# EXISTENCE AND SMOOTHNESS OF THE NAVIER–STOKES EQUATION

CHARLES L. FEFFERMAN

The Euler and Navier–Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or $3$). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\le i\le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \geq 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The *Navier–Stokes* equations are then given by

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \geq 0), \tag{1}$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \geq 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n). \tag{3}$$

Here, $u^\circ(x)$ is a given, $C^\infty$ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given, externally applied force (e.g. gravity), $\nu$ is a positive coefficient (the viscosity), and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The *Euler equations* are equations (1), (2), (3) with $\nu$ set equal to zero.

Equation (1) is just Newton's law $f = ma$ for a fluid element subject to the external force $f = (f_i(x,t))_{1\le i\le n}$ and to the forces arising from pressure and friction. Equation (2) just says that the fluid is incompressible. For physically reasonable solutions, we want to make sure $u(x,t)$ does not grow large as $|x| \to \infty$. Hence, we will restrict attention to forces $f$ and initial conditions $u^\circ$ that satisfy

$$|\partial_x^\alpha u^\circ(x)| \leq C_{\alpha K}(1+|x|)^{-K} \qquad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K \tag{4}$$

and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \quad \text{on } \mathbb{R}^n \times [0,\infty), \text{ for any } \alpha, m, K. \tag{5}$$

We accept a solution of (1), (2), (3) as physically reasonable only if it satisfies

$$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{6}$$

and

$$\int_{\mathbb{R}^n} |u(x,t)|^2 dx < C \quad \text{for all } t \geq 0 \quad \text{(bounded energy)}. \tag{7}$$

Alternatively, to rule out problems at infinity, we may look for spatially periodic solutions of (1), (2), (3). Thus, we assume that $u^\circ(x), f(x,t)$ satisfy

$$u^\circ(x+e_j) = u^\circ(x), \quad f(x+e_j,t) = f(x,t) \quad \text{for } 1 \leq j \leq n \tag{8}$$

($e_j = j^{\text{th}}$ unit vector in $\mathbb{R}^n$).

In place of (4) and (5), we assume that $u^\circ$ is smooth and that

$$(9) \qquad |\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|t|)^{-K} \quad \text{on } \mathbb{R}^3 \times [0,\infty), \text{ for any } \alpha, m, K.$$

We then accept a solution of (1), (2), (3) as physically reasonable if it satisfies

$$(10) \qquad u(x,t) = u(x+e_j,t) \quad \text{on } \mathbb{R}^3 \times [0,\infty) \text{ for } \quad 1 \le j \le n$$

and

$$(11) \qquad p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)).$$

A fundamental problem in analysis is to decide whether such smooth, physically reasonable solutions exist for the Navier–Stokes equations. To give reasonable leeway to solvers while retaining the heart of the problem, we ask for a proof of one of the following four statements.

**(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$.** Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

**(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$.** Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

**(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$.** Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions $(p,u)$ of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

**(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.** Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions $(p,u)$ of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

These problems are also open and very important for the Euler equations ($\nu = 0$), although the Euler equation is not on the Clay Institute's list of prize problems.

Let me sketch the main partial results known regarding the Euler and Navier–Stokes equations, and conclude with a few remarks on the importance of the question.

In two dimensions, the analogues of assertions (A) and (B) have been known for a long time (Ladyzhenskaya [4]), also for the more difficult case of the Euler equations. This gives no hint about the three-dimensional case, since the main difficulties are absent in two dimensions. In three dimensions, it is known that (A) and (B) hold provided the initial velocity $u^\circ$ satisfies a smallness condition. For initial data $u^\circ(x)$ not assumed to be small, it is known that (A) and (B) hold (also for $\nu = 0$) if the time interval $[0,\infty)$ is replaced by a small time interval $[0,T)$, with $T$ depending on the initial data. For a given initial $u^\circ(x)$, the maximum allowable $T$ is called the "blowup time." Either (A) and (B) hold, or else there is a smooth, divergence-free $u^\circ(x)$ for which (1), (2), (3) have a solution with a finite blowup time. For the Navier–Stokes equations ($\nu > 0$), if there is a solution with a finite blowup time $T$, then the velocity $(u_i(x,t))_{1\le i\le 3}$ becomes unbounded near the blowup time.

Other unpleasant things are known to happen at the blowup time $T$, if $T<\infty$. For the Euler equations ($\nu=0$), if there is a solution (with $f\equiv 0$, say) with finite blowup time $T$, then the vorticity $\omega(x,t)=\mathrm{curl}_x\,u(x,t)$ satisfies

$$\int_0^T \left\{\sup_{x\in\mathbb{R}^3} |\omega(x,t)|\right\} dt = \infty \qquad \text{(Beale–Kato–Majda)},$$

so that the vorticity blows up rapidly.

Many numerical computations appear to exhibit blowup for solutions of the Euler equations, but the extreme numerical instability of the equations makes it very hard to draw reliable conclusions.

The above results are covered very well in the book of Bertozzi and Majda [1].

Starting with Leray [5], important progress has been made in understanding *weak solutions* of the Navier–Stokes equations. To arrive at the idea of a weak solution of a PDE, one integrates the equation against a test function, and then integrates by parts (formally) to make the derivatives fall on the test function. For instance, if (1) and (2) hold, then, for any smooth vector field $\theta(x,t)=(\theta_i(x,t))_{1\le i\le n}$ compactly supported in $\mathbb{R}^3\times(0,\infty)$, a formal integration by parts yields

$$\begin{aligned}(12)\quad \iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\frac{\partial\theta}{\partial t}dxdt - \sum_{ij}\iint_{\mathbb{R}^3\times\mathbb{R}} u_i u_j \frac{\partial\theta_i}{\partial x_j}dxdt \\ = \nu\iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\Delta\theta dxdt + \iint_{\mathbb{R}^3\times\mathbb{R}} f\cdot\theta dxdt - \iint_{\mathbb{R}^3\times\mathbb{R}} p\cdot(\mathrm{div}\,\theta)dxdt.\end{aligned}$$

Note that (12) makes sense for $u\in L^2$, $f\in L^1$, $p\in L^1$, whereas (1) makes sense only if $u(x,t)$ is twice differentiable in $x$. Similarly, if $\varphi(x,t)$ is a smooth function, compactly supported in $\mathbb{R}^3\times(0,\infty)$, then a formal integration by parts and (2) imply

$$(13)\qquad \iint_{\mathbb{R}^3\times\mathbb{R}} u\cdot\nabla_x\varphi dxdt = 0.$$

A solution of (12), (13) is called a *weak solution* of the Navier–Stokes equations.

A long-established idea in analysis is to prove existence and regularity of solutions of a PDE by first constructing a weak solution, then showing that any weak solution is smooth. This program has been tried for Navier–Stokes with partial success. Leray in [5] showed that the Navier–Stokes equations (1), (2), (3) in three space dimensions always have a weak solution $(p,u)$ with suitable growth properties. Uniqueness of weak solutions of the Navier–Stokes equation is *not* known. For the Euler equation, uniqueness of weak solutions is strikingly false. Scheffer [8], and, later, Schnirelman [9] exhibited weak solutions of the Euler equations on $\mathbb{R}^2\times\mathbb{R}$ with compact support in spacetime. This corresponds to a fluid that starts from rest at time $t=0$, begins to move at time $t=1$ with no outside stimulus, and returns to rest at time $t=2$, with its motion always confined to a ball $B\subset\mathbb{R}^3$.

Scheffer [7] applied ideas from geometric measure theory to prove a partial regularity theorem for suitable weak solutions of the Navier–Stokes equations.

Caffarelli–Kohn–Nirenberg [2] improved Scheffer's results, and F.-H. Lin [6] simplified the proofs of the results in Caffarelli–Kohn–Nirenberg [2]. The partial regularity theorem of [2], [6] concerns a parabolic analogue of the Hausdorff dimension of the singular set of a suitable weak solution of Navier–Stokes. Here, the *singular set* of a weak solution $u$ consists of all points $(x^\circ, t^\circ) \in \mathbb{R}^3 \times \mathbb{R}$ such that $u$ is unbounded in every neighborhood of $(x^\circ, t^\circ)$. (If the force $f$ is smooth, and if $(x^\circ, t^\circ)$ doesn't belong to the singular set, then it's not hard to show that $u$ can be corrected on a set of measure zero to become smooth in a neighborhood of $(x^\circ, t^\circ)$.)

To define the parabolic analogue of Hausdorff dimension, we use *parabolic cylinders* $Q_r = B_r \times I_r \subset \mathbb{R}^3 \times \mathbb{R}$, where $B_r \subset \mathbb{R}^3$ is a ball of radius $r$, and $I_r \subset \mathbb{R}$ is an interval of length $r^2$. Given $E \subset \mathbb{R}^3 \times \mathbb{R}$ and $\delta > 0$, we set

$$\mathcal{P}_{K,\delta}(E) = \inf \left\{ \sum_{i=1}^{\infty} r_i^K : Q_{r_1}, Q_{r_2}, \cdots \text{ cover } E, \text{ and each } r_i < \delta \right\}$$

and then define

$$\mathcal{P}_K(E) = \lim_{\delta \to 0+} \mathcal{P}_{K,\delta}(E).$$

The main results of [2], [6] may be stated roughly as follows.

**Theorem.** *(A) Let $u$ be a weak solution of the Navier–Stokes equations, satisfying suitable growth conditions. Let $E$ be the singular set of $u$. Then $\mathcal{P}_1(E) = 0$.*

*(B) Given a divergence-free vector field $u^\circ(x)$ and a force $f(x,t)$ satisfying (4) and (5), there exists a weak solution of Navier–Stokes (1), (2), (3) satisfying the growth conditions in (A).*

In particular, the singular set of $u$ cannot contain a spacetime curve of the form $\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \colon x = \phi(t)\}$. This is the best partial regularity theorem known so far for the Navier–Stokes equation. It appears to be very hard to go further.

Let me end with a few words about the significance of the problems posed here. Fluids are important and hard to understand. There are many fascinating problems and conjectures about the behavior of solutions of the Euler and Navier–Stokes equations. (See, for instance, Bertozzi–Majda [1] or Constantin [3].) Since we don't even know whether these solutions exist, our understanding is at a very primitive level. Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas.

## References

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## Errata

The further condition $p(x+e_j,t)=p(x,t)$ should be made explicit in Eqn (8).

Eqn (10) should read:

$$\begin{aligned}
&-\iint\limits_{\mathbb{R}^3\times\mathbb{R}} u\cdot\frac{\partial\theta}{\partial t}\,\mathrm{d}x\mathrm{d}t-\sum_{ij}\iint\limits_{\mathbb{R}^3\times\mathbb{R}} u_i u_j\frac{\partial\theta_i}{\partial x_j}\,\mathrm{d}x\mathrm{d}t\\
&= \nu\iint\limits_{\mathbb{R}^3\times\mathbb{R}} u\cdot\triangle\theta\,\mathrm{d}x\mathrm{d}t+\iint\limits_{\mathbb{R}^3\times\mathbb{R}} f\cdot\theta\,\mathrm{d}x\mathrm{d}t+\iint\limits_{\mathbb{R}^3\times\mathbb{R}} p\cdot(\operatorname{div}\theta)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$