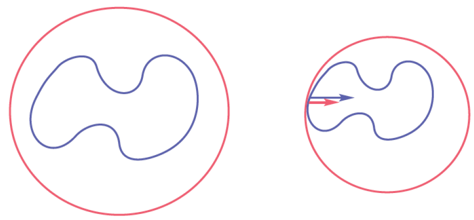
William Minicozzi

William Minicozzi's talk covered geometric and analytic aspects of the mean curvature flow problem. This is the problem of understanding the behavior of a family of hypersurfaces M_t evolving under the differential equation

$$\frac{\partial \boldsymbol{x}}{\partial t} = -H\boldsymbol{n}$$

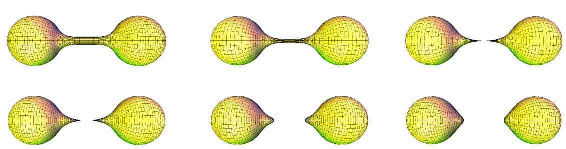
where H is the mean curvature of M_t and $\boldsymbol{n}$ is the unit normal. Two examples of solutions in $\mathbb{R}^3$ are 2-spheres of radius $\sqrt{-4t}$ and cylinders of radius $\sqrt{-2t}$ $(t < 0)$. In these cases the evolution equations are ordinary differential equations. In both the surfaces are 'extinquished' in a finite time, in the first with the spheres contracting to a point, in the second, with the cylinders contracting to a line.



A general feature of mean curvature flow is that a hypersurface enclosing another cannot overtake the inner one, because at the point at which they first touched, the inner surface, with larger mean curvature, would have to be moving faster than the outer one. This is the 'avoidance principle': if hypersurfaces start disjoint, then they must remain disjoint. Any closed hypersurface must shrink to a point: it must remain inside any enclosing sphere, which must itself shrink to a point. So singularities must develop. The problem is to understand them.

In 1984, Huisken dealt with the convex case by proving that a closed convex surface remains convex and flows smoothly until it disappears to a point. Just before the extinction time, it looks completely round.

The non-convex case is less straightforward. A thin symmetrical torus, for example, collapses to a circle. A more involved example is 'Grayson's dumbell'—two large 'bells' connected by a thin bar. Under the flow, the bells shrink more slowly than the bar. The result is that the neck first pinches off; the bells then shrink to points. Because the flow continues past the first singularity, one needs the notion of a 'weak solution' to understand examples of this sort. One such is given by focusing not a single hypersurface but on the level sets of a function, all evolving simultaneously under the flow. Because of the avoidance principle, they remain level sets of a function.



In the *mean convex case* (i.e. $H > 0$), an appropriate function is $v(x,t) = u(x) - t$, where $u(x)$ is the *arrival time*, the time at which M_t reaches x. If M_0 is mean convex then M_t moves monotonically inwards and $u(x)$ is well-defined in the domain swept out by M_t. It satisfies the degenerate elliptic equation

$$\Delta\, u - \mathrm{Hess}_u \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) = -1\,.$$

The critical points of u are the singularities of the flow. By using viscosity solutions, it had been shown that there exists a solution, and that it is Lipschitz. In 2016, Colding and Minicozzi (CM) showed that u is twice differentiable everwhere and that at critical points the Hessian matches that of the flow of the cylinder or of the sphere. That left open the question of whether or not u was actually C^2. This CM resolved in 2016: the solution is C^2 if and only if the critical set is either a single point where the Hessian is spherical or a simple closed C^1 curve where the Hessian is cylindrical.

The key to understanding the detailed structure of the singularities is that they are self-similar under rescaling. This is clear in the cases of spheres and cylinders, but is true more generally. As we zoom in on a critical point, the level sets are spherical or cylindrical. For the second derivative to exist, the orientation of the cylinders must not jump around from point to point, in particular their axes must be aligned. This 'uniqueness' was established by CM in 2015, the essential ingredient being the application of a Lojasiewicz inequality inspired by algebraic geometry.

In the examples (the sphere, the cylinder, the torus, and the dumbbell), the singular set is made up of points and curves. It is never more than one dimensional. By using the same ideas White showed in 2000 that generally in the mean convex case it never exceeds one, in the sense of Hausdorff measure. Again in 2016, CM proved 'rectifiability' by showing that the singular set is in fact contained in a union of compact C^1 curves and a countable set of points.

The lecture concluded with a brief review of higher-dimensional results and of the tools used to prove them. There is a richer set of possibilities for the singularities in higher dimensions. For example, in the mean convex case it is possible to have the product of a sphere of any dimension with a plane of complementary dimension. But otherwise much of the three-dimensional theory extends in a natural way. So, for example, the singular set is rectifiable with finite measure and contained in the finite union of compact C^1 $(n-1)$-manifolds together with a set of dimension not greater than $n-2$. The lower strata are themselves countable unions of C^1 manifolds of lower dimension.