LEVEL SET FLOW

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ABSTRACT. We will be concerned with the regularity of solutions to a classical degenerate nonlinear second order differential equation on Euclidean space. A priori solutions were only defined in a weak sense but it turns out that they are always twice differentiable classical solutions. The proof weaves together analysis and geometry. Without deeply understanding the underlying geometry, it is impossible to prove fine analytical properties.

The level set method has been used with great success the last thirty years in both pure and applied mathematics to describe evolutions of various physical situations. Given an initial interface (front) M_0 bounding a region in $\mathbf{R}^{n+1}$, the level set method is used to analyze its subsequent motion under a velocity field. The idea is to represent the evolving front as a level set of an evolving function $v(x,t)$ where $x \in \mathbf{R}^{n+1}$ and t is time. The initial front M_0 is given by

$$M_0 = \{x \,|\, v(x,0) = 0\}$$

and the evolving front is described for all later time as the set where $v(x,t)$ vanishes. This is significant because topological changes of the evolving front, such as components breaking apart or merging together, are well defined.

In mean curvature flow, the velocity vector field is the mean curvature vector and the evolving front is the level set of a function that satisfies a nonlinear degenerate parabolic equation. Solutions are defined weakly in the viscosity sense; in general, they may not even be differentiable (let alone twice differentiable). However, it turns out that for a monotonically advancing front viscosity solutions are in fact twice differentiable and satisfy the equation in the classical sense. Moreover, the situation becomes very rigid when the second derivative is continuous.

Suppose $\Sigma \subset \mathbf{R}^{n+1}$ is an embedded hypersurface and $\mathbf{n}$ is the unit normal of Σ. The **mean curvature** is given by $H = \operatorname{div}_\Sigma(\mathbf{n})$. Here

$$\operatorname{div}_\Sigma(\mathbf{n}) = \sum_{i=1}^{n} \langle \nabla_{e_i} \mathbf{n}, e_i \rangle \; ;$$

where e_i is an orthonormal basis for the tangent space of Σ. If $\Sigma = u^{-1}(s)$ is the level set of a function u on $\mathbf{R}^{n+1}$ and s is a regular value, then $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ and $H = \operatorname{div}_{\mathbf{R}^{n+1}} \left(\frac{\nabla u}{|\nabla u|} \right)$.

A one-parameter family of smooth hypersurfaces $M_t \subset \mathbf{R}^{n+1}$ flows by the **mean curvature flow**, or MCF for short, if

$$x_t = -H\, \mathbf{n} \, ,$$

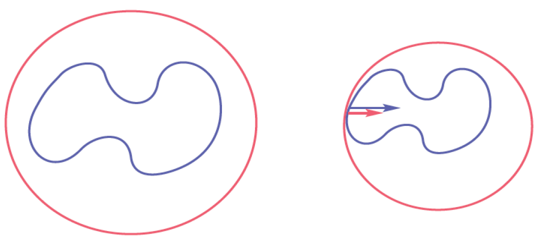
The authors were partially supported by NSF Grants DMS 1404540 and DMS 1206827. This material is based upon work supported by the NSF DMS 1440140, while T.H.C. was in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, CA, during the Spring of 2016.

where H and $\mathbf{n}$ are the mean curvature and unit normal, respectively, of M_t at the point x. The earliest reference to the mean curvature flow we know of is in the work of George Birkhoff in the 1910s, where he used a discrete version of this, and independently in the material science literature in the 1920s.

Two key properties:

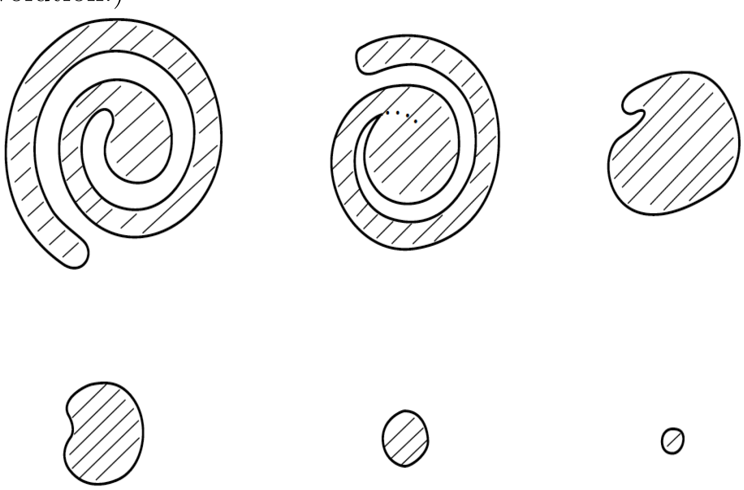
- H is the gradient of area, so MCF is the negative gradient flow for volume (Vol M_t decreases most efficiently).
- Avoidance property: If M_0 and N_0 are disjoint, then M_t and N_t remain disjoint.

The avoidance principle is simply a geometric formulation of the maximum principle. An application of it is illustrated in the following drawing that shows that if one closed hypersurface (the red one) encloses another (the blue one), then the outer one can never catch up with the inner. The reason for this is that if it did there would be a first point of contact and right before that the inner one would contract faster than the outer, contradicting that the outer was catching up.



Curve shortening flow: When $n = 1$ and the hypersurface is a curve, the flow is the curve shortening flow. Under the curve shortening flow, a (round) circle shrinks through (round) circles to a point in finite time. A remarkable result of Matt Grayson from 1987 (using earlier work of Richard Hamilton and Michael Gage) shows that any simple closed curve in the plane remains smooth under the flow until it disappears in finite time in a point. Right before it disappears, the curve will be an almost round circle.

The evolution of the snake like simple closed curve that is the boundary of the shaded region in the figure below illustrates this remarkable fact. (The six figures are time shots of the evolution.)



Level set flow: The analytical formulation of the flow is the level set equation that can be deduced as follows: Given a closed (embedded) hypersurface $\Sigma \subset \mathbf{R}^{n+1}$, choose a function $v_0 : \mathbf{R}^{n+1} \to \mathbf{R}$ so that Σ is the level set $\{v_0 = 0\}$.

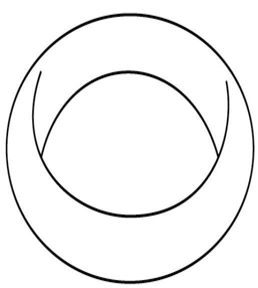
- If we simultaneously flow $\{v_0 = s_1\}$ and $\{v_0 = s_2\}$ for $s_1 \neq s_2$, then avoidance implies they stay disjoint.
- In the level set flow, we look for $v : \mathbf{R}^{n+1} \times [0, \infty) \to \mathbf{R}$ so that each level set $t \to \{v(\cdot, t) = s\}$ flows by MCF and $v(\cdot, 0) = v_0$.
- If $\nabla v \neq 0$ and the level sets of v flow by MCF, then

$$v_t = |\nabla v| \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right).$$

This is degenerate parabolic and undefined when $\nabla v = 0$. It may not have classical solutions.

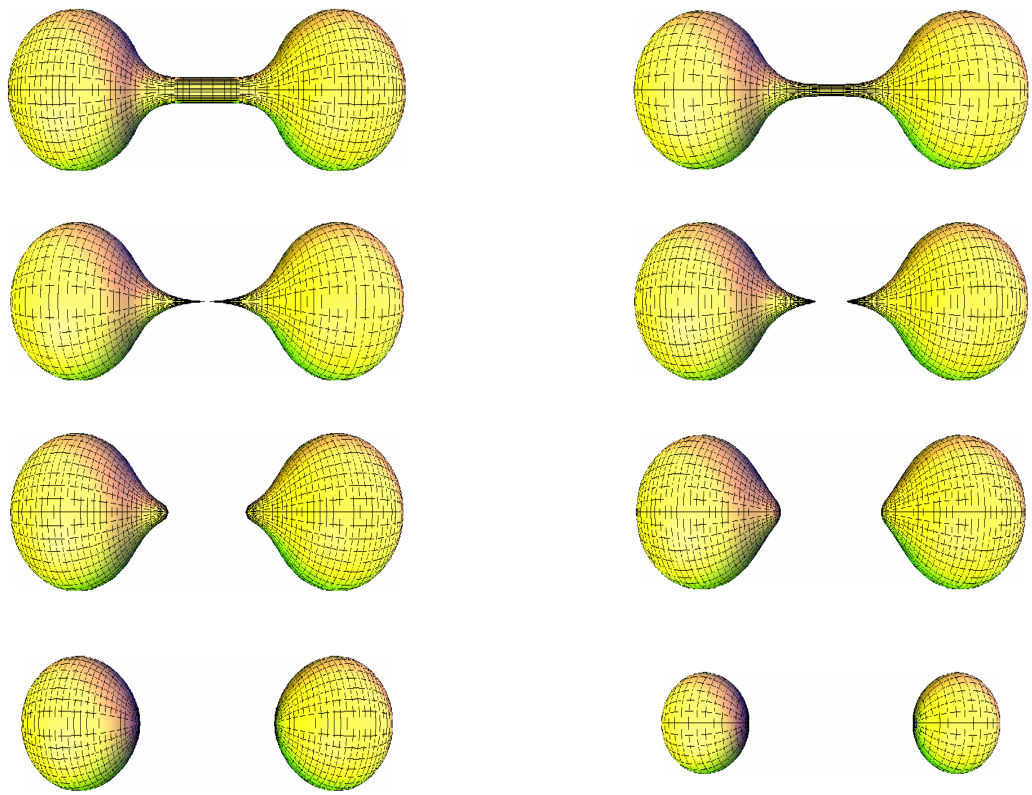
In a widely cited paper (cited more than 12000 times) from 1988, Stanley Osher and Jamie Sethian studied this equation numerically. The analytical foundation was provided by Craig Evans and Joel Spruck in a series of four papers in the early 1990s and, independently and at the same time, by Yun Gang Chen, Yoshikazu Giga, and Shunichi Goto. Both of these two groups constructed (continuous) viscosity solutions and showed uniqueness. The notion of viscosity solutions had been developed by Pierre-Louis Lions and Michael G. Crandall in the early 1980. The work of these two groups on the level set flow was one of the significant applications of this theory.

Examples of singularities: Under MCF a round sphere remains round but shrinks and eventually becomes extinct in a point. A round cylinder remains round and eventually becomes extinct in a line. The marriage ring is the example of a thin torus of revolution in $\mathbf{R}^3$ (see figure below). Under the flow the marriage ring shrinks to a circle then disappears.



Dumbbell: Under the mean curvature flow of the rotationally symmetric mean convex dumbbell in $\mathbf{R}^3$ (see the figure below) with a sufficiently thin neck, the neck pinches off first and the surface disconnects into two components. Later each component (bell) shrinks to a round point. This example falls into a larger category of rotationally symmetric surfaces that are rotationally symmetric around an axis. Because of the symmetry, then the solution reduces to a one-dimensional heat equation. This was analyzed already in the early 1990s by Sigurd Angenent, Steven Altschuler and Giga. A key tool in their arguments was a parabolic Sturm-Liouville theorem of Angenent that holds in one spatial dimension.

Singular set: Under MCF closed hypersurfaces contract, develop singularities and eventually become extinct. The **singular set** $\mathcal{S}$ is the set of points in space and time where the flow is not smooth.



In the first 3 examples (the sphere, the cylinder and the marriage ring): $\mathcal{S}$ is a point, a line, and a closed curve, respectively. In each case, the singularities occur only at a single time. In contrast, the dumbbell has two singular times with one singular point at the first time and two at the second.

Mean convex flows: A hypersurface is convex if every principal curvature is positive. It is mean convex if $H > 0$, i.e., if the sum of the principal curvatures is positive at every point. Under the MCF a mean convex hypersurface moves inward and, since mean convexity is preserved, it will continue to move inward and eventually sweep out the entire compact domain bounded by the initial hypersurface.

Level set flow for mean convex hypersurfaces: When the hypersurfaces are mean convex, the equation can be rewritten as a degenerate elliptic equation for a function u called the arrival time defined by

$$u(x) = \{t \,|\, x \in M_t\} .$$

We say that u is the **arrival time** since it is the time the hypersurfaces M_t arrive at x as the front sweeps through the compact domain bounded by the initial hypersurface. It follows easily that if we set $v(x,t) = u(x) - t$, then v satisfies the level set flow. From this, we see that the arrival time u satisfies

$$-1 = |\nabla u| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) .$$

This is a degenerate elliptic equation that is undefined when $\nabla u = 0$. Note that if u satisfies this equation, then so does u plus a constant. This just corresponds to shifting the time when the flow arrives by a constant. A particular example of a solution to this equation is the function $u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)$ that is the arrival time for shrinking round cylinders in $\mathbf{R}^3$. In general, Evans-Spruck (cf. Chen-Giga-Goto) constructed Lipschitz solutions to this equation and Robert Kohn and Sylvia Serfaty gave a game theoretic interpretation of the arrival time.

Singular set of level set flow for mean convex: The singular set of the flow is the critical set of u. Namely, $(x, u(x))$ is singular **iff** $\nabla_x u = 0$. For instance, in the example of the shrinking round cylinders the arrival time is given by $u = -\frac{1}{2}\left(x_1^2 + x_2^2\right)$ and the flow is singular in the line $x_1 = x_2 = 0$ that is exactly where $\nabla u = 0$.

We will next see that even though the arrival time was only a solution to the level set equation in a weak sense it turns out to be always twice differentiable classical solution.
Differentiability: CM, 2015:

- u is twice differentiable everywhere and smooth away from the critical set.
- u satisfies the equation everywhere in the classical sense.
- At each critical point the hessian is symmetric and has only two eigenvalues 0 and $-\frac{1}{k}$; $-\frac{1}{k}$ has multiplicity $k+1$.

Obviously this result is equivalent to saying that at a critical point, say $x = 0$ and $u(x) = 0$, the function u is (after possibly a rotation of $\mathbf{R}^{n+1}$) up to higher order terms equal to the quadratic polynomial

$$-\frac{1}{k}\left(x_1^2 + \cdots + x_{k+1}^2\right) .$$

This second order approximation is simply the arrival time of the shrinking round cylinders. It suggests that the level sets of u right before the critical value and near the origin should be approximately cylinders (with an $n-k$ dimensional axis). This has indeed been known for a long time[1]. It also suggests that those cylinders should be nearly the same (after rescaling to unit size). That is, the axis of the cylinders should not depend on the value of the level set. This last property however was only very recently established and is the key to proving that the function is twice differentiable[2]. The proof that the axis is unique, independent on the level set, relies on a key new inequality that draws its inspiration from real algebraic geometry although the proof is entirely new. This kind of uniqueness is a famously difficult problem in geometric analysis and no general case had previously been known.

Regularity of solutions: We have seen that the arrival time is always twice differentiable and one may wonder whether there is even more regularity. Gerhard Huisken showed already in 1990 (using an earlier result of himself from 1984) that the arrival time is C^2 for **convex** M_0. However, in 1992 Tom Ilmanen gave an example of a rotationally symmetric **mean convex** M_0 in $\mathbf{R}^3$ where u is not C^2. This result of Ilmanen shows that the above theorem about differentiability can not be improved to C^2. We will see later that in fact one can entirely characterize when the arrival time is C^2. In the plane, Kohn and Serfaty (2006)

[1]This follows from work of Brian White, Huisken, Huisken and Carlo Sinestrari, Robert Haslhofer and Bruce Kleiner, and Ben Andrews; cf. work of Simon Brendle, and Colding and Minicozzi.

[2]Uniqueness of the axis is parallel to that a function is differentiable at a point precisely if it on all sufficiently small scales at that point looks like the *same* linear function.

showed that u is C^3 and for $n > 1$ Natasa Sesum gave (2008) an example of a **convex** M_0 where u is not C^3. Thus Huisken's result is optimal for $n > 1$.

The next result show that one can entirely characterize when the arrival time is C^2.

M_0 **mean convex**: CM, 2016: u is C^2 **iff**:

- There is exactly one singular time T.
- The singular set $\mathcal{S}$ is an embedded closed C^1 k-dimensional submanifold of cylindrical singularities.

In $\mathbf{R}^3$ when u is C^2 the singular set $\mathcal{S}$ is either:

(1) A single point with a spherical singularity.
(2) A simple closed C^1 curve of cylindrical singularities.

The examples of the sphere and the marriage ring show that each of these examples can happen, whereas the example of the dumbbell does not fall into either showing that in that case the arrival time is not C^2.

We can restate this result for $\mathbf{R}^3$ in terms of the structure of the critical set and Hessian: u is C^2 **iff** u has exactly one critical value T and the critical set is either:

(1) A single point where Hess_u is $-\frac{1}{2}$ times the identity.
(2) A simple closed C^1 curve where Hess_u has eigenvalues 0 and -1 with multiplicities 1 and 2, respectively.

In case (2), the kernel of Hess_u is tangent to the curve.

Concluding remarks: We have seen that for one of the classical differential equations in order to understand the analysis it is necessary to understand the underlying geometry. There are many tantalizing parallels to other differential equations both elliptic and parabolic.

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