliptic configuration belonging to the elliptic integral, that is the two-sheeted Riemann surface one has to construct above the $z-$plane in to make the integrand, single-valued on it. The situation is illustrated in figure 20 for any one of the quadrilaterals.

figure 20 here

After two reflections one has a rectangle of twice the size and attached to it along a ramification cut $\pi \ldots \pi$ a full plane. (In the figure we think of the rectangle as lying on the full plane.) Corresponding to the pinching-off of a quarter-plane we have here the pinching-off of the full plane. What remains is a rectangle with sides $2\Omega, 2\Omega'$, as it should be.

Ninth meeting, January 17, 1906

Continuation of the talk of Professor Klein

I will turn today to those $\zeta-$functions that can be traced back to elementary functions and their integrals. We will gain some insight thereby into how the considerations we made so far fit within the framework of the usual theory of functions. We prefer to express this using the monodromy group. We will consider $\zeta-$functions with three singular places $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$,

$$\zeta\begin{pmatrix} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \\ & & & z \\ \lambda & \mu & \nu & \end{pmatrix}$$

whose monodromy group

[13] Legendre's relation says that the determinant does not vanish.

1) consists of the identity substitution only,

2) consists of finitely many substitutions only,

3) takes all its substitutions from some subgroup of the group of all linear substitutions, that is either

$$\zeta' = \zeta + n, \quad \text{or}$$

$$\zeta' = m\zeta, \quad \text{or}$$

$$\zeta' = \zeta + n.$$

ad 1. $\zeta(z)$ is single-valued in the entire sphere and has the character of a rational function, in short is itself a rational function. λ, μ, ν are integers and their sum is odd:

$$\lambda + \mu + \nu = 2k + 1.$$

We have moreover

$$\lambda < \mu + \nu, \quad \mu < \nu + \lambda, \quad \nu < \lambda + \mu.$$

If not, logarithmic terms would appear upon integration of the differential equation of order 3. As far as the *derivation of the rational function from the differential equation* is concerned, we have

$$\zeta - \alpha : \zeta - \beta : \zeta - \gamma = (z - \mathfrak{a})^{\lambda}\varphi(z) : (z - \mathfrak{b})^{\mu}\chi(z) : (z - \mathfrak{c})^{\nu}\psi(z).$$

The degree of the function is calculated as

$$k = \frac{\lambda + \mu + \nu - 1}{2}$$

The φ, χ, ψ are found to be hypergeometric series with finitely many terms. We omit the detailed calculation since we will later on have to deal with these hypergeometric series in all generality.

The *conformal map* effected by $\zeta(z)$ has the following form: the positive $z-$half-plane is transformed into the interior of a circle in the $\zeta-$plane, so into a circular arc triangle with each angle equal to π, or more specifically into the positive $\zeta-$half-plane.

figure 21 here

To it the right number of lateral attachments (polar attachments give nothing new here) have to be made so that λ, μ, ν reach their true values, that is

$$-\frac{\lambda + \mu + \nu - 1}{2}, \quad \frac{\lambda - \mu + \nu - 1}{2}, \quad \frac{\lambda + \mu - \nu - 1}{2}$$

half-planes to resp. $\overline{\beta\gamma}, \overline{\gamma\alpha}, \overline{\alpha\beta}$. This means that

$$\frac{\lambda + \mu + \nu - 3}{2}$$

half-planes in total have been attached to the original one. After a reflection in the real $\zeta-$axis we obtain the $k-$fold covered $\zeta-$plane as image of the full $z-$plane. This way it becomes evident that $\zeta(z)$ is a rational function of degree k.

I put next here what I have to say

ad 3. I treat generally $\zeta-$functions whose monodromy group is composed exclusively of substitutions of the form $\zeta' = m\zeta+n$. These $\zeta-$functions can be represented in an elementary way as

$$\zeta = \int \frac{(z-\mathfrak{a})^{\lambda-1}(z-\mathfrak{b})^{\mu-1}(z-\mathfrak{c})^{\nu-1}}{\varphi_k(z)^2}dz.$$

Here $\varphi_k(z)$ is a finite polynomial with zeros arranged so that no logarithmic terms appear when the integrand, decomposed into partial fractions, is integrated. The sum of λ, μ, ν again has to be odd,

$$\lambda + \mu + \nu = 2k + 1,$$

but they no longer have to be integers. To gain some geometric insight into this case we will assume that λ, μ, ν are real. As far as the core is concerned, the special features of our case are then characterized by the fact that its center is on the sphere, and indeed is the point ∞. One then has a rectilinear triangle in the $\zeta-$plane that is obtained from a minimal triangle with $\lambda_0 + \mu_0 + \nu_0 = 1$ by lateral and polar attachments. Zero, one, two of the vertices of this triangle are at infinity according to whether resp. zero, one, two among the numbers λ, μ, ν are negative. From the shape of this triangle (its overlaps) it can be read off how the real zeros of the real polynomial $\varphi_k(z)$ are distributed among the intervals $\mathfrak{bc}, \mathfrak{ca}, \mathfrak{ab}$ of the real axis and what their number is. This question was once answered by Hilbert in a different way.

ad 2. Here the situation is much more intriguing. In essence we are looking for all finite groups of linear substitutions. For we will initially drop the restriction to $\zeta-$functions with only three singular places, only to find a posteriori that in reality only these have to be considered.

I will follow the method I used in my paper in Math. Ann. 9 and that I already mentioned several times. To begin with, a substitution in a finite group can have only finitely many distinct iterates. It follows that it can only be one of the non-Euclidian motions of the $\zeta-$sphere which we call rotations by an integral fraction of 2π. I then show that the axes of two such rotations must meet if their product is again to be a rotation. Moreover their common point is in the interior of the sphere. Continuing this line of argument I find that the axes of all the rotations in the group meet in one and the same point in the interior of the sphere. The result of moving this point to the center of the sphere then is that all substitutions of the group become ordinary rotations of the sphere.

After a suitable transformation is applied to the $\zeta-$sphere, every finite group of linear substitutions in ζ consists solely of ordinary rotations of the $\zeta-$sphere. In this way the entire problem is reduced to a problem of elementary geometry which has long been solved.

Another very simple proof makes use of Hermitian forms. Let $\zeta = \frac{\zeta_1}{\zeta_2}$. If $\overline{\zeta}_1, \overline{\zeta}_2$ are the complex conjugates of the quantities ζ_1, ζ_2, then

$$\zeta_1\overline{\zeta}_1 + \zeta_2\overline{\zeta}_2$$

is a definite Hermitian form. If one has a finite group of substitutions, applies all of them to the above form and adds all the resulting forms, then this sum is again a definite Hermitian form and it remains unchanged under all substitutions of the group. Now a definite Hermitian from equated to zero defines a point in the interior of the sphere. So there is such a point fixed under all motions of the group. - This reasoning will not work in the presence of an infinite group. For the finite sum of all transforms we used above becomes an infinite series that need not converge. This is the way Loewey put it during the Frankfurter Naturforschertag.

The determination of all finite rotation groups leads to solving the diophantine equation

$$(1-\frac{1}{k_1})+(1-\frac{1}{k_2})+\cdots=2-\frac{2}{N},$$

where $k_1, k_2, \ldots$ give the integral fractions of 2π by which the various substitutions rotate the sphere, and N is the number of all substitutions in the group, including the identity. It has five systems of solutions:

k_1	k_2	k_3	N	
n	n	1	n	
2	2	n	$2n$	dihedron
2	3	3	12	tetrahedron
2	3	4	24	octahedron - cube
2	3	5	60	ikosahedron - pentagondodekahedron

A more detailed discussion yields the following result:

All finite groups that can be formed with linear substitutions in one variable are isomorphic with the rotation groups that move the regular solids into themselves.

So precisely the regular solids, these “allegories of perfection”, appear here as representing all possible groups that exhibit certain simple and beautiful properties of a character much more concrete than the one Plato praises in the regular solids. And these same regular solids that are praised by Plato, and that in Euclid, in chapter XIII, represent quasi the cap-stone and high point of the work, appear here in a new light. At the same time we are given the opportunity to establish analoga by progressing to groups of linear substitutions in several variables or to infinite groups.

Now as far as the $\zeta-$functions are concerned that belong to these monodromy groups, the circular arc triangles associated with them will have the center of the sphere as center of the core, and they will be cut out by the symmetry planes of these regular solids. Since the $\zeta-$function is essentially characterized by specifying the circular arc triangle (through its shape) we will give a brief overview of these shapes. With each regular body several such triangles have to be considered. We omit here the first two, more trivial, of the five groups.

First one has for each of the three solids minimal triangles that together cover the $\zeta-$sphere simply. On the ikosahedron, for instance the triangle with angles $\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi$ has the sum of angles $\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{5}=\frac{31}{30}\pi$ and so the excess $\frac{\pi}{30}$. The whole sphere has area 4π, so 120 such triangles are needed to cover the sphere once. On the other hand the triangle $\frac{\pi}{3}, \frac{2\pi}{5}, \frac{3\pi}{5}$ that also appears in connection with the ikosahedron has excess $\frac{\pi}{3}$. With the 120 replicas produced by the group this gives 40, so the sphere is covered $\frac{40}{4}=10$ times by these triangles. So $z(\zeta)$ is an algebraic function, not a rational one.

We list the triangles in a table:

tetrahedron	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	|	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	
octahedron	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	|	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{3}$	
ikosahedron	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	|	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{5}$	and 8 others

We altogether have 14 possibilities and thus 14 strands of $\zeta-$functions, together with those related to them.

Tenth meeting, January 24, 1906

Continuation of the talk of Professor Klein

Corresponding to the smallest triangles one obtains in each case z as a rational function of degree resp. $12, 24, 60$. In the last case (icosahedron) one has in particular

$$z-\mathfrak{a} : z-\mathfrak{b} : z-\mathfrak{c} = \varphi_{30}^2(\zeta) : \chi_{20}^3(\zeta) : \psi_{12}(\zeta)^5,$$

where the indices give the degree of the polynomials $\phi(\zeta), \chi(\zeta), \psi(\zeta)$. More details can be found in "Ikosaeder" and in Otto Fischer's dissertation (Leipzig 1885).

We asked during the last meeting in which cases the $\zeta-$function is of elementary character, when it can be traced back to known functions, in particular, when it is algebraic. I will close this series of questions by briefly treating the following:

When are two distinct functions $\zeta_1(z), \zeta_2(z)$ algebraically dependent?

When are two distinct functions $z_1(\zeta), z_2(\zeta)$ algebraically dependent?

I will talk in particular about the last question. It is evident to begin with that functions $z(\zeta)$ with the same fundamental domain are algebraically dependent. In fact, $\zeta(z_1)$ maps the full (cut up) z_1-sphere onto the common fundamental domain of z_1 and z_2 which is then mapped to a full sheet of the z_2-sphere by $z_2(\zeta)$. To an orbit in the z_1-sphere about a ramification point of the function $\zeta(z_1)$ there corresponds a path from a point in the fundamental domain in the $\zeta-$sphere to a homologous point in a neighboring fundamental domain that is obtained from the first by the symmetry principle. To this path, however, there again corresponds, via the function z_2-sphere. Since essential singularities do not occur, we see that z_2 is a single-valued function of z_1, and analogously z_1 single valued in z_2. So $z_1(z_2)$ is a fractional linear expression in z_2. This way of reasoning can in essence be repeated if $z_1(\zeta), z_2(\zeta)$ do not have exactly the same fundamental domain, but if a larger domain $\mathfrak{B}$ can be found that can on the one hand be subdivided into a finite number m_1 of fundamental domains of z_1, and ont the other into m_2 fundamental domains of z_2. In that case m_1 sheets of the infinitely layered Riemann surface of the function $\zeta(z_1)$ will always be mapped one-to-one and conformally onto m_2 sheets of the Riemann surface of $\zeta(z_2)$. To a path in (z_1) which orbits a ramification point once there will now in general not correspond a closed path in (z_2), but if one orbits the ramification point more than m_1 times, one can be certain to also trace out a closed path on the z_2-sphere. One can now close up the Riemann surfaces of m_1 and resp. m_2 sheets just described, after separating them above the cuts from the initially infinitely layered Riemann surfaces. One obtains that way an algebraic relation between z_1 and z_2 of degree m_1 in z_1 and of degree m_2 in z_2.

A similar conclusion is possible for functions with related fundamental domains.

If it is possible to form a larger domain out of finitely many (m_1) fundamental domains, located next to each other, of the function $z_1(\zeta)$ so that it is also composed of finitely many (m_2) fundamental domains of the function $z_2(\zeta)$, then there is an algebraic relation between z_1 and z_2 of degree m_1 in z_1 and degree m_2 in z_2.

These are in particular the results and arguments that Riemann worked with.

Because of its somewhat abstract character I will illustrate what I said above by an example that will show at the same time how the *theory of the so called spherical functions* fits into the picture here.

It is customary to define the spherical functions in two different ways. To put it right away into our language: we are speaking either of

$$\zeta\begin{pmatrix} 0 & \infty & 1 & \\ & & & z_2 \\ \lambda & \mu & 1/2 & \end{pmatrix} \quad \text{or of} \quad \zeta\begin{pmatrix} 0 & \infty & 1 & \\ & & & z_1 \\ \lambda & 2\mu & \lambda & \end{pmatrix}.$$

These are not themselves spherical functions, but if one writes them as quotients of hypergeometric functions, then these special hypergeometric functions are spherical functions acording to one or the other definition.

The connection between these two parallel definitions now is seen to be precisely the one we just discussed in all generality and thus becomes part of a general principle. For the fundamental domain of the function $z_1(\zeta)$ consists of an isosceles triangle with angles $\lambda\pi, \lambda\pi, 2\mu\pi$ and its mirror image with respect to the base,

figure 22 here

and the fundamental domain of $z_2(\zeta)$ of a rectangle with angles $\lambda\pi, \mu\pi, \frac{\pi}{2}$ and one of its mirror images. Only a further reflection of this fundamental domain gives that of $z_1(\zeta)$. This then is the domain $\mathfrak{B}$ that contains the fundamental domain of $z_1(\zeta)$ $m_1 = 1$ times and that of $z_1(\zeta)$ $m_2 = 2$ times. According to our general principle, z_2 has to be a rational function of degree 2 in z. This function is to vanish at $z_1 = 0$ and $z_1 = 1$ and takes the value ∞ for $z_1 = \infty$. So we have

$$z_2 = 4z_1(1 - z_1),$$

with $\theta = \frac{1}{2}$.

In recent times, Goursat and Papperitz have worked in this area.

It will be preferable to deal with the algebraic dependence of related functions in a few weeks only when I will have decomposed the $\zeta-$function into numerator and denominator and so will be talking about Riemann's $\mathfrak{P}-$function.

I will now turn to a quite different question. I will not investigate already familiar functional relations in connection with the $\zeta-$function, but will expressly look for new phenomena. I will impose the restriction, however, that $z(\zeta)$ and all its analytic continuations be single valued. So I am asking for *all linear-polymorphic functions whose inverse function is single-valued*, in other words for *all single valued linear-automorphic functions.*

A first fact is that λ, μ, ν must be real, and indeed the reciprocals of posititve integers, $\lambda = \frac{1}{l}, \mu = \frac{1}{m}, \nu = \frac{1}{n}$. Indeed, for $z(\zeta)$ to be single valued, the replicas of its fundamental domain have to assemble in a way never to cover the $\zeta-$sphere in multiple fashion. If we focus on a vertex of the fundamental domain, then in a neighbourhood of the vertex

figure 23 here

the replicas will assemble as the two parts of figure 23 show. The left figure is meant to illustrate how the replicas could overlap if $2\lambda\pi$ is not an integral fractional part of 2π, i.e. if $\frac{1}{\lambda}$ is not an integer. The figure on the right shows how no overlap occurs if $\frac{1}{\lambda}$ is an integer.

I will now report on the subsequent grand theory. To begin with, the integrality of $\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}$ we just discussed will also be sufficient for the single-valuedness of $z(\zeta)$.

The necessary and sufficient condition for $z(\zeta)$ to be single-valued is that λ, μ, ν are reciprocals of rational integers.

I now make distinctions according to the type of the core, that is according to whether $\lambda + \mu + \nu$ is $>, <$ or $= 1$.

1) *For $\lambda + \mu + \nu > 1$ (cores of type 1)* the center of the core can be placed at the center of the sphere and one has ordinary relation groups. One obtains the regular solids.
2) *For $\lambda + \mu + \nu = 1$ (cores of type 3)* the center of the core can be placed at the north pole of the sphere. So in the $\zeta-$plane we will be dealing with rectilinear triangles. I will illustrate with the help of the isosceles right triangle and the equilateral triangle how in all cases with $\lambda + \mu + \nu = 1$ the triangles fit together into parallelograms that cover the plane in lattice fashion, as one is used to in the theory of double periodic functions. Moreover, $z(\zeta)$ will take the same value at homologous points, in short $z(\zeta)$ in this second case will always be an elliptic function.

figure 24 here

Eleventh meeting, January 31, 1906

Continuation of the talk of Professor Klein

3) *For $\lambda+\mu+\nu < 1$ (cores of type 3)* the center $\mathfrak{M}$ of the core lies outside the $\zeta-$sphere. We consider the tangential cone T from $\mathfrak{M}$ to the sphere and the circle N in which it touches the sphere. All planes of the core and of the cores obtained from it by the reflection process are perpendicular to the circle N, N is a normal circle perpendicular to all sides of the circular arc triangle and its reflections. The symmetric reproductions of the spherical arc triangle accumulate (inside the calotta of the sphere bounded by N) towards the normal circle N. They are in fact dense at all parts of it, and so this circle becomes a natural boundary, an essential singular line for the function $z(\zeta)$.

figure 25 here

Here we regard the substitutions of our group as non-Euclidian rotations about the point $\mathfrak{M}$. There are two other ways in which we can consider them. First again as non-Euclidian motions, but not in the non-Euclidian geometry based on the $\zeta-$sphere as absolute configuration, but in a non-Euclidian geometry that uses the cone T as fundamental surface, so a ternary non-Euclidian geometry. The one we used so far in contrast belongs to the quartenary realm.

On the other hand we can return to the $\zeta-$plane by stereographic projection and specially choose the real axis as normal circle N, as shown in figure 26.

figure 26 here

All circles are then perpendicular to the real axis and accumulate towards it. One then has the non-Euclidian geometry that is used by Poincaré in his investigations on the theory of automorphic functions. In it the circles perpendicular to the real axis are the shortest lines, the element of arc is

$$ds = \frac{\sqrt{(dx)^2 + (dy)^2}}{y}$$

The angle therefore agrees with the angle in the elementary geometry of the $\zeta-$plane. Our circular arc triangles are for this geometry what the rectilinear triangles are in elementary geometry and all triangles of our figure, which fills the positive half-plane, become congruent in the sense of this non-Euclidian geometry.

Finally it can be advantageous, as Professor Minkowski in particular emphasizes, to pass to a calotta of one sheet of a two sheeted hyperboloid. The hyperboloid here serves a function for the purposes of our non-Euclidian geometry similar to the one the sphere has in elementary geometry in the following context: the Euclidian area of a piece of surface bounded by an arbitrary countour on the sphere of radius 1 is the volume of the cone from the center of the sphere to this piece of surface, multiplied by 3.

There remain the cases with exponents of value 0, that is

$$\zeta \begin{pmatrix} 0 & 1 & \infty & \\ & & & z_1 \\ \frac{1}{2} & \frac{1}{3} & 0 & \end{pmatrix} \quad \text{and} \quad \zeta \begin{pmatrix} 0 & 1 & \infty & \\ & & & z_2 \\ 0 & 0 & 0 & \end{pmatrix}.$$

Here the triangles will have cusps. The replicas produced by the symmetry operations will assemble next to each other in infinite numbers.

figure 26.1 here

The cusps therefore will always belong to the natural boundary of the domain of $z(\zeta)$, that is the normal circle. First in the case of $z_2(\zeta)$ a more detailed picture in figures 27 and 28.

figures 27 and 28 here

Here we consider an area bounded by 3 circles tangent in pairs. If one subdivides this triangle by its three (non-Euclidian) altitudes, the six individual triangles have precisely the shape required by $z_1(\zeta)$. Hence z_1 is closely related to z_2, more precisely, it is a rational function of z_2. The precise calculation gives

$$z_1 = \frac{4}{27} \frac{(1 - z_2 + z_2^2)^3}{z_2^2(1 - z_2)^2},$$

or

$$z_1 : (z_1 - 1) : 1 = 4(z_2^2 - z_2 + 1)^3 : (4z_2^3 - 3z_2^2 - 3z_2 + 1)^2 : 27z_2^2(1 - z_2)^2.$$

As far as the replication of the triangles for $z_2(\zeta)$ is concerned one has the picture in figure 29 or 30,

figures 29 and 30 here

depending on whether one chooses the normal circle as a circle at finite distance or as the real axis. The cusps of the replicated triangles reach precisely all rational points of the real axis.

For if we choose weights appropriately, the group of substitutions belonging to $z_2(\zeta)$ consists of all integral substitutions $\zeta' = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}$ with determinant $\alpha\zeta - \beta\gamma = 1$. These map the real axis to itself, and one can show that they transform $0, 1, \infty$ into any given rational point of the real axis.

Besides the connection with number theory that is apparent here our functions have a very close relationship with the theory of elliptic functions. It is in fact from this direction that they and automorphic functions were first encountered. For this reason this field is called theory of *elliptic modular functions*. I will come back to this only later.

If one approaches the real axis along an arbitrary path, $z_2(\zeta)$ will not in general approach any specific value. This is the case, though, if the approach takes place completely within a fundamental domain. The same is true if one gets close to a rational point on the real axis on a path that is not tangent to it, for instance in the manner of the curves that are drawn in red in figure 30. The situation is different for the curve drawn in blue which is tangent to the axis. One sees easily that this is the case if instead of the real axis one takes the point ∞ and a path that approaches it. To the blue curve there would then correspond a path that runs more and more parallel to the axis as we approach ∞, whether maintaining finite distance from it or approaching it ever more closely.

figure 31 here

Twelfth meeting, February 3, 1906

Continuation of the talk of Professor Klein

I will endeavor today to explain the connection of the two transcendental functions $z_1(\zeta)$ and $z_2(\zeta)$ that I introduced a week ago with the theory of elliptic functions and integrals. It is not only the historical interest that motivates me to dwell on this subject somewhat at length. In fact, the splitting of ζ into $\frac{\zeta_1}{\zeta_2}$, and at the same time that of z into the quotient of two quantities, in short the passage from analytic functions in one variable, that is from conformal maps, to forms and pairs of forms is strongly suggested by the theory of elliptic integrals. To carry through this idea of splitting in all generality will be the aim of the remaining meetings of our seminar this semester.

Let

$$u = \int \frac{d\xi}{\sqrt{\xi(1-\xi)(1-\kappa^2\xi)}}$$

be the elliptic integral of the first kind. κ is called its *modulus* and its periods are

$$\omega_1 = 2\int_0^{1/\kappa^2} \ldots, \quad \omega_2 = 2\int_0^1 \ldots.$$

If we put $\kappa^2 = z_2, \omega = \frac{\omega_1}{\omega_2}$, we have

$$\omega = \zeta \left(\begin{array}{cccc} 0 & 1 & \infty & \\ & & & z_2 \\ 0 & 0 & 0 & \end{array} \right).$$

This is the oldest part of the theory and historically the starting point for the elliptic modular functions. The older of the functions is $z_2(\zeta)$.

The theory of elliptic integrals did not stop with the above normal form of u. Strictly speaking, u is not yet an integral of the first kind in a normal form to which each elliptic integral of the first kind

$$\int \frac{dx}{\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e}}$$

can be reduced by fractional linear substitutions, u can acquire a factor dependent on a, b, c, d during such a transformation. Such a factor is not material if one treats simple questions, such as the possibility to reduce u to elementary functions, and similar questions. But if we turn to more refined questions concerning the transformation and the dependence between a, b, c, d and the periods of the integral, then such a factor can be no means be neglected. It is quite obvious anyway that a single parameter under the integral sign just will not do. The bi-quadratic form has two independent parameters, a quantity of degree 2 and a quantity of degree 3 in the coeffients:

$$g_2 = ae - 4bd + 3c_2,$$

$$g_3 = ace + 2bcd - ad^2 - eb^2 - c^3 = \left| \begin{array}{ccc} a & b & c \\ b & c & d \\ c & d & e \end{array} \right|.$$

If one puts $a = 0, b = 1, c = 0, d = -\frac{1}{4}g_2, e = g_3$, then one obtains a bi-quadratic expression with invariants precisely g_2, g_3. So each bi-quadratic expression can be transformed into one of the form

$$4x^3 - g_2x - g_3.$$

So when Weierstrass takes

$$\int \frac{d\xi}{\sqrt{4\xi^2 - g_2\xi - g_3}}$$

as normal form for the integral of the first kind, he puts both essential quantities in evidence. It is not the case anymore that one, κ^2, is preferred and the other completely suppressed.

So the question arises how the invariants g_2, g_3 of Weierstrass are connected with ω_1, ω_2. On the one hand the theory of the elliptic integral provides simple developments for this connection:

$$\frac{1}{60}\, g_2(\omega_1, \omega_2) = \sum{}' \tfrac{1}{(m_1\omega_1 + m_2\omega_2)^4} \quad (Eisenstein),$$

$$\frac{1}{140}\, g_3(\omega_1, \omega_2) = \sum{}' \tfrac{1}{(m_1\omega_1 + m_2\omega_2)^6} \quad ,$$

the sums extended over all pairs (m_1, m_2) of positive or negative integers, with the exception of the pair $(0,0)$. In Weierstrass' notation this is expressed by affixing an accent at the top of the summation sign. One next finds

$$\Delta = g_2^3 - 27g_3^2$$

for the discriminant Δ of the bi-quadratic form.

If we use a transformation method of Jacobi we obtain the faster converging $q-$series

$$\begin{aligned} \frac{1}{60}\, g_2(\omega_1, \omega_2) = &\quad (\tfrac{2\pi}{\omega_2})^4 \,\{-\tfrac{1}{12} + 20 \sum_1^\infty \tfrac{m^3 q^{2m}}{1-q^{2m}}\}, \\ \frac{1}{140}\, g_3(\omega_1, \omega_2) = &\quad (\tfrac{2\pi}{\omega_2})^6 \,\{-\tfrac{1}{216} + \tfrac{7}{3} \sum_1^\infty \tfrac{m^5 q^{2m}}{1-q^{2m}}\}, \\ \Delta(\omega_1, \omega_2) = &\quad (\tfrac{2\pi}{\omega_2})^{12} \; q^2 \prod_1^\infty (1-q^{2m})^{24} \end{aligned}$$

and therefore

$$lg\Delta(\omega_1, \omega_2) = 12\; lg\frac{2\pi}{\omega_1} + 4i\pi\omega + 24 \sum_1^\infty lg(1-q^{2m}).$$

Here $q = e^{2i\pi\omega}$ ($= h$ in the notation of Weierstrass), with $\omega = \frac{\omega_1}{\omega_2}$.

To make the connection, on the other side, with modular functions we have to temporarily once more corrupt the Weierstrass normal form and put

$$\xi = \frac{g_3}{g_2}\eta.$$

Then

$$u = \sqrt{\frac{g_3}{g_2}} \int \frac{d\eta}{\sqrt{4\eta^3 - 27\frac{\mathfrak{z}}{\mathfrak{z}-1}(\eta+1)}}.$$

Here now we have <u>one</u> quantity $\mathfrak{z}$,

$$\mathfrak{z} = \frac{g_3}{\Delta} = \frac{g_2^3}{g_2^3 - 27g_3^2}, \quad \mathfrak{z} : \mathfrak{z} - 1 : 1 = g_2^3 : 27g_3^2 : \Delta.$$

Considered as a function of $\omega = \frac{\omega_1}{\omega_2}$ this $\mathfrak{z}$ is nothing but our $z_1(\zeta)$ with exponents $\frac{1}{3}, \frac{1}{2}, 0$.

Weierstrass' theory and its elegant apparatus of formulas now suggests to split ζ as well as z into two quantities. ζ (alias ω) is decomposed into the quotient $\frac{\zeta_1}{\zeta_2}$. The splitting of $\mathfrak{z}$ is more complicated.

Let us observe that the splitting of $\mathfrak{z}$, as presented by Weierstrass' theory, obeys the following rules. We have

$$\mathfrak{z} : \mathfrak{z} - 1 : 1 = g_2^3 : 27g_3^2 : e^{lg\Delta}.$$

The reciprocals of the exponents appearing here, that is $\frac{1}{3}, \frac{1}{2}, 0$, are precisely the characteristic exponents of $z_1(\zeta)$. We observe further that g_2 is a quantity of degree $-4, g_3$ a quantity of degree -6 in ω_1, ω_2. So $g_2^3, 27g_3^2$ are both quantities of degree -12. The number -12 agrees with

$$\Omega = \frac{2}{\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1}$$

for $l = 3, m = 2, n = \infty$.

The same rules for the splitting are forced upon the icosahedron function. I have to remind you only of the continuous proportionalities

$$\mathfrak{Z} : \mathfrak{Z} - 1 : 1 = \varphi_{20}^3 : \chi_{30}^2 : \psi_{12}^5.$$

So we are ready here to conjecture a general splitting principle for z that will have to occupy us in the sequel.

Thirteenth meeting, February 14, 1906

Continuation of the talk of Professor Klein

The splitting principle used successfully in the last meeting at the hand of some examples proves its validity again in a certain sense for the function

$$\omega \begin{pmatrix} 0 & 1 & \infty & \\ & & & \kappa^2 \\ 0 & 0 & 0 & \end{pmatrix}.$$

Here $\Omega = -2, \frac{\Omega}{l} = 0, \frac{\Omega}{m} = 0, \frac{\Omega}{n} = 0$. If we consider the postulation

$$\kappa^2 : \kappa^2 - 1 : 1 = e^{\Phi} : e^{\chi} : e^{\Psi}$$

with expressions of degree -2 as equivalent to it, we can regard the following formulas in Weierstrass' theory as supporting our splitting rule:

$$\begin{aligned} e_2 - e_3 = & \quad (\tfrac{\pi}{\omega_2})^2 \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 + q^{2m-1})^8, \\ e_1 - e_3 = & \quad (\tfrac{\pi}{\omega_2})^2 \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 - q^{2m-1})^8, \\ e_2 - e_1 = & \quad (\tfrac{\pi}{\omega_2})^2 16q \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 + q^{2m})^8 \end{aligned}$$

The logarithms of these expressions are precisely what Riemann dealt with in his fragment on the modular function and its behaviour near the natural boundary.

It is unfortunate here that $e_2 - e_3, e_1 - e_3, e_2 - e_1$ cannot be expressed as Eisenstein series of weight -2. In fact

$$\sum{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^2}$$

is not an absolutely convergent series. Only by a reordering of its terms is it at all possible to give it a suitable meaning. In all cases as well where Ω, or only one of $\frac{\Omega}{l}, \frac{\Omega}{m}, \frac{\Omega}{n}$, isn't an integer, a development of ϕ, χ, ψ as Eisenstein series will not be possible. One is forced to restrict oneself to such developments for certain expression in ϕ, χ, ψ. The series established by Poincaré suffer from the same complications.

Poincaré calls these developments $\Theta-$series. This name is justified not only through the analogy with the $\Theta-$series in the theory of elliptic functions, where they appear in the splitting process in a similar way Poincaré's series appear here. But in addition Poincaré's $\Theta-$series are identified directly with the $\Theta-$series in the theory of elliptic functions in the cases where our triangle functions belong to the class of elliptic functions. So they represent a true generalization of this well known notion.

In the case $l = 4, m = 2, n = 4$, for instance, one has

$$\zeta\begin{pmatrix} 0 & 1 & \infty & \\ & & & z \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \end{pmatrix} = \int \frac{dz}{\sqrt[4]{z^3(z-1)^2}} = \int \frac{z_2 dz_1 - z_1 dz_2}{\sqrt[4]{z_1^3(z_1-z_2)^2 z_2^3}}$$

and the theory of elliptic functions gives a representation by $\Theta-$series, or, which amounts to the same, by $\sigma-$series:

$$z : z-1 : 1 = \sigma_1(u)^4 : \sigma_2(u)^2\sigma_3(u)^2 : \sigma(u)^4.$$

The concept of *uniformizing auxiliary variable* will now lead to numerically effective developments and will enable us to carry out the *general splitting principle.* Instead of investigating directly the dependence of z, or of the ϕ, χ, ψ, on ζ or ζ_1, ζ_2, we now aim to represent both z and ζ as depending on a third auxiliary variable.

$$\omega\begin{pmatrix} 0 & 1 & \infty & \\ & & & z = \kappa^2 \\ 0 & 0 & 0 & \end{pmatrix}.$$

offers itself for this purpose. All our functions $\zeta(z)$ are *single valued* analytic functions of ω, as are quite generally all functions of z ramified only in $0, 1, \infty$, and z itself is single valued in ω. I proved this result in Math. Ann. 14. That Riemann knew it we learn from the last sentences of his lectures of 1859 (cf. supplements) that I told you about during our first meeting. Poincaré then developed his general uniformization theorem from it.

When considered as depending on ω we will call ζ a *linear-homomorphic* function since it undergoes its own substitutions while ω is subjected to those of the modular group. A similar relationship exists between the integrals of the first kind u and the second kind Z in the theory of elliptic integrals. Because of this analogy Poincaré gives the name $\zeta-$*series* to the series he uses in representing ζ as a function of ω. Poincaré's $\zeta-$series, though, do not give ζ_1, ζ_2 themselves, but only certain composites of them that are cogredient (related) with the ζ_1, ζ_2.

So here as well there remains the task to find series more intimately attuned to the subject. Wirtinger succeeded in doing so in 1902 (cf. Wiener Ber. 1902). Wirtinger's idea is simple, but had not been observed before. He noted that in the representation of ζ_1, ζ_2 through definite integrals with z as parameter the mere substitution of known elliptic formulas will lead to the desired representation by ω_1, ω_2. It will be my last aim this term to report to you on this result. So the objective will be to carry through the general splitting idea.

The general splitting idea

I begin with what I already touched briefly, the *theory of linear differential equations of the second order.* Let

$$y'' + p(z)y' + q(z)y = 0$$

be a homogeneous linear differential equation of order 2, with p and q rational functions of z. I recall some well known fundamental results, namely that all integrals are linear combinations of two of them, y_1 and y_2, that at a place of the independent variable z where p and q are regular all integrals $y = \alpha y_1 + \beta y_2$ will exhibit regular behavior as well whereas a pole of p or q can become a ramified place for y. I recall in addition that the different branches of y_1, y_2 one obtains by analytic continuation around a singular place must be homogeneous linear combinations of y_1, y_2, let us say

$$\overline{y}_1 = \alpha y_1 + \beta y_2, \overline{y}_2 = \gamma y_1 + \delta y_2.$$

So the quotient $\zeta = \frac{\zeta_1}{\zeta_2}$ is transformed into

$$\overline{\zeta} = \frac{\overline{y}_1}{\overline{y}_2} = \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \delta y_2} = \frac{\alpha \frac{y_1}{y_2} + \beta}{\gamma \frac{y_1}{y_2} + \delta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}.$$

Hence the quotient of the two particular integrals of the differential equation is ramified in the manner of our linear-polymorphic functions. If p, q are rational functions of z, then ζ is directly a linear-polymorphic function and indeed

$$[\zeta]_z = 2q - \frac{dp}{dz} - \frac{p^2}{2}$$

is a rational function of z. At the same time it becomes evident that, in the opposite direction, one has a lot of leeway in choosing the rational functions p, q if one wants to split a given $\zeta-$function into the quotient of two particular solutions of a homogeneous linear differential equation of order 2. One can, for instance, set $p = 0$. We will in the sequel prefer a normalization that is computationally more complicated but better attuned to the quantities inherent to our task, the ramification points and their orbital substitutions.

I would like to mention another fundamental fact of Riemann-Fuchs theory.

Fourteenth meeting, February 21, 1906

Continuation of the talk of Professor Klein

What we are concerned with is the behavior of the integrals of our linear differential equation of order 2 in the neighborhood of a ramification point. Since all integrals are linearly composed of them, the type of ramification is characterized completely by the coefficients $\alpha, \beta, \gamma, \delta$ of the linear substitutions that two linearly independent particular integrals y_1, y_2 are subject to there. In general one can achieve through appropriate choice of y_1, y_2 that these two integrals are, quite specially, changed only by multiplicative constants:

$$\overline{y}_1 = \rho_1 y_1, \quad \overline{y}_2 = \rho_2 y_2.$$

This is the case if the so called *fundamental equation* for the ramification point under consideration,

$$\begin{vmatrix} \alpha - \rho & \beta \\ \gamma & \delta - \rho \end{vmatrix} = 0,$$

has two distinct roots. The factors ρ_1, ρ_2 are then precisely the roots of this equation.

Any path around the ramification point will change the determinant

$$y_2\frac{dy_1}{dz} - y_1\frac{dy_2}{dz}$$

by a constant factor only. For the derivatives of y_1, y_2 are of course subject to substitutions with exactly the same coefficients, and so the above determinant is changed only by the factor

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}.$$

So its logarithmic derivative is a rational function of z, and in fact $-p$. Now

$$\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} = e^{-\int p dz} \quad \text{(Abel's theorem)},$$

a formula found in Abel, and maybe earlier. Usually this formula is used to reduce the integration of the second order homogeneous differential equation to the integration of a first order equation if <u>one</u> particular integral is known. For us the significance of Abel's theorem is the following: If ζ is given, and if the task is to split ζ, then it gives such a splitting. For we obtain

$$\zeta' = \frac{y_2y_1' - y_1y_2'}{y_2^2} = \frac{e^{-\int p dz}}{y_2^2},$$

and therefore

$$y_2 = \frac{e^{\frac{1}{2}\int p dz}}{\sqrt{\zeta'}}, \quad y_1 = \frac{\zeta e^{-\frac{1}{2}\int p dz}}{\sqrt{\zeta'}}.$$

Here p can still be chosen arbitrarily. On the other hand, if our starting point is not the differential equation of order 2, but a pencil of linear functions $c_1y_1 + c_2y_2$ that ramifies in the linear way we always assume here, then we can use the above argument which showed that $y_2y_1' - y_1y_2'$ is a multiplicative function and p rational.

The main theorem of Fuchs' theory now is as follows. In the neighborhood of a singular (yet single valued) place of the coefficients of the differential equation of order 2, and regardless of its nature, the canonical fundamental system y_1, y_2 of two multiplicative functions that I mentioned above can be set up in the form

$$y_1 = (z - z_0)^{\lambda_1}\mathfrak{L}_1(z - z_0), \quad y_2 = (z - z_0)^{\lambda_2}\mathfrak{L}_2(z - z_0).$$

Here $\mathfrak{L}_1, \mathfrak{L}_2$ are Laurent series - with logarithmic terms added in the cases where the fundamental equation has repeated roots. Moreover, the Laurent series reduce to power series $\mathcal{P}_1(z - z_0), \mathcal{P}_2(z - z_0)$ (with non vanishing constant terms) *if and only if* z_0 is at most a simple pole for p and a double pole for q.
For $z = \infty$ something similar holds, with certain modifications. The theorem is *not exactly the same, but rather just the opposite.* (Something similar occurs with the definition of the residue at the place ∞ of an analytic function. Contrary to what one might expect, it is not the coefficient of the first positive power of z in the development in powers of $\frac{1}{z}$, but precisely the coefficient of z^{-1}.)

We evidently have

$$\rho_1 = e^{2i\pi\lambda_1}, \quad \rho_2 = e^{2i\pi\lambda_2},$$

and the equation that has λ_1, λ_2 as roots is called by Fuchs the determining fundamental equation. This name was simplified by Frobenius to "*determining equation*".

Now logarithmic terms occur when λ_1, λ_2 differ by an integer $\lambda_1 - \lambda_2 = n$. Even then it can occur that no logarithmic terms are present. I will then talk of an *auxiliary point* of the differential equation of order 2 since $\zeta = \frac{y_1}{y_2}$ has no ramification there. If specifically $n = 1$, I will talk of a *parasitic auxiliary point* of the differential equation, there is no evidence of its presence in ζ itself.

If Fuchs' necessary and sufficient condition is satisfied everywhere on the $z-$sphere, so if p and q are rational functions with singularities of limited orders, then Fuchs' students say the differential equation of the second order is of *"Fuchsian type"*. If p, q have only 3 poles $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we obtain for the quotient $\zeta = \frac{y_1}{y_2}$ precisely the conditions to which we gave first place in the definition of ζ in our first meeting. Formulated for λ_1, λ_2 they are also used by Riemann as the starting point in his paper of 1857. The exponents λ, μ, ν appear here as the differences $\lambda_1 - \lambda_2, \mu_1 - \mu_2, \nu_1 - \nu_2$ of the roots of the determining equations belonging to $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.

Conversely it is possible, starting with ζ, to arrange the splitting, that is dispose of p, in a way such that λ, μ, ν are split into differences in a prescribed way and the differential equation is precisely of Fuchsian type if and only if

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \nu_1 + \nu_2 = 1.$$

This strange additional conditions comes about in the following way. We first compute from the conditions we impose

$$\begin{aligned} p &= \frac{1-\lambda_1-\lambda_2}{z-\mathfrak{a}} + \frac{1-\mu_1-\mu_2}{z-\mathfrak{b}} + \frac{1-\nu_1-\nu_2}{z-\mathfrak{c}} \\ q &= \frac{1}{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})}\{\frac{\lambda_1\lambda_2(\mathfrak{a}-\mathfrak{b})(\mathfrak{a}-\mathfrak{c})}{z-\mathfrak{a}} \\ &\quad + \frac{\mu_1\mu_2(\mathfrak{b}-\mathfrak{c})(\mathfrak{b}-\mathfrak{a})}{z-\mathfrak{b}} + \frac{\nu_1\nu_2(\mathfrak{c}-\mathfrak{a})(\mathfrak{c}-\mathfrak{b})}{z-\mathfrak{c}}\}. \end{aligned}$$

At infinity the two canonical particular solutions take the form

$$(\frac{1}{z})\frac{1-\Sigma}{2}\mathfrak{P}_1(\frac{1}{z}), \quad (\frac{1}{z})\frac{3-\Sigma}{2}\mathfrak{P}_2(\frac{1}{z})$$

and there will be a resulting parasitic auxiliary point of the differential equation of order 2 unless $\Sigma = 1$. But this Σ is precisely $\lambda_1 + \lambda_1 + \mu_1 + \mu_2 + \nu_1 + \nu_2$.

Riemann designates the functions $y = c_1 y + c_2 y_2$, that is the whole pencil of functions, by the letter $\mathcal{P}$. More precisely he puts all the characteristic exponents into the definition of

$$\mathcal{P}\left\{\begin{array}{cccc} a & b & c & \\ \lambda_1 & \mu_1 & \nu_1 & z \\ \lambda_2 & \mu_2 & \nu_2 & \end{array}\right\}.$$

The name "$\mathcal{P}-$function" is not very illuminating and has not become common. I use the designation "*hypergeometric function*" since the representation through the hypergeometric series of Gauss is a primary concern. We can summarize as follows:

The $\zeta-$ function with 3 *singular points* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *can always be written as the quotient of two* $\mathcal{P}-$*functions with the exponents* $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ *whose sum is* 1 *and whose differences give* $\lambda = \lambda_1 - \lambda_2, \mu = \mu_1 - \mu_2, \nu = \nu_1 - \nu_2$.

Let us note further that the canonical choice of p we just made with splitting in mind leads to an expression for $e^{-\int p dz}$ that is after all not very complicated. (If we choose $p = 0$ it is simply 1). It becomes

$$e^{-\int p dz} = (z-\mathfrak{a})^{\lambda_1+\lambda_2-1}(z-\mathfrak{b})^{\mu_1+\mu_2-1}(z-\mathfrak{c})^{\nu_1+\nu_2-1}.$$

One can successively simplify the $\mathcal{P}-$functions and make them more *canonical*. This is the basis for their analytic representation.

By the substitution

$$Z = \frac{z-\mathfrak{a}}{z-\mathfrak{b}} : \frac{\mathfrak{c}-\mathfrak{a}}{\mathfrak{c}-\mathfrak{b}}$$

one can <u>first</u> locate the ramification points at $0, \infty, 1$. One then deals with

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda_1 & \mu_1 & \nu_1 & z \\ \lambda_2 & \mu_2 & \nu_2 & \end{array}\right\}.$$

One can <u>further</u> consider $\mathcal{P}-$functions as belonging to each other and being not essentially different if they belong to the same ζ. This means that they differ by a factor $(z-\mathfrak{a})^{\alpha}(z-\mathfrak{b})^{\beta}(z-\mathfrak{c})^{\gamma}$ (with $\alpha+\beta+\gamma = 1$), which of course cancels if we form the quotient $\frac{\mathcal{P}_1}{\mathcal{P}_2}$ or $\frac{y_1}{y_2}$, as we always write. This way we transform $\mathcal{P}$ into

$$\mathcal{P}\left\{\begin{array}{cccc} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \\ \lambda_1+\alpha & \mu_1+\beta & \nu_1+\gamma & z \\ \lambda_2+\alpha & \mu_2+\beta & \nu_2+\gamma & \end{array}\right\} = \mathcal{P}(z-\mathfrak{a})^{\alpha}, (z-\mathfrak{b})^{\beta}(z-\mathfrak{c})^{\gamma},$$

and more particularly into

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda_1+\alpha & \mu_1-\alpha-\gamma & \nu_1+\gamma & z \\ \lambda_2+\alpha & \mu_2-\alpha-\gamma & \nu_2+\gamma & \end{array}\right\} = \mathcal{P}\left\{\begin{array}{ccc} 0 & \infty & 1 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{array}\right\} Z^{\alpha}(Z-1)^{\gamma}.$$

The multiplication formula for the $\mathcal{P}-$*function groups together all those* $\mathcal{P}-$*functions which belong to the same* ζ.

Among the $\mathcal{P}-$functions grouped together this way we now wish to select certain canonical ones that can be represented analytically in a simple way. The main tool for this analytic representation is the hypergeometric series.

$$\mathcal{F}(l,m,n;Z) = 1 + \frac{l.m}{1.n}Z + \frac{l.l+1.m.m+1}{1.2.\cdots n.n+1}Z^2 + \cdots$$

The connection with our theory is made by the fact that we can put

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda_1 & \mu_1 & \nu_1 & Z \\ 0 & m_2 & 0 & \end{array}\right\} = \mathcal{F}(m_1, m_2, 1-e; Z).$$

More precisely:
The branch of the $\mathcal{P}-$function which has λ_2, ν_2 normalized to 0, and which is a power series $\mathcal{P}(Z)$ at $Z = 0$, can be developed into a hypergeometric series.

This not very interesting apparatus of formulas almost leads one to suspect that Gauss used it to hide from the public his most essential idea, the use of complex variables - his publication from 1812 stems from a time when Cauchy's theory of functions was not yet known. *One cannot immediately infer from his paper, however, that Gauss was in full command of the theory of linear differential equations of order* 2 *and knew in detail about the three singular points, as became clear from the writings in his scientific estate.*

With Riemann the situation is just the opposite. Riemann puts his innermost idea first, namely that the $\mathcal{P}-$function is determined completely by its monodromy behaviour at $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. He offers next the multiplication theorem and several other results from which he in the end deduces the existence of a differential equation of the second order. From this the existence of the $\mathcal{P}-$function and its representation by hypergeometric series then follows as a corollary.

The *theory of related functions* gains a simplified aspect for split ζ and a considerable importance for the calculations. We said that

$$\zeta\begin{pmatrix} a & b & c & \\ & & & z \\ \lambda & \mu & \nu & \end{pmatrix} \quad \text{and} \quad \zeta\begin{pmatrix} a & b & c & \\ & & & z \\ \lambda' & \mu' & \nu' & \end{pmatrix}$$

are related if $\lambda' - \lambda, \mu' - \mu, \nu' - \nu$ are integers and

$$(\lambda' - \lambda) + (\mu' - \mu) + (\nu' - \nu) \equiv 0 \pmod 2.$$

This now means that for related functions the $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ can differ by integers only. The condition that the sum is even becomes superfluous since it is always 0 in view of $\lambda_1 + \lambda_2 + \cdots + \nu_2 = \Sigma = 1$.

Applied to the $\mathcal{P}-$functions with three singular points the notion of related $\zeta-$functions simply becomes the following: Two such $\mathcal{P}-$functions are related if corresponding exponents differ by integers only.

If y_1, y_2, Y_1, Y_2 are two pairs of related functions they will be subject to the same substitutions by orbits around the ramification points. Since

$$\begin{vmatrix} ay_1 + by_2 & cy_1 + dy_2 \\ aY_1 + bY_2 & cY_1 + dY_2 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} y_1 & y_2 \\ Y_1 & Y_2 \end{vmatrix}$$

we find that $y_1 Y_2 - y_2 Y_1$ is a multiplicative function. If we write $\mathcal{P}_1', \mathcal{P}_2'$ for y_1, y_2 and $\mathcal{P}_1'', \mathcal{P}_2''$ for Y_1, Y_2, then

$$\mathcal{P}_1'\mathcal{P}_2'' - \mathcal{P}_2'\mathcal{P}_1'' = (z-a)^{\lambda'+\lambda''}(z-b)^{\mu'+\mu''}(z-c)^{\nu'+\nu''}.Rat(z).$$

If now $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1', \mathcal{P}_2', \mathcal{P}_1'', \mathcal{P}_2''$ are three pairs of related $\mathcal{P}-$functions and if we have

$$\mathcal{P} = \alpha\mathcal{P}_1 + \beta\mathcal{P}_2, \mathcal{P}' = \alpha\mathcal{P}_1' + \beta\mathcal{P}_2', \mathcal{P}'' = \alpha\mathcal{P}_1'' + \beta\mathcal{P}_2'',$$

then

$$\begin{vmatrix} \mathcal{P} & \mathcal{P}' & \mathcal{P}'' \\ \mathcal{P} & \mathcal{P}'_1 & \mathcal{P}'_2 \\ \mathcal{P} & \mathcal{P}''_1 & \mathcal{P}''_2 \end{vmatrix} = 0$$

and therefore

$$\mathcal{P}.\mathfrak{Rat}(z) + \mathcal{P}'.\mathfrak{Rat}'(z) + \mathcal{P}''.\mathfrak{Rat}''(z) = 0.$$

All related $\mathcal{P}-$functions are linear combinations of two of them with rational coefficients.

Such relations do not appear with $\zeta-$functions, it is the splitting that causes them. Here lies the reason for the increased sophistication of the theory of split functions compared to just conformal maps.

The considerable advantage of passing to a homogeneous situation lies in encountering simple functions, not only if one forms the cross-ratio of four related functions, but already if one forms a small determinant (bracket factor) out of two corresponding branches of related functions.

We obtain an application of the above theorem if we interpret the accents we affixed to the $\mathcal{P}$ as differentiations. Quite obviously a series of $\mathcal{P}-$functions is related with its derivatives. So we obtain a linear homogeneous differential equation of order 2 with rational coefficients, constructed exclusively out of the ramification properties of $\mathcal{P}$.

In this way Riemann makes the transition from the $\mathcal{P}-$functions, which are defined by their ramification properties, to the differential equation of order 2. In our days one has in fact succeeded to provide a direct argument for Riemann's existence problem with the help of integral equations. (cf. Hilbert, Verh. d. Heidelberger Congresses p. 233 ff. and Gtt. Nachr. 1904, Heft 3, p. 213 ff.).

Fifteenth meeting, February 28, 1906
Continuation of the talk of Professor Klein

Following Riemann's line of thought one can derive from the ramification properties of the $\mathcal{P}-$function not only the existence of the linear differential equation of order 2 that it satisfies, but in all detail also the shape of the hypergeometric differential equation.

To this end one only has to observe that the pencil of functions related to $\mathcal{P}$,

$$y\begin{pmatrix} a & b & c & \\ \lambda_1 + a_1 & \mu_1 + b_1 & \nu_1 + c_1 & z \\ \lambda_2 + a_2 & \mu_2 + b_2 & \nu_2 + c_2 & \end{pmatrix},$$

does not consist of $\mathcal{P}-$functions, but of functions with auxiliary points, precisely

$$\kappa = a_1 + a_2 + b_1 + b_2 + c_1 + c_2$$

of them, part of which can of course lie at $z = \infty$.

The derivatives of a pencil of $\mathcal{P}-$functions are not themselves $\mathcal{P}-$functions properly speaking, but related functions with 6 auxiliary points, since their exponents in each instance differ by $+1$ form those of the $\mathcal{P}-$function. By the same reasoning the second derivatives have 12 auxiliary points. Hence the calculation for the 2×2 determinants which become the coefficients of the second order differential equation yields the following:

$$\begin{aligned}
(\mathcal{P}'\mathcal{P}'') &= (z-\mathfrak{a})^{\lambda_1+\lambda_2-3}(z-\mathfrak{b})^{\mu_1+\mu_2-3}(z-\mathfrak{c})^{\nu_1+\nu_2-3}\varphi,\\
(\mathcal{P}''\mathcal{P}') &= (z-\mathfrak{a})^{\lambda_1+\lambda_2-2}(z-\mathfrak{b})^{\mu_1+\mu_2-3}(z-\mathfrak{c})^{\nu_1+\nu_2-3}\chi,\\
(\mathcal{P}\mathcal{P}') &= (z-\mathfrak{a})^{\lambda_1+\lambda_2-1}(z-\mathfrak{b})^{\mu_1+\mu_2-1}(z-\mathfrak{c})^{\nu_1+\nu_2-1}\psi.
\end{aligned}$$

Here

$$\begin{array}{lll}
\varphi \ \text{ is of degree } \frac{\kappa'+\kappa''}{2}-1=8, & \text{but with 6 zeros at } z=\infty,\\
\chi \ \text{ is of degree } \frac{\kappa''+\kappa}{2}-1=5, & \text{but with 3 zeros at } z=\infty,\\
\psi \ \text{ is of degree } \frac{\kappa+\kappa'}{2}-1=2, & \text{but with 2 zeros at } z=\infty.
\end{array}$$

Hence φ and χ become polynomials of degree 2 and ψ a constant $=1$.
One obtains the Gaussian differential equation

$$\mathcal{P}''+\frac{\chi_2}{(z-\mathfrak{a})(z-\mathfrak{b})(z-\mathfrak{c})}\mathcal{P}'+\frac{\varphi_2}{(z-\mathfrak{a})^2(z-\mathfrak{b})^2(z-\mathfrak{c})^2}\mathcal{P}=0.$$

We now complete the general splitting process by replacing $z,\mathfrak{a},\mathfrak{b},\mathfrak{c}$ by homogeneous quantities as well. We replace $z,\mathfrak{a},\mathfrak{b},\mathfrak{c}$ by the quotients z_1/z_2, $\mathfrak{a}_1/\mathfrak{a}_2$, $\mathfrak{b}_1/\mathfrak{b}_2$, $\mathfrak{c}_1/\mathfrak{c}_2$ and instead of $\mathcal{P}$ consider the pencil of binary forms

$$\mathcal{P}\left(\begin{array}{cccc} \mathfrak{a} & mathfrakb & \mathfrak{c} & \\ \lambda & \mu & \nu & z\\ \lambda_2 & \mu_2 & \nu_2 & \end{array}\right)(z\mathfrak{a})^{-\lambda_2}(z\mathfrak{b})^{-\mu_2}(z\mathfrak{c})^{-\nu_2}=\Pi\left(\begin{array}{cccc} \mathfrak{a}_1/\mathfrak{a}_2 & \mathfrak{b}_1/\mathfrak{b}_2 & \mathfrak{c}_1/\mathfrak{c}_2 & \\ \lambda & \mu & \nu & z_1/z_2\\ 0 & 0 & 0 & \end{array}\right),$$

where (uv) stands for the bracket factor $u_1v_2-u_2v_1$. As opposed to $\mathcal{P},\Pi$ is no more a function homogeneous of degree 0, but of degree $-\lambda_2-\mu_2-\nu_2$. If one adds

$$0=\frac{1-\lambda_1-\lambda_2-\mu_1-\mu_2-\nu_1-\nu_2}{2}$$

one finds

$$\frac{\lambda+\mu+\nu-1}{2}$$

for the degree of the pencil of forms Π. So the normalization of $\mathcal{P}$ becomes more elegant through the introduction of homogeneous variables.

This process of normalization still leaves 8 choices, corresponding to the fact that the signs of λ,μ,ν can be chosen arbitrarily. Among these 8 normalizations I always can find one for which λ,μ,ν have non-negative real parts. Not in all cases, however, is the normalization uniquely determined by this requirement.

The form Π normalized in this fashion has the advantage that all its branches are everywhere finite and never vanish simultaneously. The meaning of this is as follows: the homogeneous quantities z_1,z_2 can take as values all *finite pairs besides* $0,0$. They give a representation of all values of $z=\frac{z_1}{z_2}$, including $z=\infty$, without the use of the symbol ∞. In this way the introduction of homogeneous quantities puts into evidence the equal status of the place $z=\infty$ with all other places for calculations as well. The great elegance and simplicity of the apparatus of formulas written in homogeneous form result precisely from this. We have now succeeded in normalizing Π in such as way that Π_1,Π_2 as well take only pairs of values that are meaningful for Π_1, Π_2 as homogeneous quantities, that is both are finite and not simultaneously 0. The effect is that now conversely z_1, z_2 are entire homogeneous forms in Π_1, Π_2.

Π_1, Π_2 *are entire forms in* z_1, z_2 *and* z_1, z_2 *are entire forms in* Π_1, Π_2. *Here I apply Weierstrass' notion of an entire function to forms with suitable modifications: an entire pencil of forms is one whose members are nowhere infinite and never vanish simultaneously.*

In the short time still available to me I cannot explain in detail how the apparatus of formulas for the $\mathcal{P}-$function, the *relationes inter functiones contiguas*, the construction of the second order linear differential equation, is simplified by carrying out completely the programm of homogeneization. I just want to mention the following: Let

$$\Pi' \begin{pmatrix} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \\ & & & z_1/z_2 \\ \lambda' & \mu' & \nu' & \end{pmatrix}, \quad \Pi'' \begin{pmatrix} \mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \\ & & & z_1/z_2 \\ \lambda'' & \mu'' & \nu'' & \end{pmatrix}$$

be two related pencils of forms, i.e. let $\lambda' - \lambda'', \mu' - \mu'', \nu' - \nu''$ be integers whose sum is even. Let λ be the smaller of the two numbers λ', λ'', μ the smaller of the numbers μ', μ'', and similarly for ν. In each case let $\lambda + A, \mu + B, \nu + C$ be the other, larger number. So A, B, C are never negative, and not all zero since otherwise Π', Π'' would not be different pencils. We then have

$$(\Pi'\Pi'') = \Pi'_1\Pi''_2 - \Pi'_2\Pi''_1 = (z - \mathfrak{a})^\lambda (z - \mathfrak{b})^\mu (z - \mathfrak{c})^\nu . \varphi(z_1, z_2).$$

The degree of the form equals that of the determinant:

$$\frac{\lambda' + \mu' + \nu' - 1}{2} + \frac{\lambda'' + \mu'' + \nu'' - 1}{2} = \frac{\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu''}{2} - 1 =$$
$$\lambda + \mu + \nu + \frac{A + B + C}{2} - 1,$$

diminished by

$$\frac{A + B + C}{2} - 1.$$

This is a non-negative integer. If we knew up to now from the inhomogeneous representation that φ is rational we know now that $\varphi(z_1, z_2)$ is an entire form of degree $\frac{A+B+C}{2} - 1$. This is quite an advantage in the calculation of φ and hence for the construction of the second order differential equation. We are saved from having to consider the auxiliary points appearing initially and unnecessarily at ∞, and φ right away appears with the appropriate low degree. In particular if we are dealing with "*neighbouring*" functions (it functiones contiguae), that is if $A + B + C = 2$, then φ is a constant. The *relationes inter functiones contiguas* follow from this in a particularly straightforward way.

I will give preference now now to touching briefly on the

connection of our subject with the theory of Eulerian integrals.

This is again a well established area of mathematical research, just like several we were already able to connect with our circle of ideas. Just as the theory of the Gaussian differential equation, the Eulerian integrals can be used as a base on which to build the theory of linearly ramified functions, the $\mathcal{P}-$functions of Riemann. Again in the opposite direction, just as the Gaussian differential equation, the relation with the Eulerian integrals can be deduced solely from the ramification behaviour of the $\mathcal{P}-$function. The seeds of this can again be found with Riemann.

What are Eulerian integrals? Integrals of the form

$$\int (x-\mathfrak{a})^{\alpha} \ldots (x-\mathfrak{n})^{\nu} dx$$

where $\alpha+\beta+\cdots+\nu=-2$. This condition insures that there is no singularity at $x=\infty$. We will in short order discuss the paths of integration.

But before that I may be allowed to remind you of what can be found in the standard textbooks regarding these integrals. Usually what is written about are the *Eulerian integrals of the first kind with* 3 *singular points* - with two the whole affair would be too simple. More precisely

$$B=\int x^{\alpha}(1-x)^{\beta} dx.$$

This means that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are placed at $0, 1, \infty$. Then there are the integrals of the *second kind*:

$$T=\int (x-\mathfrak{a})^{\alpha} e^{-\frac{1}{x-\mathfrak{b}}} dx$$

where in the limit two of the ramification points have coalesced into an essential singularity. The general Eulerian integral with n singular points quite naturally also permits a large variety of such limit cases, analogous to the $T-$functions, but we will not consider this here.

Another point in which the newer theory, beginning with Riemann, has abandoned a not always suitable specialization, is the question of the path of integration. In the old theory, and in most textbooks, the integration is performed form 0 to 1 along the real axis. This is natural if one does not have recourse at all to the complex domain. $\int x^{\alpha} dx$, however, has a permissible behaviour at $x=0$ only if the real part of α is larger than $-1: Re(\alpha)>-1$.

It is permissible to take the Eulerian integral on a straight line from one singular point to another only if $Re(\alpha)>1, Re(\beta)>1, \ldots, Re(\nu)>1$.

Just as with the moduli of elliptic and Abelian integrals, Riemann here as well introduced a fitting generalization by replacing the straight path with a path that encloses the singular points. More precisely, not only is this path closed as such, but closed also on the Riemann surface of the integrand

$$(x-\mathfrak{a})^{\alpha} \cdots (x-\mathfrak{n})^{\nu}.$$

Two questions arise here:

1) Is it possible at all to find a path that is closed on this Riemann surface with, in general, infinitely many sheets?
2) If such a path is possible, will it be a generalization of the straight paths from one singular point to another?

Both questions are answered positively by the design of the so-called *double loop* (trefoil loop).

figure 32 here (bottom of p. 111)

It is drawn so that $\mathfrak{a}$ and $\mathfrak{b}$ are both orbited once in each direction and the other points not at all. The integrand then is multiplied by

$$e^{2i\pi\alpha}e^{-2i\pi\alpha}e^{2i\pi\beta}e^{-2i\pi\beta} = 1$$

One then indeed has a closed path on the Riemann surface as well. Its shape ca be read off from figure 33.

figure 33 here (first figure on p. 112)

The question now is how the double circuit, or rather the Eulerian integral taken along it, depends on $\alpha, \beta, \ldots$. It obviously remains meaningful as long as these quantities are at finite distance. It is an entire transcendental function of $\alpha, \beta, \ldots, \nu$.

A single double loop produces an entire transcendental function of $\alpha, \beta, \ldots, \nu$.

To study how the integral from $\mathfrak{a}$ to $\mathfrak{b}$ (taken along a straight path or a double loop) depends on $\mathfrak{c}$ when $\mathfrak{c}$ moves along a path in its plane, one has to imagine that the path of integration has ends fixed at $\mathfrak{a}, \mathfrak{b}$, but always avoids the moving $\mathfrak{c}$ and may be pulled aside by it.

figure 34 here (second figure on p. 112)

As an example, $\int_{\mathfrak{a}}^{\mathfrak{b}}$ (taken along a straight path) is transformed into $\int_{\mathfrak{a}}^{\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{b}}$ by an orbit of $\mathfrak{c}$ around $\mathfrak{b}$.

figure 35 here (third figure on p. 112)

As $\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{n}$ *move along arbitrary closed paths the result is that the fundamental integrals that one has chosen, by specifying double loops for instance, become linear combinations of these fundamental integrals.*

From here we will be able to make the connection with the theory of the $\mathcal{P}-$function.

Sixteenth meeting, March 3, 1906

Continuation of the talk of Professor Klein

To finish I will today touch on two points:

1) The connection of the Eulerian integrals to the $\mathcal{P}-$function.
2) The representation of Π through ω according to Wirtinger.

I begin with the Eulerian integral with 3 singular points, but will now use its homogeneous form:

$$\int (x\mathfrak{a})^{\alpha}(x\mathfrak{b})^{\beta}(x\mathfrak{c})^{\gamma}(x\ dx).$$

The substitution

$$u_1 = (x\mathfrak{b})(a\mathfrak{c}), u_2 = (x\mathfrak{a})(b\mathfrak{c})$$

transforms it into

$$\frac{\int u_2^{\alpha}u_1^{\beta}(u_1-u_2)^{\gamma}\ u\ du}{(\mathfrak{b}\mathfrak{c})^{\alpha+1}(\mathfrak{c}\mathfrak{a})^{\beta+1}(\mathfrak{a}\mathfrak{b})^{\gamma+1}}.$$

Here on the one hand the full symmetry vis a vis $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ becomes evident - the path of integration as well, the trefoil loop, does not favor one of the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ - , and on the other the simple form of the dependence on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, they are no more present under the integral sign.

We can finally express our integral in terms of the $\Gamma-$function as follows:

$$= \frac{-4\pi^2}{\Gamma(-\mathfrak{a})\Gamma(-\mathfrak{b})\Gamma(-\mathfrak{c})(\mathfrak{b}\mathfrak{c})^{\alpha+1}(\mathfrak{c}\mathfrak{a})^{\beta+1}(\mathfrak{a}\mathfrak{b})^{\gamma+1}}.$$

Only beginning with the Eulerian integral with four singular points is the dependence on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ no more of elementary nature. This corresponds to the fact that 3 points can be transformed linearly into any other 3, but not 4 points into any other 4. Correspondingly we have the choice to move $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ to $0, 1, \infty$ and consider $\mathfrak{d}$ as essential variable ramification point, or to keep in evidence the symmetric roles of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ and regard the cross-ratio

$$z = \frac{z_1}{z_2} = \frac{(\mathfrak{d}\mathfrak{b})(\mathfrak{a}\mathfrak{c})}{(\mathfrak{d}\mathfrak{a})(\mathfrak{b}\mathfrak{c})}$$

as the essential variable.

Here now we encounter the connection with the $\mathcal{P}-$function. We have

$$\int (x\mathfrak{a})^\alpha (x\mathfrak{b})^\beta (x\mathfrak{c})^\gamma (x\mathfrak{d})^\delta (xdx) = \frac{\int u_2^\alpha u_1^\beta (u-u_2)^\gamma (z_2u_1 - z_1 - u_2)^\delta (u\ du)}{(\mathfrak{b}\mathfrak{c})^{\alpha+\delta+1}(\mathfrak{c}\mathfrak{a})^{\beta+\delta+1}(\mathfrak{a}\mathfrak{b})^{\gamma+\delta+1}}$$

and can say:

As to the dependence of the integral on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$*, it depends, apart from bracket factors that enter in an elementary way, only on the numerator and denominator of the above cross-ratio, and this with degree* δ*.*

By first gaining some insight into the totality of paths possible in a four-term integral, by secondly recalling that during an orbit of z, or of one of the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, each of these points pushes ahead of itself a path of integration barring its way, one comes to the conclusion that the countably many values that our integral can attain along all possible paths of integration are contained in a pencil of $\mathcal{P}-$functions, or, written homogeneously, of $\Pi-$forms.

Let

$$\lambda = \alpha + \delta + 1, \mu = \beta + \delta + 1, \nu = \gamma + \delta + 1$$

or, which amounts to the same in view of $\alpha + \beta + \gamma + \delta = -2$,

$$\lambda = -(\beta + \gamma + 1), \mu = -(\gamma + \alpha + 1), \nu = -(\alpha + \beta + 1).$$

Then

$$\Pi \left(\begin{array}{cccc} \infty & 0 & 1 & \\ \lambda & \mu & \nu & z_1/z_2 \\ 0 & 0 & 0 & \end{array} \right)$$

is the form Π we have to consider and its degree

$$\frac{\lambda + \mu + \nu - 1}{2} = \frac{\alpha + \beta + \gamma + \delta + 2\delta + 3 - 1}{2}$$

indeed equals δ. In the opposite direction, α, β, γ are expressed in terms of λ, μ, ν as follows:

$$\alpha = \frac{\lambda - \mu - \nu - 1}{2}, \quad \beta = \frac{-\lambda + \mu - \nu - 1}{2},$$
$$\gamma = \frac{-\lambda - \mu + \nu - 1}{2}, \quad \delta = \frac{\lambda + \mu + \nu - 1}{2}.$$

If our point of departure is not Π, but $\mathcal{P}$, then the representation through definite integrals can still be effected in 8 different ways. For since in the $\mathcal{P}-$function the exponent differences λ, μ, ν are determined only up to sign, and so a normalized Π can be introduced in 8 different ways, we finally have 8 different choices to represent the $\mathcal{P}-$function as an Eulerian integral. The exponents $\alpha, \beta, \gamma, \delta$ are differentiated precisely by the signs of λ, μ, ν. Following through with what we said during the last meeting we will give preference among these 8 forms to those where the real parts of λ, μ, ν are non-negative.

Another point now becomes apparent which gives the representation of the $\mathcal{P}-$function by Eulerian integrals a particular importance. In our earlier definitions of the $\mathcal{P}-$function only the dependence on $z = \mathfrak{d}$ was specified for the fundamental branches that belong to the singular points, but their dependence on the constants $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, or even the exponents $\alpha, \beta, \gamma, \delta$, was left open.

This has to be considered a gap if one endeavours as much as possible in function theory to give all constants again the role of variables. As Clebsch once said: Constants are variables that one accidentally considers as fixed. Here we can now claim for the representation through definite integrals:

The definition of the $\mathcal{P}-$function, or the $\Pi-$form, by Eulerian integrals is in some ways better than the definitions we considered so far. For our earlier definitions took into account only the dependence of the one point $\mathfrak{d}$, or z, whereas the definition via the definite integral also considers the dependence on the $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and the $\alpha, \beta, \gamma, \delta$, or λ, μ, ν.

Specifically, the definitions of the two fundamental branches that arise when $\mathfrak{d}$ coincides with $\mathfrak{a}$ are given by two double circuits, one that encloses $\mathfrak{a}$ and $\mathfrak{b}$, and it belongs to the exponent 0, and another that encloses $\mathfrak{c}$ and $\mathfrak{d}$, and it belongs to the exponent λ.

Representation of Π_1/Π_2 by the elliptic modular functions ω_1/ω_2.

In essence I have to reproduce Wirtinger. Using the notation of Weierstrass, however, I end up with considerably shorter and better organized formulas. This means that instead of

$$\zeta(0,0,0,z) = \frac{\omega_1}{\omega_2}, \text{with } z = \kappa^2,$$

I use

$$z = \frac{e_2 - e_3}{e_1 - e_3}.$$

The Eulerian integral belonging to $\zeta(0,0,0,z) = \omega$ has all exponents equal to $-\frac{1}{2}$ and therefore is the elliptic integral of the first kind

$$\int \frac{(udu)}{\sqrt{u_2 u_1 (u_1 - u_2)[(e_1 - e_3)u_1 - (e_2 - e_3)u_2]}}.$$

Now ω_1, ω_2 are of degree $\frac{1}{2}$ in z_1, z_2 or $e_1 - e_3, e_2 - e_3$. On the other hand the general integral belonging to (Π_1, Π_2) is of degree $\frac{\lambda+\mu+\nu-1}{2}$ in $z_1, z - 2$ or $e_1 - e_3, e_2 - e_3$. If we desire the possibility to represent Π by ω, then Π_1, Π_2 have to be of degree $1 - \lambda - \mu - \nu$ in ω_1, ω_2:

If a representation of Π_1, Π_2 *by* ω_1, ω_2 *is possible at all, then it will be homogeneous of degree* $1 - \lambda - \mu - \nu$.

As to the possibility of this representation we have

$$\Pi = \Pi \begin{pmatrix} 0 & \infty & 1 & \\ & & & e_1 - e_2/e_1 - e_3 \\ \lambda & \mu & \nu & \end{pmatrix}$$
$$= \frac{1}{2}\int (x-e_1)\frac{-\lambda+\mu+\nu-1}{2}(x-e_2)\frac{\lambda-\mu+\nu-1}{2}(x-e_3)\frac{\lambda+\mu-\nu-1}{2}dx.$$

It may be regarded as Wirtinger's main idea to write for this

$$\int (x-e_1)\frac{-\lambda+\mu+\nu}{2}(x-e_2)\frac{\lambda-\mu+\nu}{2}(x-e_3)\frac{\lambda+\mu\cdot\nu}{2}\frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$
$$= \int \sigma_1(u)^{-\lambda+\mu+\nu}\sigma_2(u)^{\lambda-\mu+\nu}\sigma_3(u)^{\lambda+\mu-\nu}\sigma(u)^{-\lambda-\mu-\nu}du,$$

the integral taken along a period path in the $u-$plane that corresponds to a double loop. This formula now contains everything that is necessary.

For $\sigma(u)$ is a homogeneous form of degree 1 in u, ω_1, ω_2, and $\sigma_1(u), \sigma_2(u), \sigma_3(u)$ are homogeneous of degree 0 in the same quantities. So the integral, which we realize is also a form in ω_1, ω_2, has weight $(-\lambda - \mu - \nu) \cdot 1 + 1 = -\lambda - \mu - \nu + 1$, and this is precisely the degree that we had determined beforehand. Here the $\sigma-$series are power series in u with coefficients that are Eisenstein series in ω_1, ω_2. So in principle the integrand can also be written as a power series $\mathfrak{P}(u)$, whose coefficients are aggregates of Eisenstein series in ω_1, ω_2. If I integrate these power series term by term with limits of integration $u_0, u_0 + 2\omega$, or $u_0, u_0 + 2\omega_2$, I obtain forms in ω_1, ω_2 for Π_1, Π_2 which are homogeneous of degree $1 - \lambda - \mu - \nu$, and that are built from powers of the ω multiplied with aggregates of Eisenstein series.

Now that a representation of the Π_1, Π_2 by ω_1, ω_2 has been found in this way, the question arises whether one can also arrive at the same final result with relative ease by following the route of Poincaré. This would mean to represent in terms of ω_1, ω_2, with the help of Poincaré's $\zeta-$series, at first not Π_1, Π_2 themselves, but certain forms related to Π_1, Π_2. One would then not have to rely, as Wirtinger does, on the theory of elliptic functions.

As far as passing from the terminology of Weierstrass to that of Jacobi is concerned, the integrand (apart from a power of ω_2 that appears for homogeneity reasons) changes into a power series in ν with coefficients that are polynomials in q. If we then integrate between given bounds we obtain series in q for Π_1, Π_2. Here we have

$$e^{\frac{ie\pi u}{\omega_2}} = \nu, \quad e^{\frac{i\pi\omega_1}{\omega_2}} = q$$

in the usual way.

I refer to Papperitz (Math. Ann. 34) in connection with such series.

Weierstrass, by the way, uses $2\omega, 2\omega'$ for our ω_1, ω_2.

I have tried this term to provide you with an overview of a field to which I would like to give the name: *Riemannian functions*, and to explain to you its connections with various mathematical disciplines. Doing this, I tried to keep matters simple: functions with 3 ramification points defined on the single-sheeted plane, and without auxiliary points. It might be appropriate to compare the field of Riemannian functions with the high peak area in a mountain range. This semester we explored the foothills bordering it. We climbed some well known heights that provided a glimpse of the highest, unscaled peaks. Let us hope that in the coming semesters we will succeed in conquering some of these as well.