[Ir] My dear Lady Lovelace

I have of course but little to say on your report of progress up to the 21st. With regard to your music, if you have any wish to begin the study of acoustics, you may find an elementary compendium of the most material points directly connected with music, in the following articles of the Penny Cyclopaedia

*Acoustics, Cord, Harmonics, Pipe*

The last will appear at the end of this month and contains a correction to be made in each of the first and second. I should recommend to you, out of the same Cyclopædia the articles

*Infinite; Nothing; Limit; Negative and Impossible Quantities;*

*Fractions, Vanishing;*

[IV] With regard to the only point to which you have alluded your finding that ‘I do not insist in zero being something’ the matter is as follows. Zero is something though not some quantity, which is what you here mean by thing. Some writers on the differential calculus use not 0, but \( \frac{0}{0} \), in an absolute sense, as standing for a quantity. The difference between them and others consists mostly in phraseology. One person will say that \( \frac{a^2-a^2}{a-a} \) is \( 2a \); a second will say that the appearance of \( \frac{a^2-a^2}{a-a} \) denotes a misapprehension to have taken place during the process, which being avoided in a repetition of the process, \( 2a \) comes out rationally as the answer. If the first be pressed to prove his assertion, he will tell you that the two words underlined are the abbreviation of all that is afterwards underlined.

You will see more of this abbreviation in the article *Infinite* to which I have referred. Absolute modes of speaking, which are false, are conventionally used in abbreviation of circumlocutions.

I forwarded to Lord Lovelace the other day a prospectus of a Society now in process of formation for [2r] printing scientific manuscripts which have not been hitherto printed. The works of this society are not to be published, but only distributed
among the members. As you will certainly take
an interest in the results of its labors, if you
continue your studies, I should recommend
your being a member, with Lord Lovelace as
your proxy.
There is another Society also forming, of the same
kind, and on the same terms, the Percy Society
for the publication of ancient inedited
songs and ballads.

I remain
Faithfully yours
ADeMorgan

3 Grotes’ Place
Blackheath
Thursd’y m’s

My wife desires kind remembrances.
The solution of the $t^3 + t^2 + t$ problem is correct and I have no doubt from it that you fully understand the problem of the stone.

The difficulty you meet with in the variable coefficient would be fatal to the process if the coefficient increased without limit as the value of $k$ diminishes without limit.

But in this case $t$ which you call a variable, is really a given constant, for it is not $t$ which varies, but the time which is first $t$, then $t + k$, then $t + 2k$, &c.

There is a little incorrectness in the phraseology of variable quantities. A quantity varies, it is first $x$, and then $x + a$; here it is usual to say that $x$ varies, whereas it is not $x$, but the magnitude which $x$ represents, which changes and is no longer represented by $x$, but by $x + a$.

Thus if we pass in thought from 10 seconds to 11 eleven [sic] seconds, we say let 10 vary, and become 11. Now 10 is a fixed symbol, and so is 11; it is we ourselves who vary our supposition, and pass from one to the other.

But even if the coefficients of $k$ were variable, it would not vitiate the result, as may be thus shown.

Let $a + vk$ be an expression in which $a$ is a constant $k$ diminishes without limit, and $v$ at the same time varies, say increases. Let $v$ always remain finite, that is, let it not increase without limit while $k$ diminishes.

[3v] Suppose for instance, that it never exceeds a certain number (no matter how great) say a million. Then $vk$ never exceeds 1000,000$k$. Now if $k$ may be made as small as we please, so may 1000,000$k$, and still more $vk$, which is less, or at least not greater. That is $vk$ diminishes without limit, and $a + vk$ has the limit $a$.

But if $v$ increased without limit as $k$ diminished, the case might be altered (not necessarily would). For example

1. Let $v = \frac{1}{k}$ crossed out $\frac{1+k}{k}$
   $a + vk = a + 1 + k$ and the limit is $a + 1$

2. Let $v = \frac{1+k}{k^2}$
   $a + vk = a + \frac{1+k}{k}$, and increases without limit
3. Let \( v = \frac{1+k}{\sqrt{k}} \)

\[
a + vk = a + (1 + k)\sqrt{k}, \text{ and the limit is } a, \text{ as at first.}
\]

And \( \frac{1+k}{k}, \frac{1+k}{k^2}, \frac{1+k}{\sqrt{k}} \) all increase without limit as \( k \) diminishes without limit.

[4r] Your correction of the press is right. On the cover of no 12 you will see a list of errata.

When two lines at right angles are made standards of position, and when the position of a point is determined by its perpendicular distance from the lines, as \( PA, PB, \)

then \( PA \) and \( PB \) are called co-ordinates of \( P \). One of them, as \( BP \) is usually found by its equal \( OA, \) and called the abscissa of \( P \), the other, \( AP \), is called the ordinate.

I have written something on the paper which I return.

My wife desires kind remembrances, and with our united remembrance to Lord Lovelace

I remain

Yours very truly

A De Morgan

3 Grotes’ Place Blackheath

August 1, 1840

Should you decide on supporting our old printing Society, I shall be very happy to have Lord Lovelace’s name inserted. We are beginning with an Old Saxon treatise on astronomy with a translation.
My dear Lady Lovelace

I should be as able as willing to see you in town on Friday, but have first heard that Mr. Frend is not so well as he has been, and am going to Highgate to day to see how he is. In consequence, having various matters to complete definitively by the 16th instant, I shall find it impossible to go to town again this week.

With regard to the second chapter, pray remember that you are not supposed to know, or to want to know, what differentiation is, but only that there is a process of that name, which is to be learnt by rule for the present, as an exercise in algebraical work.

With regard to the logarithms, in the first place, Bourdon is too long. If you will look at the chapter in my algebra, you will find it shorter.

In the equation

\[ a^b = c \]

\( b \) is called the logarithm of \( c \) to the base \( a \). This is the meaning of the term. But for convenience the series \( 1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \&c \) ad inf or \( 2.7182818 \cdots \) (called \( \varepsilon \)) is the base always used in theory; while when assistance in calculation is the object, 10 is always the base; thus if

\[ \varepsilon^x = y \]

\( x \) is the logarithm of \( y \)

[6r] Thus \( a = \log b \) is by definition synonymous with \( b = \varepsilon^a \) \( \varepsilon \) being 2.7182818 \cdots

I remain

Yours very truly

ADeMorgan

3 Grotes’ Place, Blackheath
Wednesday M
My dear Lady Lovelace

The Theorem in page 16 can be easily proved when the following is proved
\[ \frac{a + a'}{b + b'} \text{ lies between } \frac{a}{b} \text{ and } \frac{a'}{b'} \]
\[ \frac{a + a'}{b + b'} = \frac{a(1 + \frac{a'}{b})}{b(1 + \frac{a'}{b'})} = \frac{a}{b} \times \frac{1 + \frac{a'}{b'}}{1 + \frac{a'}{b}} \]
Now if \( \frac{a}{b} \) be greater than \( \frac{a'}{b'} \)
\[ \frac{a}{b} \times \frac{1 + \frac{a'}{b'}}{1 + \frac{a'}{b}} \text{ is less than } \frac{a}{b} \]
or \( \frac{a + a'}{b + b'} \) is less than \( \frac{a}{b} \)
Similarly, it may be shown that if \( \frac{a}{b} \) be less than \( \frac{a'}{b'} \)
\[ \frac{a + a'}{b + b'} \text{ is greater than } \frac{a}{b} \]
You will now I think, not have much difficulty in proving the whole. Page 48 [or 28?] contains the general view of this theorem

Page 29. Our conclusions are really the same. To say that \[ \text{diagram in original} \] is a right angled triangle, is to say that \( OP \) is straight and not curved. The following however will explain

[7v] \[ \text{diagram in original} \] By the tangent of \( \angle POM \) is meant the fraction \( \frac{PM}{OM} \), which is, by similar triangles, the same thing for every point of \( OP \).
If then \( PM = \frac{2}{3} OM \), always, we have \( \frac{PM}{OM} = \frac{2}{3} \) always, or the direction \( OP \) is always such as to make the angle \( POM \) the same, namely that angle which has \( \frac{2}{3} \) for its tangent.
To see all this fully something of Trigonometry and the application of algebra to geometry is required.
The Differential and Integral Calculus deal in the same elements, but the former separates one element from the mass and examines it, the latter puts together the different elements to make the whole mass.
The examination of \( PQMN \) (p. 29) with a view to the relation between \( OM \) and \( MP \) is a case of the first: the summation of the rectangles in page 30, of the second.
Page 32. The reference is unnecessary.
The first series \( 1 + 4 + \&c \) is finite, the second infinite.
It is not easy to see à priori why one problem should be attainable with given means and another not
so. It is stated here with a view to the following common misapprehension.

[8r] It is thought that Newton and Leibnitz had some remarkable new conception of principles, which is not true. Archimedes and others ['and others’ inserted] had a differential and integral calculus, but not an algebraical system of sufficient power to express very general truths.

Many persons before Newton knew, for instance that if \((x+h)^n-x^n\) could be developed for any value of \(n\), the tangents of a great many curves could be drawn and they knew this upon principles precisely the same as Newton and Leibnitz knew it. But Newton did develop \((x+h)^n-x^n\) and did that which they could not do.

It was the additions made to the powers of algebra in the seventeenth century, and not any new conceptions of quantity, which made it worth while to attempt that organization which has been called the Differential Calculus.

I should recommend your decidedly continuing the Differential Calculus, warning you that you will have long digressions to make in Algebra and Trigonometry. I should recommend you to get my Trigonometry, but not to attempt anything till I send you a sketch of what to read in it. The Algebra you must go through at some time or [8v] other, adding to it the article

“Negative and impossible quantities in the Penny Cyclopaedia.

I have no doubt of being able to talk this matter over with you in town when you arrive.

In the mean while, as mechanical expertness in differentiation is of the utmost consequence, and as it is the most valuable exercise in algebraical manipulation which you can possibly have, I should recommend your thoroughly acquiring and keeping up the Chapter you are now upon.

Yours very truly

ADeMorgan

3 Grotes’ Place
Monday Aug\textsuperscript{st} 17/40
Mr Frend is rather better. I will add Lord Lovelace’s name to my list of members.
My dear Lady Lovelace

I am in the middle
of arranging my books and can
only just get room to write a
short note.
I received yours relative to the
inquiry about the study of
Math\(^\text{cs}\) but did not answer as
you would have left town and
I did not know your country
address.
I don’t think you will want
the ‘Study of Math\(^\text{cs}\)’. The
corresponding subject in the
Algebra or Trigonometry will
be what you want.
The continuity question
you must consider well (because
you will not allow yourself to
skip) and perhaps your ultimate
difficulties will not be
altogether the same as those
you commence with. The
curve is laid down right
and I have appended
a remark to what
I return.

Yours very truly

ADeMorgan

69 G.S.

I have now returned to town.
My wife is gone to Highgate
for a fortnight
[in Lovelace’s hand] The successive values of \( x, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4 \) are selected & the corresponding function \( x^2 \) represented on perpendicular lines drawn from the extremities of the respective line representing \( x \).

By writing the successive perpendicular extremities \( \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1, \frac{9}{4}, 4 \) &c &c, a curve appears to be produced.

In lines 13 & 14, this curve (according to my interpretation) is alluded to as “the representation of a function, or functions”.

This does not appear to me to hold good, as I should have said that the perpendicular straight lines are the representation of the function, & I do not see any precise relation that the existing curve holds to them.

[in De Morgan’s hand] The precise relation is that this one curve, and no other, belongs to \( y = x^2 \). Of course there could be no visible relation unless to a person whose eye was so good a [Judge crossed out] judge of length that he could see the ordinate increasing with the square of the abscissa.
[12r] My dear Lady Lovelace

You have got through the matter about which you write better than I should have expected.

I have finished what you sent as you will see

With regard to the curve, I drew it as containing every possible sort of singular point. Its equation would be enormously complex. There must be an infinite number of different equations which belong to a curve of a similar form, but the question ‘given the more general form of a curve, required the equations which may belong to such form’ is a very difficult one.

I will merely give you a glimpse.

Required an equation to a curve such that it passes through the following points $P Q R$

[diagram in original] at $P$ let $x = a, y = A$

$Q \quad x = b \quad y = B$

$R \quad x = c \quad y = C$

[the next formula and the following line of text stretch across 12v and 13r]

$$y = A \frac{ (x-b)(x-c) }{ (a-b)(a-c) } + B \frac{ (x-c)(x-a) }{ (b-c)(b-a) } + C \frac{ (x-a)(x-b) }{ (c-a)(c-b) } + \left\{ \begin{array}{c} \text{any function of } x \text{ which does not become infinite} \\ \text{when } x = a, \text{ or } b, \text{ or } c \end{array} \right\} \times (x-a)(x-b)(x-c)$$

Here is an infinite number of equations which you will find to satisfy the conditions.

I have to thank you for very good partridges received from Ockham.

With kind remembrances to Lord Lovelace

I am Yours very truly

ADeMorgan

I have heard of Lady Byron by Mr Phitton [?] who left her safe at Fountainebleu.
My dear Lady Lovelace

Your inquiries were received just after I
had dispatched the receipt for Lord L’s subscription to the Hist. Soc.

While I think of it (the Hist. Soc. reminds me) Nicolas Occam, or
Ockham, or of Ockham, who flourished about 1350, took his name I
rather think, from the same place as your little boy. He was a mathe-
matician, and one of the most remarkable English metaphysicians
before Locke. It is very likely that the late Ld King may on both
accounts, Ockham and metaphysics, [‘, Ockham and metaphysics,’ inserted] have collected
something about him, or that Lord Lovelace
may be in possession of something relating to him. If so, it can
certainly be made useful. His logic was printed very early but
is so scarce that I have never been able to get sight of a copy.

Now to your queries. Festina lente, and above all never estimate
progress by the number of pages. You can hardly be a judge of the
progress you make, and I should say that it is more likely you
progress rapidly upon a point that makes you think for an
hour, than upon an hour’s quick reading, even when you
feel satisfied. That which you say about the comparison of what
you do with what you see can be done was equally said by Newton
when he compared himself to a boy who had picked up a few pebbles
from the shore; and the last words of Laplace were ‘Ce que nous
connaissons est peu de chose; ce que nous ignorons est immense’
So that you have respectable authority for supposing that you
will never get rid of that feeling; and it is no use trying
to catch the horizon.

Peacocks examples will be of more use than any
book.

As to the functional equation. You must distinguish
in algebra questions of quantity from questions of
form. For example “given $x + 8 = 10$, required $x$,” is
a question of quantity but “given $x$, an arbitrary
variable, required a function of $x$ in which if the function
itself be substituted for $x$, $x$ shall be the result”
is a question of form, independent of value, for it is to
be true for all values of $x$. One solution is

$$
\frac{1-x}{1+x}
$$

for $x$ substitute the function itself, this gives

$$
\frac{1+x}{1+x} = \frac{1-x}{1+x}
$$

or

$$
\frac{1+x-(1-x)}{1+x+(1-x)} = \frac{2x}{2} = x.
$$

Another solution is $1 - x$, since $1 - (1 - x)$ is $x$;
a third is $-x$, since $-(-(x))$ is $x$. 
Now suppose \( \varphi(x + y) = \varphi x + \varphi y \)

- \( x^2 \) does not satisfy this; \( x + y \) is not \( x^2 + y^2 \)
- \( ax \) does \( a(x + y) \) is \( ax + ay \)

\[ \varphi x = x^a \quad \varphi(xy) = (xy)^a = x^a y^a = \varphi x + \varphi y \]
\[ \varphi x = a^x \quad \varphi x \times \varphi y = a^x \times a^y = a^{x+y} = \varphi(x + y) \]

| \( \varphi x = ax + b \) | \( \frac{\varphi x - \varphi y}{\varphi x - \varphi z} = \frac{ax+b-(ay+b)}{ax+b-(az+b)} = \frac{ax-ay}{ax-az} = \frac{x-y}{x-z} \) |

A functional equation is one which has for its unknown the form proper to satisfy a certain condition.

Example. What function of \( x \) is that which is not altered by changing \( x \) into \( 1 - x \), let \( x \) be what it may. Or, required \( \varphi x \) so that

\[ \varphi x = \varphi(1 - x) \]

One solution is \( \varphi x = 1 - 2x + 2x^2 \)

- \( \varphi(1 - x) = 1 - 2(1 - x) + 2(1 - x)^2 \)
- \( = 1 - 2 + 2x + 2 - 4x + 2x^2 \)
- \( = 1 - 2x + 2x^2 \) as before.

The equation of a curve means that equation which must necessarily be true of the coordinates of every point in it, and obviously depends upon 1. The point chosen from which to measure coordinates 2.

The direction chosen for the coordinates. 3. The nature and position of the curve. For example let the curve be a circle, the point chosen its center, and the axes of coordinates two lines at right angles. Let the [diagram in original] radius be \( a \); then at every point \( x \) and \( y \) must be the two sides of a right angled triangle whose hypotenuse is \( a \); or

\[ x^2 + y^2 = a^2 \]

which being an equation true at every point [15v] of the circle, is called the equation of the circle.

My wife returns to day from Highgate.
Mr Frend continues very comfortable, and neither mends nor grows worse. I hope Ld Lovelace and the little people are well. The old Ockham will be a poor example for the young one, though
he was a monk, as I suppose. I would have
been nothing else had I lived in his day
Yours very truly
ADeMorgan

69 Gower St.
Monday Sep† 15/40
My dear Lady Lovelace

You have taken a proper time to begin with
Incommensurables and if the subject interests you, I should recommend
you to continue. You understand of course that your Diff
calculus must be delayed from time to time while you make up
those points of Algebra and Trigonometry which you have
left behind.

D. C. p. 53. As in page 22 refers to the method of proving that
if \( P = 2Q \), lim. of \( P \) = 2, lim. of \( Q \)
In similar way it may be shown that if
\( \frac{\Delta u}{\Delta x} \cdot \frac{\Delta x}{\Delta u} = 1 \) lim. of \( \frac{\Delta u}{\Delta x} \times \) lim. of \( \frac{\Delta x}{\Delta u} = 1 \)
With reference to your remark remember that
\( \frac{\Delta u}{\Delta x} \cdot \frac{\Delta x}{\Delta u} = 1 \) and \( \frac{a}{b} \times \frac{b}{a} = 1 \) are the same proposition
But \( \frac{du}{dx} \times \frac{dx}{du} = 1 \) and \( \frac{a}{b} \times \frac{b}{a} = 1 \) are not the same

\( \frac{\Delta u}{\Delta x} \times \frac{\Delta x}{\Delta u} = 1 \) by common algebra \( \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1 \)
But we cannot say \( \frac{du}{dx} \times \frac{dx}{du} = \frac{du}{dx} \frac{dx}{du} = 1 \)
because \( \frac{du}{dx} \) is a mere symbol to denote limit of \( \frac{\Delta u}{\Delta x} \) and \( du \) and \( dx \)
have no separate meaning

N. & M. p. 17

The erratum exists ['but the misprint is’ crossed out] and must
be set right as you propose
['for \( \frac{a}{p_1} - \frac{a}{p_2} \)’ crossed out]
The lengthiness of the proof arises from the necessity
of adapting a very common algebraical theory to Euclid’s
method.

You should try some of the examples of differentiation
in Peacock’s book. Remember that there are some
misprints in it. You will not have to go through
it to try a little of everything.

When the article Proportion appears in the Penny Cycl.
which it will in a few weeks, \( \Gamma \) recommend your
attention to it

With remembrances to Lord Lovelace \hspace{1cm} I am
Yours truly \hspace{1cm} ADeMorgan

69 Gower St.

Sunday M’s Sept 27/40
My dear Lady Lovelace

First as to your non mathematical question: I do not think an English jury would have found Mad. Laffarge guilty; but the presumptions of guilt and innocence can only be perfectly made by those who have the same national opinions and feelings as the accused. I think it very possible that a Frenchman may know a Frenchman to be guilty upon grounds which an Englishman would not understand; for instance, a particular act may be such as a Frenchman may know a Frenchman would not do, unless he had committed a murder before; and such acts may have been proved for aught I (who have not read much of the trial) can tell. It is the same thing in our courts of justice: judges and counsel can by experience make things which would appear to you or me almost indifferent, carry very positive conclusions to their own minds.

Now as to the part of your difficulty contained between XX in the remarks. It matters nothing (p. 210) whether $\frac{n-p}{p+1}x$ is negative or positive. If you perfectly understand why I neglect the sign in the fractional case, the same reason applies to the negative one. When $n$ is negative then $n - p$ (p being essentially +) is necessarily negative. Consequently, (x being positive) the terms alternate in sign from the very beginning, whereas when $n$ is positive and fractional, they do not begin to alternate until $p$ passes $n$. But our matter is to determine convergency, and if a series of positive terms be convergent, so will be the series of similar terms alternating. It is on the absolute magnitude of $\frac{n-p}{p+1}x$, independent of sign, that it depends whether the terms shall ultimately diminish so as to create convergence, or not. Now the limit of this is $x$, whence follows as in the book [18v] It is important to remember in results which depend entirely on limits that they have nothing to do with any vagaries which the quantity tending to a limit chooses to play, provided that, when it has sown its wild oats, it settles down into a steady approach to its limit. The sins of its youth are not to be remembered against it. Now $\frac{n-p}{p+1}x$ when $n$ is positive, remains positive until $p$ passes $n$ and then settles into incurable negativeness. But when $x$ is [‘positive’ crossed out] negative, it is negative from the beginning. Now this matters nothing as to a result which depends only on the limit to which $\frac{n-p}{p+1}x$ approaches as $p$ is increased without limit.

As to the point marked $B$, remember that placing this doubtful assumption, namely the expansibility of functions of $x$ in whole powers of $x$, out of doubt by instances,
has been a prevailing vice of algebraical writers, and one which is to be carefully avoided. It was once thought, by instance, that \( x^2 + x + 41 \) must be a prime number, whenever \( x \) is a whole number, for

\[
\begin{align*}
  x = 0 & \quad x^2 + x + 41 = 41 \text{ a prime no} \\
  = 1 & \quad = 43 \quad \cdots \cdots \\
  = 2 & \quad = 47 \quad \cdots \cdots \\
  = 3 & \quad = 53 \quad \cdots \cdots \\
  = 4 & \quad = 61 \quad \cdots \cdots \\
  = 5 & \quad = 71 \quad \cdots \cdots \\
  = 6 & \quad = 83 \quad \cdots \cdots \\
  = 7 & \quad = 97 \quad \cdots \cdots 
\end{align*}
\]

Now \( x^2 + x + 41 \), though it gives nothing but prime numbers up to \( x = 39 \) inclusive, yet gives a composite number when \( x = 40 \); for it then is

\[
40 \times 40 + 40 + 41 \quad \text{or} \quad 41 \times 40 + 41 \text{ or } 41 \times 41
\]

and when \( x = 41 \) it is

\[
41 \times 42 + 41 \quad \text{or} \quad 43 \times 41
\]

and for higher values it gives sometimes prime numbers sometimes not, like other functions. So much for instances.

C. The supposition as to the meaning of the non-arithmetical roots is right (Read from p. 109 “We shall now proceed” to p. 113 inclusive). When we use \( \sqrt{-1} \), which we must do at present, if at all, without full explanation, it is to be remembered that we say two expressions are equal when they are algebraically the same, that is, when each side has all the algebraical properties of the other. [‘M’ crossed out] Numerical accordance must not be looked for when one or both sides are numerically unintelligible, and algebraical accordance merely means that everything which is true of one side is true of the other. It is then unnecessary to consider any restrictions which may be necessary when numerical accordance is that which is denoted by \( = \).

But this is touching on even a higher algebra than the one before you

Suppose

\[
\frac{1}{1-x^2} = 1 + x + x^2 + x^3 + \cdots \text{ ad inf.}
\]

which is certainly true in the arithmetical sense when [19v] \( x < 1 \). But if \( x > 1 \), say \( x = 2 \), we have

\[
\frac{1}{1-x^2} \text{ or } -1 = 1 + 2 + 4 + 8 + 16 + \&c
\]
which, arithmetically considered is absurd. But nevertheless

−1 and 1 + 2 + 4 + 8+ &c have the same properties

This point is treated in the chapter on the meaning

of the sign =.

My wife desires to be kindly remembered

I remain

Yours very truly

[Signature]

69 Gower St.

Thursday Ev§ Oct† 15/40

It is fair to tell you that the use of divergent

series is condemned altogether by some modern names of

very great note. For myself I am fully satisfied

that they have an algebraical truth wholly independent

of arithmetical considerations; but I am also satisfied

that this is the most difficult question in mathematics.
[20r] My dear Lady Lovelace

With regard to the error in Peacock you will see that you have omitted a sign. It is very common to suppose that if \( \varphi x \) differentiated gives \( \psi x \), then \( \varphi(-x) \) gives \( \psi(-x) \), but this should be \( \psi(-x) \times \text{diff.co.}(x) \) or \( \psi(-x) \times -1 \). Thus

\[
y = e^x \quad \frac{du}{dx} = e^x
\]
\[
y = e^{-x} \quad \frac{du}{dx} = e^{-x} \times (-1) = -e^{-x}
\]

As to the note, my copy of Peacock wants a few pages at the beginning by reason of certain thumbing of my own and others in 1825. I remember however that there is a note which I did not attend to, nor need you. But if curiosity prompts, pray sent it to me in writing.

As to \( du = \varphi(x).dx \), you should not have written it \( du = \varphi(x) \) as you proposed but

\[
\frac{du}{dx} = \varphi(x)
\]

The differential coeff. is the limit of \( \frac{\Delta u}{\Delta x} \), and is a Total symbol. Those whose [sic] write

\[
y = x^2 \quad \therefore dy = 2xdx \text{ make an error}
\]

but if \( dy = 2xdx + \alpha \) be the truth, \( \alpha \) diminishes without limit as compared with \( dx \), when \( dx \) diminishes. Consequently \( \alpha \) is of no use in finding [20v] any limit, and those who use differentials, as they are called, do not differ at the end of their process from those who make limiting ratios as they go along. You can however for the present transform Peacock’s formula \( du = Adx \) into \( \frac{du}{dx} = A \).

There is the erratum you mention in Alg. p. 225

As to p. 226

\[
\frac{1+b}{1-b} = \frac{1+x}{x}
\]

\[
(1+b)x = (1-b)(1+x)
\]

\[
x + bx = 1 + x - b - bx
\]

\[
bx = 1 - b - bx
\]

\[
2bx + b = 1
\]

\[
(2x + 1)b = 1 \quad b = \frac{1}{2x+1}
\]

Verification

\[
\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{2x+1+1}{2x+1-1} = \frac{2x+2}{2x}
\]

\[
= \frac{2(x+1)}{2x} = \frac{x+1}{x}
\]

[21r] p. 212. To shew that for instance

\[
n \frac{n-1}{2} \frac{n-2}{3} \frac{n-3}{4} \frac{n-4}{5} + m \frac{n-1}{2} \frac{n-2}{3} \frac{n-3}{4} \frac{n-4}{5} + m \frac{n-1}{2} \frac{n-2}{3} \frac{n-3}{4} \frac{n-4}{5}
\]

\[
+ m \frac{n-1}{2} \frac{n-2}{3} \frac{n-3}{4} \frac{n-4}{5}
\]
\[
= \frac{m+n}{2} \frac{m+n-1}{3} \frac{m+n-2}{4} \frac{m+n-3}{5} \frac{m+n-4}{6}
\]
without actual multiplication
\[m \text{ and } n \text{ being whole numbers}\]

1. \( \frac{m-1}{2} \frac{m-2}{3} \cdots \frac{m-(r-1)}{r} \) is the number of ways in which \( r \)
can be taken out of \( m \) (see chapter on combinations in the
Arithmetic)

If then we denote by \((a, b)\) the number of ways in
which \( a \) can be taken out of \( b \), we have to prove that
\[
(5, n) + (1, m) \times (4, n) + (2, m) \times (3, n) + (3, m) \times (2, n)

(4, m) \times (1, n) + (5, m) = (5, m + n)
\]
Suppose \([\text{the} \text{' crossed out'}]\) \( m + n \) counters to be divided into two parcels,
one containing \( m \) and the other \( n \) counters
He who would take 5 out of them must either
take

0 out of the \( m \) and 5 out of the \( n \)
or 1 \( m \cdots 4 \cdots n \)
or 2 \( m \cdots 3 \cdots n \)
3 \( m \cdots 2 \cdots n \)
4 \( m \cdots 1 \cdots n \)
5 \( m \cdots 0 \cdots n \)

Now if to take say the third of these cases, we
can take two of \( m \) in \( a \) ways and 3 out of \( n \) in \( b \) ways
\[21v\] we can do both together in \( a \times b \) ways. For if
for instance there are 12 things in one lot and 7 in another,
we can take one out of each lot in 12 \( \times \) 7 ways, since any
one of the twelve may come out with any one of the seven
Hence the number of distinct \([\text{\text{distinct} inserted}]\) ways of bringing 2 out of \( m \) and
3 out of \( n \) together is
\[
(2, m) \times (3, n)
\]
I think you will now be able to make out that the
preceding theorem is true when \( m \) and \( n \) are whole,
whence, by the reasoning in the book it must be
ture when they are fractional.

This reasoning you do not see. It is an appeal to the
nature of the method by which algebraical operations
are performed. There is no difference of operation in
the fundamental rules (addition subtr & mult & div)
whether the symbols be whole nos or fractions. Hence
if a theorem be true when the letters are any wh. nos, it
remains true when they are fractions
For example, suppose it proved that for all whole
nos
\[
(a + b) \times (a + b) = a \times a + 2a \times b + b \times b
\]
we should then, if we performed the operation \((a + b) \times (a + b)\) remembering that \(a\) and \(b\) are whole numbers find \(a \times a + \&c\)

\[
\begin{align*}
\quad & a + b \\
\quad & a + b \\
\quad & a + ab \\
\quad & + ab + bb \\
\quad & aa + 2ab + bb
\end{align*}
\]

[22r] Now in no part of this operation are you required to stop and do ['or omit' inserted] anything because the letters are whole numbers which you would not do or not omit if they were fractions. Consequently, the reservation that the letters are whole numbers cannot affect the result which if true with it is true without. This principle requires some algebraical practice to see the necessity of its truth.

The notation of functions is very abstract. Can you put your finger upon the part of Chapt. X at which there is difficulty

The equation

\[ \varphi(x) \times \varphi(y) = \varphi(x + y) \]

is supposed to be universally true for all values of \(x\) and \(y\). You have hitherto had to deal with equations in which value was the thing sought: now it is not value, but form. Perhaps you are thinking of the latter when it ought to be of the former.

With our remembrances to L. Lovelace I am

Yours very truly

ADeMorgan

69 Gower St. No^r 14/40
My dear Lady Lovelace

I send back your worked question. The second is right the first wrong in two places. I should recommend you to get out of the habit of writing \( d \) thus ['\( d' \) with flourish at top of stem] or thus ['\( d' \) with flourish and double stem]. If you have much to do with the Diff. Calculus, it will make a good deal of difference in time. The best way is to form all the letters as like those in the book as you can.

Pages 13–15 of Elementary Illustrations bear closely on the distinction of 
\[
\frac{du}{dx} = a \quad \text{and} \quad du = adx
\]

It is not because \( 1 + x^m \times 1 + x^n = 1 + x^{m+n} \) when \( m \) & \( n \) are whole numbers that the same is true for fractions but because a certain other property which is therefore true makes it necessary in the case of fractions.

For instance, the logic is as follows

\( A \) is true when \( m \) is a whole number

Whenever \( A \) is true, \( B \) is true

\( B \) is of that nature, that if true when \( m \) is a whole number, it is also true when \( m \) is a fraction.

When \( B \) is true \( C \) is true

\( \therefore C \) is true if \( A \) be true when \( m \) is a whole number

Thus, when \( m \) and \( n \) are whole numbers,
\[
\frac{1}{1 + x^m} = 1 + mx + m \frac{m-1}{2} x^2 + \cdots
\]
\[
\frac{1}{1 + x^n} = 1 + nx + \cdots
\]

But
\[
\frac{1}{1 + x^m} \times \frac{1}{1 + x^n} = 1 + x^{m+n}
\]

always

Therefore when \( m \) and \( n \) are whole numbers

\( (1 + mx + \cdots) \times (1 + nx + \cdots) = 1 + m + n + \cdots \)

but this last property (without any reference to its mode of derivation) is true of \( m \) and \( n \) fractional or negative if true of whole numbers.

Hence, if \( 1 + mx + \cdots \) be called \( \varphi m \)

\( \varphi m \times \varphi n = \varphi (m + n) \)

But this (again by an independent process) is shown to be never universally true unless

\( \varphi m = c^m, \ c \) being independent of \( m \)

whence \( c^m = 1 + mx + m \frac{m-1}{2} x^2 + \cdots \)

But since \( c \) is independent of \( m \), what it is when \( m = 1 \), it is always. Therefore

\( c^1 = 1 + 1 \cdot x + 1 \cdot \frac{1-1}{2} x^2 + \cdots \)

or \( c = 1 + x \)
Let $\varphi(xy) = x \times \varphi y$ be always true required $\varphi x$

Make $y = 1$, then $\varphi(x) = x \times \varphi(1)$

$\varphi(1)$ is not yet determined; let it be $c$. The equation is either true for all values of $c$, or for some (not all)

$\varphi x = cx$

If any particular value were needed for $c$, it would be found by making $c = 1$. Do this and we have $\varphi(1) = c$, or $c = c$, which is true for all values of $c$.

Suppose $\varphi(x) = \varphi(1).x + 2\varphi(1) - \varphi(1)^2$ to be true for all values of $x$. It is then true when $x = 1$, giving

$\varphi(1) = \varphi(1) + 2\varphi(1) - \varphi(1)^2$

or $\varphi(1)^2 - 2\varphi(1) = 0$

or $\varphi(1) = 2$ only

$\varphi(x) = 2x + 2.2 - 2^2 = 2x$

$\varphi(1) = 2$

which agree with each other.

You must remember than [sic] when a form is universally true, it is true in all particular cases. Then in $2x = x + x$, I have a perfect right to say this is true when $x = 1$ and $\therefore 2 = 1 + 1$, and true when $x = 20$, or $40 = 20 + 20$: though I do [do' crossed out] not thereby say that $x$ can be 10 and 1 both at once.

$\frac{1}{m} \log z = \left(z^{\frac{1}{m}} - 1\right) - \frac{1}{2} \left(z^{\frac{1}{m}} - 1\right)^2 + \cdots$

when $m$ increases $z^{\frac{1}{m}} - 1$ diminishes

It is not $\log z$ which $z^{\frac{1}{m}} - 1$ approaches to, but $\frac{1}{m} \log z$, which also diminishes as $m$ increases

I see that in page 219 it is thus

$log z = \frac{1}{m} \left\{z^m - 1 - \frac{1}{2} z^m - 1^m + \cdots\right\}$

and $m$ diminishes without limit. Now as $m$ diminishes [25v] $\frac{1}{m}$ increases, and the assertion is that as $m$ diminishes, and therefore $z^m - 1$, the
product \( \frac{1}{m} (z^m - 1) \)
has one factor continually increasing & the other
diminishing, so that their product approaches
without limit to \( \log z \).

In page 187, it is shown that if \( x \)
be made sufficiently small, any term of
\( a + bx + cx^2 + \cdots \) may be made to contain all the
rest as often as we please, that is may be made
as great as we please compared with the sum
of all the rest. Consequently \( m \) being small
\( z^m - 1 \) may be made as great as we please
compared with the sum of all the terms of the
series which follow, even if all were positive,
still more when they counterbalance each other
by difference of sign.

p. 205

If \( \varphi(x + y) = \varphi x + \varphi y \) be always true (hypothesis)
It is true when \( x = 0 \)
It is also true when \( y = -x \)
This equation being always true, is the representation
of a collection of an infinite number of truths
I do not say that these truths coexist
Put it thus. Let \( \varphi \) be such a function
that, if \( a, b, c, d, \&c \) be any quantities whatever
\( \varphi(a + b) = \varphi a \times \varphi b \)
\( \varphi(b + c) = \varphi b \times \varphi c \)
\( \varphi(d + e) = \varphi d \times \varphi e \) \&c \&c
That is let \( \varphi(x + y) = \varphi(x) \times \varphi(y) \)
for all values of \( x \) and \( y \)
1. Let \( a = 0 \), then \( \varphi b = \varphi(0) \times \varphi(b) \)
   or \( \varphi(0) = 1 \)
Let \( b = -c \), then \( \varphi(0) = \varphi(b) \times \varphi(c) \)
   or \( 1 = \varphi(b) \times \varphi(-b) \)
and so on.
There is a want of distinction between
an equation made true by choice of values
and one which is true of itself, independently of
all values
\( x = 3 - x \quad x = \frac{3}{2} \), and then only
\( x = 3x + a - (2x + a) \) is true for all values of \( x \),
though it cannot have more than one at a time.
There is the erratum in the Trigonometry, as you say

Yours very truly

ADERmoran

69 Gower St.
Friday Ev
My dear Lady Lovelace

I can soon put you out of your misery about p. 206.
You have shown correctly that \( \varphi(x + y) = \varphi(x) + \varphi(y) \)
can have no other solution than \( \varphi x = ax \), but the preceding question is not of the same kind; it is not show that there can be no other solution except \( \frac{1}{2} (a^x + a^{-x}) \) but show that \( \frac{1}{2} (a^x + a^{-x}) \) is a solution: that is, try this solution
\[
\varphi(x + y) = \frac{1}{2} (a^{x+y} + a^{-x-y})
\]
\[
\varphi(x - y) = \frac{1}{2} (a^{x-y} + a^{-x+y})
\]
\[
\varphi(x + y) + \varphi(x - y) = \frac{1}{2} (a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y})
\]
\[
2\varphi x.\varphi y = 2 \cdot \frac{1}{2} (a^x + a^{-x}) \cdot \frac{1}{2} (a^y + a^{-y}) = \frac{1}{2} (a^x + a^{-x})(a^y + a^{-y}) = \frac{1}{2} (a^{x+y} + a^{-x+y} + a^{x-y} + a^{-x-y})
\]
the same as before.

To prove that this can be the only solution would be above you
I think you have got all you were meant to get from the chapter on functions.
The functional equations which can be fully solved are few in number
Yours very truly

ADeMorgan

69 G.S.
Mon's M's
My dear Lady Lovelace,

I have made some additional notes on your papers.

[diagram in original] The meaning of $\frac{\theta}{\sin \theta}$ is as follows:

$\theta : 1$ and $1 : \sin \theta$ compounded give it in arithmetic

In fact $\frac{\theta}{\sin \theta}$ in arithmetic is another way of writing $a : b$.

In geometry $AB : AO$ is $\theta : 1$ and $AO$ or $OB : BM$ is $\sin \theta$

The compounded ratio is that of $AB : BM$ which approaches without limit to the ratio of $1$ to $1$ as $AB$ is diminished.

Your notion of the ratio approximating to unity is correct. The term ‘ratio approximating to $a’$ is a mixture of the geometrical and arithmetical mode of speaking, it should be ‘ratio approximating to $a : 1$.

I think you have got over the difficulty of that part of the subject.

I was sorry to have been out when Lord Lovelace called, and could not get down to St James’ Square till you had gone. With best remembrances I am

Yours very truly

ADeMorgan
My dear Lady Lovelace

You are right about
the writing down of the
terms:

\[
\frac{z}{(2n-2)(2n-3)}
\]
is the \(n\)th term divided
by the \((n - 1)\)th and the
\(n + 1\)th divided by the
\(n\)th is \(\frac{z}{2n(2n-1)}\) as you
make it.

If I understand you correctly
you are now satisfied about
all the rest

Suppose you try at what
term convergency begins in the
following series

\[
1 + \frac{x}{2.3} + \frac{x^2}{2.4.6.8} + \frac{x^3}{2.4.6.8.10.12} + \cdots
\]
when \(x = 100,000\)

With remembrances to
Lord Lovelace

I remain

Yours truly

ADEMorgan

69 G.S.
Thursday

You will see the alteration
I have made in your paper
If you do not see it
clearly, write again for
the sort of point con-
tained in it is one
of importance.
My dear Lady Lovelace

I return the papers about series which are all right, the old one is as you suppose

With reference to your remarks on the differential calculus

1. You observe that

\[
\frac{\varphi(x+n\theta)+\varphi(n\theta)}{\theta} - \frac{\varphi(x+n\theta)}{\theta} \]

differs from \( \frac{\varphi(x+\theta)-\varphi(x)}{\theta} \)

in that while \( \theta \) diminishes, \( x + n\theta \) varies. So it is, and if \( n \) be finite and fixed, it might be shown that the limits of the two are the same. But if \( n \) increase while \( \theta \) diminishes, in such manner that \( n\theta \) is either equal to or approaches the limit \( a \), then the first fraction has the same limit as \( \frac{\varphi(x+a+\theta)-\varphi(x+a)}{\theta} \)

To illustrate this, let \( \varphi x \) be the ordinate of a curve, the abscissa being \( x \). If \( x \) remains fixed, the triangle [diagram in original] (blotted) diminishes without limit with \( \theta \); but if while \( \theta \) diminishes, the point \( A \) moves in toward \( B \), so as continually to approach \( B \), and to come as near as [33v] we please to it, and yet never absolutely to reach \( B \) as long as \( \theta \) has any value, it is obvious that the small triangle would ride along the curve, perpetually diminishing its dimensions, and continually approaching in figure nearer and nearer to the figure of as small a triangle at \( B \).

All this necessarily follows from the notion of continuity [diagram in original]

2. You want to extend what I have said about continuous functions to all possible cases, not being able to imagine a function which changes its values suddenly. But for this you must wait till you come to the mathematics of discontinuous quantity. It is perfectly possible though the calculation would be laborious, to find an algebraical function which from \( x = 1 \) to \( x = 2 \) increases like the ordinate of a straight line, from \( x = 2 \) to \( x = 3 \) draws the [diagram in original] likeness of a human profile in a different place, from \( x = 3 \) to \( x = 4 \) draws a part of a circle, from \( x = 4 \) to \( x = 5 \) is nothing, and from
[34r] $x = 5$ to $x = 6$ makes any odd combination of lines or curves, perfectly irregular. None of the notions incidental to continuity must be applied to such a function

3. Your proof of the diff.co. of $x^n$ is correct, but it assumes the binomial theorem. Now I endeavor to establish the diff.calc. without any assumption of an infinite series, in order that the theory of series may be established upon the differential calculus.

Besides, if you take the common proof of the binomial theorem, you are reasoning in a circle, for that proof requires that it should be shown that $\frac{v^n - w^n}{v - w}$ has the limit $nv^{n-1}$ as $w$ approaches $v$. This is precisely the proposition which you have deduced from the binomial theorem.

Pray send your point about the exponential theorem.

And thank Lord Lovelace for pheasants and hare duly received this morning

Yours very truly

ADeMorgan

69 G.S. Wed
My dear Lady Lovelace

We shall be happy to see you on Monday Evening, and Lord Lovelace too if he be not afraid of the algebra. Your points in your letter are [should be ‘1’] think, clear enough in your own head. A little addition however may be made as follows.

You are not [something crossed out] to think that because \( x \) must be diminished without limit to prove a conclusion that conclusion is only true for small values of \( x \), or for \( x = 0 \).

For example suppose I know that 
\[(a + x)(a - x) = P + Qx + Rx^2\]
but of \( P \) \( Q \) and \( R \) I only know that they are independent of \( x \). What therefore they are for any one value of \( x \), they are for any other. I find them thus Since the preceding is by hypothesis true for all values of \( x \), and since altering \( x \) does not alter \( P \) \( Q \) or \( R \), I take \( x = 0 \) to begin with
\[a^2 = P \text{ when } x = 0\]
but \( a \) and \( P \) are independent of \( x \), therefore what relation exists when \( x = 0 \) exists always or 
\[a^2 = P\]
Let \( x = a \)
\[0 = P + Qa + Ra^2\]
Let \( x = -a \)
\[0 = P - Qa + Ra^2\]
[something crossed out] subtract \( 2Qa = 0 \) or \( Q = 0 \)

Here are two values of \( x \) made use [36r] of.
Add
\[2P + 2Ra^2 = 0\]
\[R = -\frac{P}{a^2} = -1\]
whence
\[(a + x)(a - x) = a^2 + 0.x - x^2\]
\[= a^2 - x^2\]

if it must be of the form
\[P + Qx + Rx^2\]

We are much obliged by your invitation to Ockham, but I am closely tied up by lectures & other things. Even at such times as Xmas I am generally very busy.
With kind remembrances to Lord Lovelace I am
Yours very truly

ADeMorgan
[37r] My dear Lady Lovelace

If you look back to page 48, you will there see that
\[ \frac{a + a' + a'' + \ldots}{b + b' + b'' + \ldots} \]
always lies between the greatest & least of \( \frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}, \ldots \)
whatever the signs of \( a, a', a'', \ldots \) may be, provided that \( b, b', b'', \ldots \)
&c are all of one sign. That is the reason why \( \varphi x \) need not continually increase or decrease in the next chapter.

The paper you have sent me is correct. In page 70, the reasons are given for avoiding the common proof of Taylor’s Theorem, and 71 &c contains the amended proof.

Of \( \frac{\varphi(a + h)}{\varphi(a + h)} = \frac{\varphi'(a + \theta h)}{\varphi'(a + \theta h)} \) [bracket missing in last denominator] it cannot only be said that it turns out useful. A beginner can hardly see why a diff \( \text{diff} \) coeff itself should be of any use.

Yours truly

ADeMorgan

Feb\(^6\) 6/41
My dear Lady Lovelace

I have added a note or two to your papers.

As to the subject of continuity, it must be as much as possible your object now to remember while proving the things which are true of continuity to remember that they are not false of [‘conti’ crossed out?] dis continuous [sic] functions, because true of continuous ones. Thus, you will afterwards see that

\[ \varphi(a + h) = \varphi a + \varphi'(a + \theta h).h \]

is only an algebraical translation of the following geometrical theorem

“Every continuous and ordinary arc of a curve has somewhere a tangent parallel to its chord”

But this is not always false of discontinuous curves. Neither is the algebraical theorem false of them.

The best way at present, is to mark that discontinuous functions are now excluded only because we have no language to express them in. This will come intime [sic]: you will have enough of them when you come to apply math\textsuperscript{cs} to the theory of heat.

My wife has duly received your letter & is much obliged to you & Miss King.

Yours truly

ADeMorgan

69 G.S.
Feb\textsuperscript{y} 11/41 [?]
My dear Lady Lovelace

There is a misprint in page 83, I find, line 6 from the bottom. For $x + w$ read $x + 1$. Your result is correct for an increment [sic] $w$.

My last letter was all ready when yours arrived and I had no time to make any addition. My wife begs me to give her kind regards and to say that she knows her mother will wish her to be with her on Sunday Evening (the day after Mr. Frend’s funeral) so that she will be glad if you can come any other evening in the week.

With kind remembrances to Lord Lovelace

I remain

Yours very truly

ADEMorgan

69 Gower St.

Wed Evs

[41r] [blank]

[42v] [Small note — either where letter was addressed after folding or simply a librarian’s label]

A DeMorgan to [?] Lady L
My dear Lady Lovelace

Mr. Frend’s death (which took place on Sunday Morning) has made me answer your letter later than I should otherwise have done. The family are all well, and have looked forward to this termination for some time. My wife will answer your letter on a part of this.

Number and Magn. pp. 75, 76. The use of this theorem is shown in what follows. It proves that any quantity which lies between two others is either one of a set of mean proportionals between those two, or as near to one as we please.

It is not self evident that the base of Napier’s system, as given by himself is $e$ or $1 + \frac{1}{2} + \cdot \cdot \cdot$ as we learn from the modern mode of presenting the theory. The last sentence in the book (making $V$ a linear unit) would show that Napier’s notion was to take $k$ in such a manner that $x$ shall expound $1 + x$ ['without’ crossed out] or rather that the smaller $x$ is the more nearly shall $x$ expound $1 + x$. If this were accurately done, we should have

$$k^x = 1 + x \quad \text{or} \quad \frac{k^x - 1}{x} = 1$$

and this is to be nearer to the truth the smaller $x$ is. Now when the common theory is known, it is known that $k = e$ gives

$$\frac{e^x - 1}{x} = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \cdot \cdot \cdot \quad \text{and limit of } \frac{e^x - 1}{x} = 1$$

while

$$\frac{e^{x} - 1}{x} = \log k + \frac{\log k}{2} x + \cdot \cdot \cdot \quad \text{and limit of } \frac{e^{x} - 1}{x} = \log k$$

where the log. has this very base $e$. Having proved these things, it is then obvious that, $\log k$ being never 1 except when $k$ is the base or $e$, the last paragraph cannot consist with any other value of $k$ except $e$. In this book (Num & Mag.) I must refer you to the algebra, which I do not in the Diff. Calc., many matters of series, until the whole doctrine is reestablished.

Now ['as’ crossed out] as to the Diff. Calc. You do not see that $\theta$ is a function of $a$ and $h$. Let us take the simplest case of the original theorem which is

$$\varphi(a + h) = \varphi a + h \varphi'(a + \theta h) \quad (1)$$

Now 1. Why should $\theta$ be independent of $a$ and $h$, we have never proved it to be so : all we have proved is that one of the numerical values of $\theta$ is < 1, or that this equation (1) can be satisfied by a value of $\theta$ < 1. As to what $\theta$ is, let $\psi$ be the inverse function of $\varphi'$ so that $\psi \varphi' x = x$. Then

$$\varphi(a + h) - \varphi a \quad \frac{h}{h} = \varphi'(a + \theta h)$$

$$\psi \left( \frac{\varphi(a + h) - \varphi a}{h} \right) = \psi \varphi'(a + \theta h) = a + \theta h$$

$$\theta = \frac{\psi \left( \frac{\varphi(a + h) - \varphi a}{h} \right) - a}{h} \quad \left\{ \text{Say that this is not a function} \right\}$$

of $a$ and $h$, if you dare

For example

$$\varphi x = c^x$$

$$\varphi' x = c^x \log c$$
\[ c^{a+h} = c^a + h \log c \cdot c^{a+\theta h} \]
\[ c^{a+\theta h} = \frac{c^{a+h} - c^a}{h \log c} \]
\[ (a + \theta h) \log c = \log \frac{c^{a+h} - c^a}{h \log c} \]
\[ \theta = \frac{\log \frac{c^{a+h} - c^a}{h \log c} - a \log c}{h \log c} \]
\[ = \frac{\log (c^h - 1) - \log (h \log c)}{h \log c} \]

In this particular example \( \theta \) happens to be a function of \( h \) only, not of \( a \) : but you must remember that in every case where we speak of a quantity as being generally a function of \( a \), we do not mean thereby to deny that it may be in particular case, not a function of \( a \) at all : just as when we say that there is a number \( (x) \) which satisfies certain conditions, we do not thereby exclude the extreme case in which \( x = 0 \).

Look at the question of differences in this manner. Any thing which has been proved to be true of \( u_n \) relatively to \( u_{n-1} \) \( u_{n-2} \) &c has also been proved to be true of \( \Delta u_n \) relatively to \( \Delta u_{n-1} \) \( \Delta u_{n-2} \) &c. For in the set
\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\Delta u_0 \\
\Delta^2 u_0 \\
\Delta u_1 \\
\Delta^2 u_1 \\
\Delta u_2 \\
\end{array}
\begin{array}{c}
&c
\end{array}
\]
the first column may be rubbed out and the second column becomes the first &c. It is obvious that the \( m + 1 \), \( m + 2 \), &c columns are formed from the \( m \)th precisely as the 2nd, 3rd &c are formed from the first. If then I show that up to \( n = 7 \), for instance
\[ [u \ldots \text{crossed out}] u_n = u_0 + n\Delta u_0 + \cdots \]
I also show (writing \( \Delta u_n \) for \( u_n \)) that \( \Delta u_n = \Delta u_0 + n\Delta (\Delta u_0) + \cdots \)
Perhaps you had better let the question of discontinuity rest for the present, and take the result as proved for continuous functions. You will presently see in a more natural manner the entrance of discontinuity.

The paper which I return is correct
Yours very truly

ADeMorgan

69 Gower St
Mondy Evys
[45] [in ADM’s hand]
\[ \int_{a'} f \, ds \]
\[ \int_{a'} \varphi s. \, ds \quad \varphi s \text{ meaning } f \]
\[ \int f \, ds \]
\[ \int \frac{dv}{dt} \, ds \]
\[ \int \frac{dv}{dt} \, ds \]

Negative & Impossible Qu. \[ \int dv. \frac{ds}{dt} \]
Operation \[ \int dv. v \]
Relation \[ \int v \, dv \] [diagram in original]

\[ v^2 = 2 \int_{a'} f \, ds + C \]
\[ V^2 = 0 + C \]
\[ v^2 - V^2 = 2 \int_{a'} f \, ds \]
\[ v^2 = 2 \int_{a'} f \, ds + V^2 \]
\[ v^2 = 2 \int f \, ds + C \]

\[ dy/dx \]
Ockham
Monday. 4th Sepr

Dear Mr De Morgan

Will you send on the enclosed to M'rs De Morgan. It explains the arrangements I have made in case she comes here, & also that Lord L_ & myself have delayed our own departure until Satdy next. Our household & children however are already gone.

Now to mathematical business:
I think you have hit the right nail on the head, altho' my confused notion of Differentials was not the only piece of puzzle & mistiness which constituted the impediments towards my comprehension of $X \frac{du}{dx} + Y \frac{du}{dy} = U$. The rectifi= =cation of this however, has I believe given me the key to the remaining difficulties. If you will be kind enough to read the ['following' crossed out] enclosed observations, you will be able to judge how far I now take a just view of the matter.

I find that when (a long time ago) I studied Chapter V, I never gave due importance to the conclusion

$$\Delta u = \frac{du}{dx_1} \Delta x_1 + \frac{du}{dx_2} \Delta x_2 + \frac{du}{dx_3} \Delta x_3 + \&c$$

$$+ \{ (\Delta x)^2, (\Delta x_1 \Delta x_2), \&c, \&c \}$$

deducted at the bottom of page 87, as the result of pages 86, 87. My whole attention was given to the subsequent theorem of page 90, "If $u$ be a function of $t$ in different
“ways &c, &c,” which I conceived to be the only object in view, & that the equation of page 87 was of no consequence in itself, but merely means to an end. This seems to have been an egregious blunder, since the whole theory of Differentials rests on the very part which I [‘comparatively’ inserted] neglected, from [‘fancying it’ inserted] a merely subsidiary theory. I believe I am not wrong in this present view of the matter.

I cannot help here remarking a circumstance which I [‘believe’ crossed out] think is almost invariably true respecting all my difficulties & confusions in studying. They are without any [47v] exception that I can recal [sic], from misapprehension of the meaning of some symbol, or [‘of’ inserted] some phrase or definition; & on no occasion from either any error in my reasoning, or [‘from’ inserted] any difficulty in carrying on [‘any’ crossed out] chains of deductions correctly, however complicated or profound or lengthy [‘these may be’ inserted]. I therefore have lately begun to ask myself, whenever I am stopped, whether I clearly understand what the subjects of the reasoning are; & to go carefully over every verbal [something crossed out] & symbolic representative of a thing or an idea, with the question respecting each, “now what “does it mean, & how was it got? “Am I sure of this, in each instance “involved in the subject?” This may save me much future trouble.

I will send you my remarks tomorrow, [vertical text on 46r] as I want to look over them once more.
first; & today I have had enough of these subjects.

Yours very truly

A.A.L
Dear Mr. De Morgan,

I am very much obliged by your remarks & additions. I believe I now understand as much of the points in question as I am intended to understand at present. I am much inclined to agree with the paragraph in page 48; for though the conclusions must be admitted to be most perfectly correct & indisputable, logically speaking, yet there is a something intangible & a little unsatisfactory too, about the proposition.

I expect to gain a good deal of new light, & to get a good lift, in studying from page 52 to 58; though probably I shall be a long time about this. I could wish I went on quicker. That is I wish a human head, or my head at all events, could take in a great deal more & a great deal more rapidly than is the case; and if I had made my own head, I would have proportioned it’s [sic] wishes & ambition a little more to it’s [sic] capacity. When I sit down to study, I generally feel as if I could never be tired; as if I could
go on for ever. I say
to myself constantly, “Now today
I will get through so & so”;
and it is very disappointing
to find oneself after an
hour or two quite wearied,
& having accomplished perhaps
[49v] about one twentieth part of
one’s intentions. perhaps not
that. When I compare
the very little I do, with the
very much the infinite I
may say that there is to
be done; I can only
hope that hereafter in some
future state, we shall be
cleverer than we are now.

I am
afraid I do not understand
what you were kind enough
to write about the Curve;
and I think for this reason,
that I do not know what
[164r] the term equation to a curve
means. Probably with some
study, I should deduce that
meaning myself; but having
plenty else to attend to of
more immediate consequence,
I do not like to give my
time to a mere digression
of this sort. I should
much like at some future
period, (when I have got
rid of the common Algebra
& Trigonometry which at
present detain me), to
attend particularly to this
subject. At present, you
[164v] will observe I have four
distinct things to [something crossed out]
carry on at the same
time; _the Algebra; _
Trigonometry; _Chapter 2nd of the
Differential Calculus; _& the
mere practice in Differentiation.

This last reminds me
that my bookseller has at
last & with much difficulty
got me Peacock’s Book; &
I hope it will be of
great use, for it’s [sic] cost is
£2..12..6! _It was
originally 30p. _It is
[163r] coming here next week.

By the bye I have a
question to ask upon pages
203 & 204 of the Algebra.
In consequence of a reference
to page 203, in the 9th line
of the 25th page of the
Trigonometry, I was induced
to look & see what it
related to. Reading on
afterwards to the bottom of
the page, I found
“A functional equation is an
“equation which is necessarily
“true of a function or functions
“for every value of the letter
“which it contains. Thus if,
[163v] “ϕx = ax, we have ϕ(bx) =
“abx = b × ϕx, or
“ϕ(bx) = bϕx”
“is always true when ϕx
“means ax.” _
So far I think is clear; 
but then what follows, _
“Thus &c
“If ϕx = xα ϕα × ϕy = ϕ(αy)
“ ϕx = ax ... ϕx × ϕy = ϕ(x + y)
“ ϕx = ax + b ... (ϕx−ϕy)/(ϕx−ϕz) = (x−y)/(x−z)
“ ϕx = ax ϕx + ϕy = ϕ(x + y)
I cannot trace the connection. I suppose there is something I have not understood, in the explanation of the Functional Equation. I hope before very long to have something further to send you upon Chapter 2nd of the Calculus, either of success or of enquiry.

Has Mrs De Morgan returned yet, & how is Mr Frend?

With many thanks,
Yours very truly
A. A. Lovelace
Dear Mr. De Morgan. I hope you have not by this time come to the conclusion that I have drowned the Differential Calculus at least (if not myself with it also) in the Seine or the Channel.

I am setting about work again now in good earnest. To say the truth I find myself a little bit rusty & awkward. I have been some days getting my head in again, & it is not yet even at all what it was. However this will soon mend, I doubt not. I very easily lose a habit, but then I very easily re-form it. But I am a little vexed at this interruption. I was going on so nicely. I now write because I am in need of a little help from you. Some points from page 100 to page 103 are not perfectly clear to me; & as my head is not at this moment in it's [sic] best working habits, I think instead of plaguing myself more about the misty parts, I had better apply to you; for I
am inclined to get a little worried about it, which is always to be avoided.

I should prefer seeing you to writing, on this occasion. I have at present no evening; for tomorrow (Sun^59) I happen to have some engagements at home.

But I would suggest, (if it is not trespassing too much on your time & kindness) that I will send my carriage to bring you here on Tuesday Morning at ½ past [52r] eleven o’clock. I propose sending for you thus, in order to facilitate your coming. Or would you prefer instead that I should go to Gower St at that hour on Tues^59?

Will you tell Mrs De Morgan, with my kindest remembrances, that (if agreeable to her) I rather hope to be able to spend Sunday Evening (I mean tomorrow week) with her & you. But my arrangements [‘for that time’ inserted] are not yet quite fixed; so I am not sure yet what I shall be able to do.

Pray believe me

Yours very truly

A. A. Lovelace
S' James' Square
Friday Morning

Dear Mr De Morgan. I send you a large packet of papers:
1: Some Remarks & Queries on the subjects of a portion of pages 75 & 76 (Differential Calculus)
2: an Abstract of the demonstration of the Method of finding the $n^{th}$ Differential Co-efficient by means of the Formula
$$\lim_{(\Delta x)^n} = u^{(n)}$$
3: Some objections & enquiries on the subjects of pages 83, 84, 85.
4: Two enquiries on two Formulae in page[s’ crossed out] 35 of the “Elementary Illustrations”.

In addition to all this, I have a word to say on two points in your last letter.
Firstly: that $\theta$ is a function of $a$ & $h$, (or in all cases a function of one at any rate of these quantities), is very clearly shown by you in reply to my question. But I still do not see exactly the use & aim of this fact being so particularly pointed out in the parenthesis at the top of page 80. It does not appear to me that the subsequent argument is at all affected by it.
Secondly: I still am not satisfied about the Logarithms, I mean about the peculiarity
which constitutes a Naperian Logarithm in what I call the Geometrical Method, the method in your Number & Magnitude. I am [‘now’ inserted] satisfied of the following:

that there is nothing in the Geometrical Method to lead to the precise determination of \( \varepsilon \); that \( \varepsilon \) is arrived at by other means, Algebraical means; & then identified with the \( k \) on \( HL \) of the Geometrical Method. What constitutes a Naperian Logarithm in the Geometrical view, is “taking \( k \) so that \( x \) shall expound \( 1 + x \), or rather that “the smaller \( x \) is, the more nearly shall \( x \) expound \( 1 + x \).

But in this definition there are two points that are still misty to me: I do not see in what, (beyond the mere fact itself), these Logarithms differ from those [56r] in which \( x \) does not expound \( 1 + x \). I cannot perceive how this one peculiarity in them, involves any others, or imparts to them any particular use, or simplicity, not belonging to other logarithms. Also, I do not comprehend the doubt implied as to the absolute theoretical strictly-mathematical existence of a construction in which \( x \) shall expound \( 1 + x \). It appears to me that, whether practically with a
pair of good compasses, or
theoretically with a pair of
[56v] mental compasses, I can as
easily as may be take any
[diagram in original]
line I please $MQ$ greater than
$OK$ or $V$, measure their difference
$PQ$ which call $x$, then on
$OH$ ($=OK$) lay down a portion
$OM$ equal to this difference $x$
(not that I pretend this is
correctly done in my figure,
which is only roughly inked
down at the moment), &
[‘finally’ inserted] stick up $MQ$ on the point
$M$. Then $x$ expounds $MQ$ or
[57r] $V + x$, or $1 + x$. I can see
no difficulty in accomplishing
this, or any reason why these
can be only an approximation
to it. Neither do
I very
clearly perceive that the
Base $k$ would be
necessarily influenced by this
proceeding.

In short I take the real
truth to be that this view
of Exponents being wholly new
to me, there is some little
link which has escaped me,
or to which at any rate I
have not given it’s [sic] due
importance. But I think
I have now fully explained
[57v] what it is that I do not
understand.
Believe me
Yours very truly
A. A. Lovelace
[I think this is in fact a letter to Mrs. De Morgan. The reference in the letter to Mr. De Morgan is pretty clear evidence for this. As for the salutation, the ‘r’ of what appears to be ‘Mr’ curls back on itself very slightly, in a way that is suggestive of AAL’s way of writing ‘Mrs’. Compare what we find here with, for example, her ‘Mr’ in Box 170, fol. 54r, and the ‘Mrs’ in Box 171, fol. 5r. Other points to mention, although these are by no means conclusive, are that the letter is much chattier than her usual letters to ADM, and that the letter ends ‘Yours most sincerely’, which, based on a glance through the letters from Box 171, was AAL’s usual way of ending letters to Mrs. De Morgan — her standard sign-off to Mr. De Morgan seems to have been ‘Yours most truly’.

[58r]

S’t James’ Square
6 o’clock
Sunday Evening

My Dear M* De Morgan

I am very sorry
indeed at the extremely
dispiriting account you
give of yourselves; & also
I am much disappointed
at losing the pleasure
of spending this evening
with you. ___ I was
out when your note arrived,
[58v] but I hear from the
servants that it came
just after half past four
o’clock. ___
Can I, (as far as you
at present foresee), spend
next Sunday Evening
with you? I hope so,
for I believe it will
be my last [‘Sunday’ inserted] previous to
going to Paris; & I
have no week-day evening
disengaged to offer. ___

[59r] Another of the little
mathematical epochs, that
are so interesting to me,
has arrived; I mean
another Chapter of the Differential Calculus is just satisfactorily completed; & I have a whole bundle of papers to submit to M'r De Morgan upon it, & one or two questions to ask on some trifling points in it that are not perfectly clear. But [59v] as it is not of a nature that immediately presses, all this may I think wait till I can come to you. Meantime I shall begin Chapter VI, on Integration &c. I have since I saw you had such a wonderful quantity of occupations of various sorts to get thro’, that I feel a little surprised how I have managed to accomplish, even this much of the mathematics. However I do not think anything will ever manage to oust the latter. Indeed the last fortnight is rather a convincing proof that nothing can.

I have been out either to the Opera, German Opera, or somewhere or other, every night. I have had music-lessons every morning, [60v] & practised my Harp too for an hour or two; & I have been on horseback nearly every day also. I
might add many sundries & et-ceteras to this list.

I must however maintain that the Differential Calculus is king of the company; & may it ever be so!

Believe me

Yours most sincerely

A. Ada Lovelace
Dear Mr De Morgan. The last fortnight has been spent in total idleness, mathematically at least; for we have had company & been as they say gadding about. I must set too [sic] now & work up arrears. But I have a batch of questions & remarks to send.

First — on Peacock’s Examples, which I have only now begun [62v] upon:

What does he mean by adding \( dx \) to every solution? It appears to me a work of supererogation. I take the very first example in the book as an instance, and the same applies to all:

Let \( u = ax^3 + bx^2 + cx + c : \) it’s [sic] differential, or \( du = 3ax^2dx + 2bxdx + cdx \) or \( (3ax^2 + 2bx + c)dx. \)

I should have written, & in fact did write : it’s [sic] differential or \( du = 3ax^2 + 2bx + c. \)

I suppose that this form [63r] is used under the supposition that \( x \) itself may be a function.

My result & the book’s do not agree in one particular in the 9th example, page 2, & I am inclined to think it is a misprint in the latter: the Books says:

Let \( u = x^2(a + x)^3(b - x)^4 \)

\( du = (2ab - (6a - 5b)x - 9x^2)x(a + x)^2(b - x)^3dx \)
and I say:
\[ du = \{2ab - (6a - 5b)x - x^2\}x(a + x)^2(b - x)^3dx \]
In case it may save you trouble, I enclose my working out of the whole.
I do not the least understand [63v] the note in page 2. Not one of the three theorems it contains is intelligible to me.
I conclude you to have the Book by you; but if not I can copy out the note & send it to you.
Secondly — to go to your Algebra: I think there is an evident erratum page 225, line 8 from the bottom, where
\[ 1 + x + \frac{x-\frac{1}{2}}{2} + \frac{x-\frac{1}{2}}{2} \cdot \frac{x-\frac{2}{3}}{3} + &c \]
should certainly be
\[ 1 + x + x \frac{x-\frac{1}{2}}{2} + x \frac{x-\frac{1}{2}}{2} \cdot \frac{x-\frac{2}{3}}{3} + &c. \]
I have a little difficulty in page 226, the last line, [64r] “let \( \frac{1+b}{1-b} = \frac{1+x}{x} \) which gives \( b = \frac{1}{2x+1} \).”
In the first place I do not feel satisfied that the form \( \frac{1+b}{1-b} \) is capable of being changed into the form \( \frac{1+x}{x} \). There are three suppositions we may make upon it, (supposing that it is capable of this second form). \( x \) may be less than \( b \), in which case the denominator must also be less than \( 1 - b \), and less in a certain given proportion, in order that the Fractional [64v] Expression may remain the same: \( x \) may = \( b \), in which case the second form can only be true on the
supposition that $1 - b = x = b$, or $b = \frac{1}{2}$.

$x$ may be greater than $b$, in which case the denominator of the second form must also be greater than $1 - b$, in a certain given proportion, in order that the Fractional expression may remain the same. 

But secondly supposing $\frac{1+b}{1-b}$ to be under all circumstances susceptible of the form $\frac{1+x}{x}$, I cannot deduce from this equation $b = \frac{1}{2x+1}$.

Your last letter, on the Binomial Theorem, was quite satisfactory to me, but I have some remarks to make on the second proof of it, pages 211 to 213. I think you well observe in the note page 213, that the two proofs supply each other’s deficiencies; for I like neither of them taken singly. The latter one is what I should call rather cumbrous, especially the verification of $\varphi n \times \varphi m = \varphi(n + m)$ by actual multiplication in page 212, which is an exceedingly awkward & inconvenient process in my opinion.

Then I am not at all sure that I like the assumption in the last paragraph of page 212. It seems to me somewhat a large one, & much more wanting of proof than many things which in
Mathematics are rigorously & scrupulously demonstrated. But these inconsistencies have always struck me occasionally, and are perhaps only in reality the inconsistencies [66r] in a beginner's mind, & which long experience & practice are requisite to do away with. The end of Euler's proof, page 213, is not agreeable to me, and for this reason, that I cannot feel properly satisfied as yet with the little Chapter on Notation of Functions, and upon the full comprehension of this depends the force of the latter part of this proof. 

I do not know why it is exactly, but I feel I only half understand that [66v] little Chapter X, and it has already cost me more trouble with less effect than most things have. I must study it a little more I suppose. I hope soon I may be able to return to your Differential Calculus. At the same time, I never more felt the importance of not being in a hurry. 

I fancy great proficiency in Mathematical Studies is best attained by time; constantly & continually doing a little. If this is so, surely then the University [67r] cramming system must be very prejudicial to a real
progress in the long run, particularly when one considers how very very little School-boys are [‘generally’ inserted] prepared on first going to the Universities, with anything like distinct mathematical or even arithmetical notions of the most elementary kind. __

I am now puzzling over the Composition of Ratios, but I hope in a day or two more I shall get successfully over that. __

It plagues me a good deal. [67v] I believe I thought some years ago, that I understood it; but I am inclined to think I certainly never did.

You see just at this moment I am full of unsatisfactory obstacles; but I doubt not they will soon yield. ____

With kindest remembrances to Mrs De Morgan, I am

Yours very truly

A. A. L

I think there is an erratum in your Trigonometry, page 34, line 7 from the top:

“let NOM = \theta \circ, MOP = \phi \circ &c” should be . . . NOP = \phi \circ &c
Dear Mr De Morgan. This is very mathematical weather. When one cannot exercise one’s muscles out of doors, one is peculiarly inclined to exercise one’s brains in-doors. Accordingly I have been setting vigorously to work again, with much satisfaction.

But I am sorry to say that I am sadly obstinate about the Term at which Convergence begins for the Series (A) page 231 of Algebra, and which we thought had been made clear by you on Monday Evening. I must persist in thinking Convergence begins at \( \frac{x^{33}}{2.3.4\ldots 32.33} \), and I have enclosed my Demonstration of my view of the case, & which seems to me as clear as possible.

At any rate this will show you at once where my blunder is; if I have blundered, for I can hardly think I have, so very palpable does the proof appear to me.

The other night I had unluckily been 10 days or more without thinking about anything of the kind, and consequently the [‘whole’ inserted] thread of the argument was not present to me; or I might have spared you & myself this
trouble now. __
I have thought with much
pleasure of my agreeable
evening on Monday, __
mathematical & educational.

Yours most truly

A. A. Lovelace

[69v] [a child’s scribbling?]
Dear Mr. De Morgan

I now see exactly my mistake. I had overlooked that the Series in question is not one in successive Powers of $x$ [‘like that in page 185’ inserted], but only in successive even powers of $x$.

I used once to regret these sort of errors, & to speak of time lost over them. But I have materially altered my mind on this subject. I often gain more from the discovery of a mistake of this sort, than from 10 acquisitions made at once & without any kind of difficulty.

There is still one little thing in your Demonstration not perfectly clear to me. At the end you remark that “our result gave the 503\textsuperscript{d} Term instead of the 502\textsuperscript{nd}, which arose from taking the whole number next above $\sqrt{x + \frac{1}{4}}$ instead of an intermediate fraction.”

In examining the equation

$$n = \text{next whole number above} \quad \frac{5}{4} + \frac{1}{2} \sqrt{1000,000 + \frac{1}{4}}$$

I see clearly that $\frac{5}{4} + \frac{1}{2} (1001)$ is greater than $\frac{5}{4} + \frac{1}{2} \sqrt{1000,000 + \frac{1}{4}}$, that the true answer would be $\frac{5}{4} + \frac{1}{2} \left(1000 + \frac{1}{4}\right)$, $\frac{1}{4}$ being some fraction. We should then have had,

$$n = \text{nearest whole number above}$$
\[ \frac{5}{4} + 500 + \frac{1}{2a} \text{, instead of } = \]
\[ = \frac{5}{4} + 500 + \frac{1}{2} \]

But, since \( \frac{5}{4} \) is greater than 1, the result must exceed 501 even if we neglected the \( \frac{1}{4} \) altogether; and therefore at any rate \( n \)

\( ( \text{the next whole number above } \frac{5}{4} + \frac{1}{2} \sqrt{Z + \frac{1}{4}} ) \), must be 502, & \( n + 1 \) consequently = 503 .

I do not therefore see that the fact of taking 1001 instead of the real square part of \( Z + \frac{1}{4} \) does account for the discrepancy in question.

[71v] I have now some question to put respecting certain operations with Incommensurables. Thanks to your Treatise I think I understand [‘this subject’ inserted] pretty tolerably now. But there are still one or two points of Practical Application which I am [something crossed out] busy in working up previous to leaving the subject altogether as a direct study, & which I find not quite plain sailing.

I have been writing out in the Mathematical Scrap-Book, a full explanation of the operations with Incommensurables analogous to those of Multiplication, Division, Raising of Powers &c, and a day or two ago I was [72r] about completing it with that analogous to the extraction of Roots, when I found I did not fully understand the process, that is beyond the consideration of one Mean Proportional. I have written out & enclose my explanation for one Mean Proportional,
& my difficulty in the case
of two or more Mean
Proportionals.

Also, I wished now to return
to the passage, page 29, lines 8
and 9 from the top, (Trigonometry)
which first suggested to me the
necessity of studying the subject
of Incommensurables; in order
that I might see if I could
[72v] now demonstrate the Proposition
of (46), for θ and sin θ Incom=
=mensurable quantities. _ But I do
not find that I can. I believe
I understand the example referred
to in (4), the long & short
of which I understand to be
that if in the Right-Angled
Triangle [diagram in original] A, B, C are
Incommensurables, and V be any
given linear unit, then the
Ratio compounded of A : V and A : V
added to the Ratio compounded of
B : V & B : V, is equal to the
Ratio compounded of C : V and
C : V. _

With respect to the Ratio of an
Angle with it’s [sic] Sine, I began to
[73r] write it out as follows, after the
manner of pages 68 & 69 of the
Number & Magnitude:
θ or θ : 1 is the Ratio of \( \frac{AB}{AO} \)
sin θ or sin θ : 1 is the Ratio of \( \frac{BM}{AO} \),
AB : AO, BM : AO being
Incommensurable Ratios, what
then does \( \frac{θ}{\sin θ} \) really mean? _
In the first place we may
consider it to mean
θ \( \frac{1}{\sin θ} \) : 1, or a Ratio
compounded of the Ratio θ : 1 and
\( \frac{1}{\sin θ} \), or compounded of the
Ratios AB : AO and AO : BM. _
But further than this I cannot get, nor see my way at all. I conclude that in Incommensurable language, a Ratio equal to 1 or a Ratio approximately to 1 can only mean a Ratio in which the Magnitude constituting the Antecedent is equal to the Magnitude constituting the Consequent, or is constantly approaching an equality to it, and therefore that if we take the above Ratio compounded of \( AB : AO \) and \( AO : BM \), or the Ratio \( AB : BM \), & prove that \( AB \) constantly approaches in equality to \( BM \), that is the desired Demonstration. I can only end by repeating what I have often said before, that I am very troublesome, & only wish I could do you any such service as you are doing me. Yours most truly

A. A. L
Dear Mr De Morgan. We have had company ever since I last wrote to you, so I have been at a Stand-still, & only yesterday was able to read over with attention your replies. I am reluctant to trouble you again with remarks on the Series $1 + \frac{x^2}{2} + \frac{x^4}{2.3.4} + \&c$, for it seems as if I was determined to plague you about it. However I feel I must do so. Your added remarks of last time, about $B$ &c, are quite clear in themselves, but I felt at once that they did not meet my difficulty which was that as long as $\frac{5}{4}$ (which is greater than 1) is to be added to $\frac{1}{2}\sqrt{1000,000 + \frac{1}{4}}$, it matters not whether for $\sqrt{1000,000 + \frac{1}{4}}$ we substitute the whole number next above it or “the intermediate fraction” alluded to by you in the line I have marked [mark a bit like her ‘$\sqrt{\ }$’], but we never can bring out $n = \ldots$ anything less than 502, whence $n + 1$ the required term must [75r] be 503. This, after reading over & over, remains in my mind a most obstinate fact, and I believe I have found out the real source of the discrepancy between the result at the bottom of the first page & the top of the Second
one. I am presumptuous enough to think there is certainly an error in your writing out, in the line I have marked X, & it is one which is very likely to have occurred in writing ['it' crossed out] in a hurry. The \((n+1)^{th}\) term divided by the \(n^{th}\) is I believe not \(\frac{Z}{(2n-2)(2n-3)}\), but \(\frac{Z}{2n(2n-1)}\), and I have re-written & now enclose the rest of the demonstration (exactly like yours) with this correction. The result comes out as I expected, owing to \(\frac{1}{4}\) taking the place of \(\frac{5}{4}\), & everything appears to me consistent. The \(n^{th}\) term divided by the \((n-1)^{th}\) term would be, (as you have written) \(\frac{Z}{(2n-2)(2n-3)}\), & this correction would do instead of the others, & be perhaps ['a' inserted] more simple mode of making it, as your demonstration would [76r] then remain correct, the \(n^{th}\) term being in that case the required unknown one instead of the \((n+1)^{th}\).

I am afraid all this is a little complicated to explain in letters, & perhaps I have still not succeeded very perfectly in doing so; but I feel it now all very clear in my own mind, & am only anxious to receive confirmation as to my being right, both as satisfactory to me in the present instance, & as tending to give me
[76v] confidence in future in
my own conclusions, or, (if
I am in this case puzzle-
headed), a due diffidence
of them. _____
I therefore beg your indulgence
for being so teasing. ________
Believe me
Yours most truly
A. A. Lovelace
Dear M’ De Morgan. I send you the ['Series’ crossed out] Analysis of the new Series I received in your letter yesterday morning. I believe I have made it out quite correctly. In fact, the Verification at the end proves this. But, owing to a carelessness in my ['first’ inserted] inspection of it, I have had the trouble & advantage of analysing two Series. I glanced too hastily at it, & did not observe that the factors of the Denominators (of the Co-efficients), are not powers of 2, but simply multiples of 2. If you will open my Sheet, you will find on the inside my analysis of the Series I at first mistook your’s [sic] for; and I am not sorry this has happened. I believe both are correctly made out. 

You kindly request me if I do not understand the erasure in the former ['small’ inserted] paper, again to return it &c. Now I do not agree to it; & ['I’ inserted] still fancy that we are in fact meaning exactly the same thing, only that you are speaking of the $n$th Term, & I of the $n+1$th. For convenience of reference I again return the former large paper (& at any rate H. M’s Post-Office will benefit). I quite understand that
\[ \frac{1}{4} + \frac{1}{2} \sqrt{\frac{10^6 + \frac{1}{4}}{4} + a} \text{ is less than 501. Therefore as } \frac{n}{2} \text{ is the next whole number above this fractional expression, } x = = 501. \text{ But } \frac{n}{2} \text{ is not the Term sought; the unknown term to be determined being by the conditions of the Hypothesis & Demonstrations, } n + 1, \text{ & therefore } = 502. \text{ And if you will examine [78v] your own ['former' inserted] Verification, you will see that you there determine the Term at which Convergence begins, to be } A_{502}, \text{ or the 502}^{\text{nd}} \text{ Term, which agrees with my result } n + 1 = 502. \text{ I think it is quite clear that we are both agreed, but that you were not aware at the moment you made the erasure that I was not speaking of the next whole number above } \frac{1}{4} + \frac{1}{2} \sqrt{\frac{10^6 + \frac{1}{4}}{4} + a} \text{ but of the next but one above it. } \]

So much for the three Series: _

Now I must go to other [79r] matters. I am indeed sending you a Budget. _

I have been working hard at the Differential Calculus, & am putting together some remarks upon Differential Co-efficients (which in due time will travel up to Town for your approbation), but in the progress of which I am interrupted by a slight objection to an old
matter of Demonstration, which did not occur to me at the time I was studying it before, & sent you a paper upon it ['from Ashley' inserted]. In the course of the observations I [79v] am now writing, I have ['had' inserted] occasion to refer to the old ['general' inserted] Demonstration, (pages 46 & 47 of your Differential Calculus), as to the finite existence of a Differential Co-efficient for all Functions of $x$; & a slight flaw, or rather what appears to me a flaw, in the conclusions drawn, has occurred to me. It is most clearly proved that, $\theta$ being supposed to diminish without limit, the Fractions $Q_1, Q_2 &c$ must have finite limit, for some value or other at all events of $n\theta$ or $h$. But the fractions in question do not [80r] appear to me to be strictly speaking analogous to $\frac{\varphi(a+\theta) - \varphi a}{\vartheta}$, except the first of them $\frac{\varphi(a+\theta) - \varphi a}{\vartheta}$ and the last of them $\frac{\varphi(a+n\theta) - \varphi(a+n-\theta)}{\vartheta}$, and for this reason.

In the expansion $\frac{\Delta u}{\Delta x}$ or $\frac{\varphi(x+\theta) - \varphi x}{\vartheta}$, as $\theta$ alters $x$ does not alter, but remains the same. In these fractions on the contrary, which all have the form $\frac{\varphi(a+k\theta) - \varphi(a+k-\theta)}{\vartheta}$ and in which $a + k - \bar{1}\theta$ [bar over $k - 1$ should have little downward-pointing hooks at the ends]

stands for the $x$ of the expression $\frac{\Delta u}{\Delta x}$ or $\frac{\varphi(x+\theta) - \varphi x}{\vartheta}$, not only does $\theta$ alter, but from the conditions of the
Hypothesis & Demonstrations, \( k - 1 \theta \)

[80v] & consequently \( a + k - 1 \theta \) must likewise alter along with \( \theta \).

There is therefore a double alteration in value going on simultaneously, which appears to me to make the Case quite a different one from that of \( \Delta u \Delta x \), & consequently to invalidate all conclusions deduced from the former with respect to the latter. 

The validity of the Conclusions with respect to the fractions \( Q_1, Q_2 \ &c \), you understand I do not question. What I question is the analogy between these Fractions & the Fraction \( \frac{\Delta u}{\Delta x} \) or \( \frac{\varphi(x+\theta)-\varphi x}{\theta} \) ['of' inserted] which \( \text{[81r] latter} \) it is required to investigate the Limits.

I also have another slight objection to make, not to the extent of Conclusions established respecting the Fractions \( Q_1, Q_2 \ &c \) having finite limits, but to the Conclusions on that point not going far enough, not going as far as they might: “either these are finite limits, or some increase “without limit and the rest “diminish without limit ; if the “latter, we shall have two “contiguous fractions, one of which “is as small as we please, and “the other as great as we please, “&c, &c, a phenomenon which “which [sic] can only be true when [81v] “\( Q_k \) is the fraction which is “near to some singular value “of the Fraction, & cannot be
“true of ordinary & calculable values of it &c.” Now it appears to me that in no possible case could such a phenomenon as this be true, when we consider how the fractions are successively formed one out of the others by the substitution of \( a + \theta \) for \( a \), \( \theta \) too being as small as we please. I therefore think it might have been concluded at once that there must always be finite limits to the fractions \( Q_1, Q_2 \) &c, [82r] and this whatever \( k \) or \( n\theta \) may be. I suppose it is not so, but I cannot conceive the Case in which it could be otherwise.

I do not know if in writing upon my two difficulties in these pages 46, 47, 48, I have expressed my objections (especially in the former case of the fractions \( Q_1, Q_2 \) not being similar to \( \frac{\Delta u}{\Delta x} \)) with the clearness necessary to enable you to answer them, or indeed to apprehend the precise points which I dispute. It is not always easy to write upon these things, & at best one must be lengthy. I shall be [82v] exceedingly obliged if you will also tell [‘me’ inserted] whether a little Demonstration I enclose as to the Differential Co-efficient of \( x^n \), is valid. It appears to me perfectly so; & if it is, I think I prefer it to your’s [sic] in page 55. It strikes me
as having the advantage in simplicity, & in referring to fewer ['requisite’ inserted] previous Propositions.

I have another enquiry to make, respecting something that has lately occurred to me as to the Demonstration of the Logarithmic & Exponential Series in your Algebra, but the real truth is I am quite ashamed [83r] to send any more; so will at least defer this. I am afraid you will indeed say that the office of my mathematical Counsellor or Prime-Minister, is no joke.

I am much pleased to find how very well I stand work, & how my powers of attention & continued effort increase. I am never so happy as when I am really engaged in good earnest; & it makes me most wonderfully cheerful & merry at other times, which is curious & very satisfactory. 

What will you say when [83v] you open this packet? Pray do not be very angry, & exclaim that it really is too bad.

Yours most truly

A. A. Lovelace
Dear Mr De Morgan. Many thanks for your reply to my enquiries. I believe I now understand about the limit of $\frac{\varphi(x+n\theta+\theta)-\varphi(x+n\theta)}{\theta}$ not being affected by $n\theta$ being a gradually varying quantity. I think your explanation of it amounts to this: that provided $x + n\theta$ varies only towards a fixed limit, either of increase or diminution; then the result of the Subtraction of $\varphi(x + n\theta)$ from $\varphi(x + n\theta + \theta)$ remains just the same as if, (calling $(x + n\theta) = Z$), $Z$ were a fixed quantity. Now by the conditions of the Demonstration in question, (in your pages 46 & 47), when a decrease takes place in $\theta$, a certain simultaneous increase takes place in $n$. That is to say, suppose $\theta$ has at any one moment a certain value corresponding to which $n$ has the value $k$. If I alter $\theta$ to a lesser value $\chi$, then say that the corresponding $[85r]$ value of $n$, necessary to fulfil the constant condition $n\theta = h$, is not $k$, but $k + m = p$. What happens now? Why as follows, I believe: there were, before $\theta$ became $\chi$, $k$ fractions; there are now $k + m$, or $p$ fractions. In ['each of' inserted] the $k$ former fractions, $[something crossed out] Z$ will
have diminished, towards a fixed limit \( x \); in each of the new fractions introduced, \( Z \) will be greater than in the old \( k \) fractions; but there is a fixed limit of increase, \( h \), which it can never pass, up to the very last Term of the Series of Fractions.

Therefore tho’ the quantity \( x + n\theta \) or \( Z \) varies necessarily with a variation in the value of \( \theta \), yet it varies within fixed limits either of diminution or increase, & thus the result of the subtraction \( \varphi(Z + \theta) - \varphi(Z) \) is not affected.

I hope I have made myself clear. I think it is now distinct & consistent in my head.

I see that my proof of the limit for the function \( x^n \) is a piece of circular argument, containing the enquiry which I was in fact aiming at in the former paper, but which required to be separated from the confusion attendant on my erroneous statements on other points. I merely return the old paper with the present one, because it might perhaps be convenient to compare them.

On the other side of the sheet containing the remarks on \( \frac{a^\theta - 1}{\theta} \), you will find an enquiry which struck me lately quite by accident in
referring to some old matters. ___
I ought to make many apologies I am sure for this most abundant budget. ___ I am very anxious about the matter of the successive Differential Co-efficients, & their finiteness & continuity. I think it troubles my mind more than any obstacles generally do. I have a sort of feeling that I ought to have understood it before, & ___ [87r] that it is not a legitimate difficulty. ___
With many thanks,
Yours most truly
   A. A. Lovelace
Dear Mr De Morgan. Then I shall be in Gower St on Monday Evening about 8 o’clock. Lord L is not afraid of Algebra, but he goes up on business that will occupy all his evenings, viz: Lord Melbourne’s dinner, & the House of Lords. I feel as if I had a great deal to say & to talk about; & by the bye one thing, unmathematical, is about a visit sometime from Mrs De Morgan; because it strikes me that your lectures need not always tie her down too. Lord L desires me to say he regrets not being able to accompany me.

Yours very truly

A. A. Lovelace
Ockham
Friday 22nd Jan¹
[¹'1841' added by later reader]

Dear M' De Morgan. Then I shall be in Gower St on Monday Evening about 8 o'clock. Lord L is not afraid of Algebra, but he goes up on business that will occupy all his evenings, viz: Lord Melbourne's dinner, & the House of Lords. I feel as if I had a great deal to say & to talk about; & by the bye one thing, unmathematical, is about a visit sometime from Mrs De Morgan; because it strikes me that your lectures need not always tie her down too.

Lord L desires me to say he regrets not being able to accompany me.

Yours very truly
A. A. Lovelace
Dear Mr De Morgan. I have a question to put respecting a condition in the establishment of the conclusion
\[
\varphi(a+h) = \frac{\varphi^{(n+1)}(a+\theta h)}{\psi^{(n+1)}(a+\theta h)}
\]
in page 69 of the Differential Calculus. I have written down, & enclose, my notions on the steps of the reasoning used to establish that conclusion. So that you may judge if I take in the objects & methods of it.

The point I do not understand, is why the distinction is made, (& evidently considered so important a one), of \(\psi x\) "being a function which has the property of always increasing or always decreasing, from \(x = a\) to \(x = a + h\), in other respects fulfilling the "conditions of continuity in the same manner as \(\varphi x\)."

For this, see page 68, lines 9, 10, 11, 12 from the top ; page 68, line 12 from the bottom ; page 69, lines 7, 8 from the bottom ; &c

I see perfectly that this condition must exist, & that without it we could not secure the denominators.
(alluded to in page 68, line 13 from the bottom), being all of one sign. __

But what I do not understand, is [something crossed out] why the condition is not made [90v] for \( \varphi \cdot x \) also. It appears to me to be equally requisite for this latter; because if we do not suppose it, how can we secure the numerators \( \varphi(x + k \Delta x) - \varphi(x + k - 1 \Delta x) \) being all of one sign; & unless they are all of one sign, we cannot be sure that they will [something crossed out] when added, so destroy one another as to give us \( \varphi(a + h) - \varphi a \); an expression essential to obtain. ____ I think I have explained my difficulty, & [something missing here?]

[the following written vertically on 89r]

believe me

Yours most truly

A. A. Lovelace
[beginning of letter seems to be missing]

[91r] (unless the limit for $\frac{v^n-w^n}{v-w}$ is dispensed with in the
demonstration for the Binomial
Theorem, which it is not
in your Algebra, nor am
I aware that it can be
dispensed with in any of the
elementary proofs of that
Theorem). It had not
struck me that, calling
$(x + \theta) = v$, the form
$(x+\theta)^n-x^n$ becomes $\frac{v^n-x^n}{v-x}$.

And by the bye, I may
here remark that the curious
transformations many formulae
can undergo, the unexpected
& to a beginner apparently
[91v] impossible identity of forms
exceedingly dissimilar at first
sight, is I think one of
the chief difficulties in the
early part of mathematical
studies. I am often reminded
of certain sprites & fairies
one reads of, who are at
one’s elbow in one shape
now, & the next minute in
a form the most dissimilar,
and uncommonly deceptive,
troublesome & tantalizing are
the mathematical sprites &
fairies sometimes; like the
types I have found for them
in the world of Fiction.

[92r] I will now go to the question
I delayed asking before:
In the development of the
Exponential Series
$a^x = 1 + (\log a)x + \frac{(\log a)^2x^2}{2} + &c$, and the Logarithmic Series
$log a = (a - 1) - \frac{1}{2}(a - 1)^2 + &c$
deduced from it; I object
to the necessity involved of
supposing $x$ to be diminished
without limit, a supposition
['obviously' inserted] quite necessary to the completion
of the Demonstration. It has
struck me that though this
supposition leaves the Demonstration
& Conclusions perfect for the
cases in which $x$ is supposed
to diminish without limit, yet
[92v] it makes it valueless for the
many in which $x$ may be
anything else which does not
diminish. No by the bye,
I think I begin to see it now;
I am sure I do. It is as
follows: the supposition of
$x$ diminishing without limit
is merely a parenthetical
one, by means of which a
limit for a certain expression
\[
\frac{a^x - 1}{x}
\]
is deduced under those
circumstances; & then the
argument proceeds, that having
already obtained in another
place, a ['different' inserted] limit for this same
expression under the same
[93r] circumstances, we at once
deduce the equality of these two
limits, from whence follows
&c, &c. Thus this supposition
of $x$ diminishing without limit,
is not a portion of the main
argument, but only a totally
independent & parenthetical
hypothesis introduced in order
to prove something else which
is a part of the main
argument. Yes this is
it, I am sure. I had
had the same objection to
the Demonstration in Bourdon, to which I have had the curiosity to refer. I am sometimes very much interested in seeing how the same conclusions are arrived at in different ways by different people; and I happen to have been inclined to compare you & Bourdon in this case of developing Exponential & Logarithmic Series; and very amusing has it been to me to see him begin exactly where you end, &c. Your demonstration is much the best for practical purposes. His is exceedingly general, & the vast number of substitutions of one thing for another make it lengthy, & by no means very simple to follow. But it is very well occasionally to make these comparisons.

We are going to Town on Monday the 25th, for two or three nights, & I will ask M's De Morgan's & your permission to spend Monday Evening with you, going towards 8 o'clock, as I did before. It would give me great pleasure, & may perhaps be not only agreeable to me, but of use [94v] too, as there are one or two points relating to my future plans which I rather think of speaking to you upon. By the bye, Lord Lovelace & I are both of us much vexed, at our own
negligence in letting the Xmas Vacation go bye [sic], without proposing to you & your lady & children to visit us here, which you might perhaps have been able to do during Holiday-time. I fear you may now be unable to think of it; but pray consider [95r] the question with her; if not for any immediate use, at any rate for the next occasion. The fact is, that we have so much the habit of thinking of you only in connexion with Town & engagements there, that it only suddenly occurred to us whether you might not be able to breathe country air like other people. You would come by Railway, & we would send the carriage to the Station for you.

Yours most truly
A. A. L
Dear Mr De Morgan. Had I waited a day or two longer, I need not have troubled you with my letter of Wed\textsuperscript{dy}, & I can only reproach myself now with having been a little too hasty in my examination of the Theorem in pages 68, 69, and having sent you an enquiry which certainly indicates some negligence. I fear this letter [96v] may not be in time to stop one from you. [something crossed out] However I will try to send it by an opportunity this afternoon. 

But, to show you that I now understand the matter completely: 

In the first place the question of the Denominator, or the Numerator, being all of the same sign, in such collection of expressions as
\[
\frac{a-b}{m-n}, \frac{c-a}{p-m}, \frac{d-c}{r-p}, \frac{e-d}{q-r} &c
\]
has nothing whatever to do with the letters effacing each other when the above are put into the form,
\[
\frac{(a-b)+(c-a)+(d-c)+(e-d)}{(m-n)+(p-m)+(r-p)+(q-r)} &c;
\]
whether \((a-b)\), &c be positive or negative, or some one & some the other, still
\[
\frac{a-b+c-a+d-c+e-d}{m-n+p-n+r+p+r} &c
\]
must = \[
\frac{e-b}{q-r}
\]
In the second place, the
Denominator must be all of the same sign, in order to fulfil the conditions of the Lemma in page 48; & this is the reason why the condition is made respectively \( \psi x \) always increasing or \( [97v] \) always decreasing &c. For \( \varphi x \), it matters not whether it alternately increases & decreases (provided always that it be continuous). I believe I now have the whole quite clear; & I shall be more careful in future.

I enclose a paper upon pages 70, 71, 72, 73. It is merely the general argument, put into my own order & from; & I send it in order to know if you think I understand as much about the matter as I am intended to do. You know I always have so many metaphysical enquiries & speculations which intrude themselves, that I never am really satisfied that I understand anything; because, understand it as well as I may, my comprehension can only be an infinitesimal fraction of all I want to understand about the many connexions & relations which occur to me, how the matter in question was first thought of \( [98r] \) or arrived at, &c, &c. I am particularly curious
about this wonderful Theorem. However I try to keep my metaphysical head in order, & to remember Locke’s two axioms. __
You should receive this about 6 o’clock this evening, if not before. I fear you will have written to me today however. _ Believe me

Yours most truly

A. A. Lovelace
Dear Mr. De Morgan. I have one or two queries to make respecting the “Calculus of Finite Differences” up to page 82.

Page 80, line 4 from the top, “remembering . . . . that in \( \phi''(x + \theta \omega) \), \( \theta \) itself is a function of \( x \) and \( \omega \), &c”; Now, neither on examining \( \theta \) as here used & introduced, nor on referring to the first rise & origin of \( \theta \) in this capacity, (see page 69), can I discover that it is a function of \( x \) and \( \omega \) here, or a function of the analogous \( a \) and \( h \) in page 69. I neither see the truth of this assertion, nor do I perceive the importance of it (supposing it is true) to the rest of the argument & demonstration in page 80.

There is also a point of doubt I have relating to the conclusion in lines 15, 16 from the top of page 79:

It is very clear that the law for the Co-efficients being proved for \( u_n \), and for \( \Delta u_n \), follows immediately & easily for \( u_{n+1} \), or \( u_n + \Delta u_n \).

But if we now wish to establish it for \( u_{n+2} \), we must prove it true not only for \( u_{n+1} \), but also for \( \Delta u_{n+1} \):

To retrace from the beginning: the object in the first half of page 79 evidently is to prove firstly, that any order of \( u \), say \( u_n \), can be expressed in term of \( \Delta u \), \( \Delta^2 u \), \( \Delta^3 u \), . . . . . . . . . \( \Delta^n u \);

Secondly, that the Co-efficients for this Series follow the law of those in the Binomial Theorem.

Now the first part is evident from the law of formulation of the Table of Differences; Since all the Differences \( \Delta u \), \( \Delta^2 u \), \( \Delta^3 u \) &c are made out of \( u \), \( u_1 \), \( u_2 \) &c, it is obvious that by exactly retracing & reversing the process, we can make \( u \), \( u_1 \), \( u_2 \) &c [101r] out of \( \Delta u \), \( \Delta^2 u \), \( \Delta^3 u \) &c.
For the second part of the above; if we can show that the law for the Co-efficients holds good up to a certain point, say $u_4$; and also that being true for any one value, it must be true for the next value too; the demonstration is effected for all values:

Now the fact is shown that it is true up to $u_4$. (I must not here enquire why the fact is so. That is I suppose not your arranging, or any part of your affairs).

It is shown that the two parts $u_3$, $\Delta u_3$ of which $u_4$ is made up are under this law, & therefore that $u_4$ is so. And next it is shown that any other two parts $u_n$, $\Delta u_n$ being under this law, their sum $u_{n+1}$ must be so. But this proves nothing for a continued succession. $u_{n+1}$ being under this law does not prove that $\Delta u_{n+1}$ is under it, & therefore that $u_{n+2}$ is under it.

There seems to me to be a step or condition omitted.

I am sorry still to be obliged to trouble you about $f x$, $f' x$, $f'' x$. I cannot yet agree to the assertion that the result would not be affected by discontinuity or singularity in $f' x$, $f'' x$, &c. The result it is true would not be directly affected; but it surely would be ["indirectly inserted"] affected, inasmuch as the conditions of page 69, necessary to prove that result, could not be fulfilled unless we suppose $f' x$, $f'' x \ldots \ldots f^{(n+1)} x$ continuous & ordinary as well as $f x$. To arrive at the equation

$$\frac{\varphi(a+h)}{\psi(a+h)} = \frac{\varphi^{(n+1)}(a+\theta h)}{\psi^{(n+1)}(a+\theta h)}$$

page 69, it is a necessary condition that $\varphi x$, $\varphi' x$, $\varphi'' x \ldots \ldots \varphi^{(n+1)} x$ be all continuous & without singularity from $x = a$ to $x = a + h$. And the $\varphi' x$, $\varphi'' x \ldots \ldots \varphi^{(n)} x$, $\varphi^{(n+1)} x$ of page 71, could not fulfil this condition unless $f' x$, $f'' x \ldots \ldots f^{(n)} x$, $f^{(n+1)} x$ did so [102r] also. I fear I am very troublesome about this.
I have remarks to make respecting some of the conclusions of the Chapter on Algebraical Development; but they will keep, and therefore I will delay them, as I think I have send abundance, & I have also some questions to put on the last 8 pages of your “Number & Magnitude” on Logarithms. 

On the Differential Calculus I will only now further say that on the whole I believe I go on pretty well with it; and that I suppose I understand as much about it, [something crossed out] as I am intended to do; possibly more, for I spare no pains to do so.

Now for the Logarithms: I had not till now read the last pages of your Number & Magnitude, & there are certain points I do not fully understand. The last line of the whole, on the natural logarithms is one. I cannot identify the constituent quality of the natural logarithms there given, with the constituent qualities I am already acquainted with thro’ other relations & means: I know [‘for instance’ inserted] that the natural logarithms must have 2.717281828 for their Base; that is to say that the line $HL$, or $A\ (OK$, or $V$ being the linear unit) should be 2.717281828 $V$ units.

Now I do not see [‘but’ inserted] that the condition in the last paragraph of the book is one that might perfectly consist with any Base whatever.

To prove that I understand the previous part, at least to a considerable degree, I enclose a Demonstration I wrote out of the property to be deduced by the Student, (see second paragraph of page 79), & which I believe is quite correct.

Pray of what use is the Theorem (page 75, [‘& which’ inserted] continues in page 76)? I do not see that it is subservient to anything that [103r] follows; and it appears to me, to say the truth, to be rather a useless & cumbersome addition to a subject already sufficiently
complicated & cumbersome. The passage I mean is from line 13 (from the top) page 75, to the middle of page 76. 
Believe me

Yours very truly

A. A. Lovelace
Dear Mr De Morgan. The reply to one of my queries to you, dispatched on Friday, has I believe just occurred to me. Probably this letter will cross one from you tonight, but the remaining points continue still unsolved, so that I shall be equally glad if I do receive an answer tomorrow morning.

The difficulty I have solved is the one relating to the law for the Co-efficients of \( \Delta u_n \). I remarked that the law for the Co-efficients of the Series for \( u_n \) being ascertained, did not ascertain those for \( \Delta u_n \) as a necessary consequence. But I see I am wrong. If a Series is obtained for \( u_n \),

\[
\Delta u_n = a_m x^{m+1} - A x^{m-2} + \cdots + P x + Q
\]

and in which I cannot help thinking there is a mistake [in the first Term’ inserted]. I make out that it ought to be

\[
\Delta u = a_m \omega x^{m-1} + A x^{m-2} + \cdots
\]

But I enclose my developments and observations therefore, on a longer & more convenient sheet. I will only add here, that we move to Town on Thursday; and that I should much like to spend Sunday Evening with Mrs De Morgan & you, if this arrangement is suitable & agreeable to you. I [105v] should arrive as usual, about 8 o’clock

I believe I shall have by the end of this week several papers ready to discuss. You see I do not waste my time, at any rate; and I only hope that I am
not the means of wasting yours either.
Believe me
    Yours very truly
    A. A. Lovelace
Ockham Park  
Ripley  
Surrey  
4th July

Dear Mr De Morgan. You are perhaps surprised that I have not sooner troubled you again. And you may think it a very bad reason to give, that I have done nothing. We returned here on Tuesday, & now I am working away famously, & hope I have before me 7 or 8 months of ditto. You left me at page 106. I remember your enquiry if I were sure that I understood \( \int b + k f x \times \frac{dx}{dt} \) as developed \(sic\) in pages 102, 103. I answered confidently, that I did. I now enclose you my own development of this Integration, that we may be quite certain of my comprehension of [something crossed out] it. On the other page of my sheet, is the application of it to \( \int udv = uv - \int vdu \) (page 105); & to \( \int \frac{1}{v} \frac{dv}{dx} dx \) (page 107).

I have now two questions to propose. I differ from you in my development of \( \int \frac{1}{1-x} dx \) (see page 107)

\[ \int \frac{1}{1-x} dx = \int \frac{1}{1-x} \times -(-1)dx \] (which is only another way of writing \( \int \frac{1}{1-x} 1 dx \))

And as \( \frac{dv}{dx} \) or \( \frac{d(1-x)}{dx} \) = -1, we may in the above substitute \( \int \frac{1}{1-x} dx = \int \frac{1}{1-x} \times -\left( \frac{d(1-x)}{dx} \right) dx \)

Or \( \int \frac{1}{v} dv = \int \frac{1}{v} \times -\frac{dv}{dx} dx \)

\[ = \int \frac{1}{v} dv \cdot (-1)dx \] which by \( \int b v dx = b \int u dx \) (see page 105) is \( = (-1) \int \frac{1}{v} \frac{dv}{dx} dx \)

or \( = \int \frac{1}{v} dv \) dx

Now since by line 4, \( \int \frac{1}{v} \frac{dv}{dx} dx = \int \frac{1}{v} dv = \)

\[ = \log v + C, \] it follows that

\[ = -(\log(1-x) + C) = - \log(1-x) - C \]

\[ = \log \frac{1}{1-x} - C \]

[107r] Now how do you get rid of \(-C\)? __
My second question is unconnected with any of your books. But I think I may venture to trouble you with it. In the two equations,

\[ V = gT \]  \hspace{1cm} (1)

\[ S = \frac{1}{2}gT^2 \]  \hspace{1cm} (2)

which you will at once recognise, I want to know how (2) is derived from (1). Will you refer to Mechanics (in the Useful Knowledge Library), page 10, Note, which is as follows, “Let \( S \) be the space described by the falling body. \( V = \frac{dS}{dT} = gT \). Hence \( dS = gT \, dT \), “which being integrated gives \( S = \frac{1}{2}gT^2 \).”

Now can I [‘as yet’ inserted] understand this application of Differentiation & Integration? I conclude that \( \frac{dS}{dT} \) here means Diff. co of \( S \) with respect to \( T \), \( S \) being (by Definition & Hypothesis) a function of \( T \), & of \( V \) I know that \( V = gT \) And that \( V = \frac{S}{T} \) But I neither see how \( V = \frac{dS}{dT} \), nor how the subsequent Integration applies. [107v] The object, I need not say, is the solution of \( S \). I mean to work very hard at my Chapter on Integration &c, now. And I hope this summer & autumn will see me progressing at no small rate.

How is the Baby? And does Mrs De Morgan enjoy Highgate? I [‘am’ inserted] enjoying the country not a little, I assure you.

Yours most truly

A. A. Lovelace
Dear Mr De Morgan. It is perhaps unfair of me to write again with a batch of observations & enquiries, before you have had time to reply to the previous one. But I am so anxious to get the present matters off my mind, that I cannot resist dispatching them by this post.

I have two series of observations to send, one relating to the passage from page 107, (line 8 from the bottom), to the last line of page 108; the other to certain former passages in pages 99, 100 & 103, concerning which some questions have suddenly occurred to me quite recently.

I shall begin with pages 107 & 108: I enclose you my development & explanation of \( \int x^n \, dx \sqrt{a^2 - x^2} \) up to \( \int x^n \, dx \sqrt{a^2 - x^2} = -x^{n-1} \sqrt{a^2 - x^2} + (n-1) a^2 \int \frac{x^{n-2} \, dx}{\sqrt{a^2 - x^2}} - (n-1) \int \frac{x^n \, dx}{\sqrt{a^2 - x^2}} \) from which you will judge if I understand it so far. I should tell you that I have not yet begun page 109.

I will now ask two or three questions: 1\(^{st}\)ly: page 107, [108v] (line 3 from the bottom): “the diff. co of \( a^2 - x^2 \) being \( (-2xdx) \) &c”. This surely is incorrect; & you will see that in my development I have written it as I fancy it should be “being = \( (-2x), \) &c.”

2\(^{nd}\)ly: page 108, (lines 8, 9, 10 form the top): “By \( \int UdV \)” we mean \( \cdots \cdots \cdots \) p. 102, where “the values of \( \Delta V \) in the several terms are “different, but comminuent.” I do not see that this is a case of page 102 rather than of page 100; in other words, that the increments in this Integration are “unequal but comminuent”.

3\(^{rd}\)ly: the subtraction in line 15 from the top, of \( (n-1)x^{n-2} \times dx \) for \( d.(-x^{n-1}) \) appears to me quite inconsistent with the inseparable indivisible nature of a diff. co.

4\(^{th}\)ly: Lines 9, 10 from the bottom, “We have therefore “\&c \cdots \cdots \cdots \cdots \) that of \( \sqrt{a^2 - x^2} x^{n-2} \, dx \)”.

Admitted, most fully. But \( \int \sqrt{a^2 - x^2} x^{n-2} \, dx \) does not answer exactly to \( \int v \, dx \) or \( \int \sqrt{v} \, 2u \), and
therefore it appears to me that this Integration is not strictly an example of lines 5, 6, 7 (from the bottom) of page 107. You will remember that \(-x^{n-1}\) was \(= 2V\), therefore the \(x^{n-2}\) of \((\sqrt{a^2-x^2}x^{n-2})\) is equal to \((-1) \times \frac{2V}{x}\) or \(-\frac{1}{x} 2V\). So that another factor \(-\frac{1}{x}\) enters into the expression which was, as I understand it, to answer strictly to \(\int vdu\) or \(\int \sqrt{v}d2u\)

5thly (line 5 from the bottom) page 108: I think there is an Erratum. Surely \(\int \left(\frac{a^2x^{n-2}}{\sqrt{a^2-x^2}} - \frac{x^n dx}{\sqrt{a^2-x^2}}\right)\)

ought to be \(\int \left(\frac{a^2x^{n-2}dx}{\sqrt{a^2-x^2}} - \frac{x^n dx}{\sqrt{a^2-x^2}}\right)\)

I don’t know if my pencil Sheet enclosed will be very intelligible, for it is as I wrote it down at the time quite roughly, & without any very great amplitude or method.

I now proceed to my series of observations relating to former pages, beginning with page 102, (line 10 from the bottom)

"+ less than \(nC\frac{\Omega^2}{2}\), or \(Ch\frac{\Omega^2}{2}\);

now in order to [‘effect’ inserted] the substitution of \(Ch\frac{\Omega^2}{2}\) for \(nC\frac{\Omega^2}{2}\) the latter is resolved into \(C.n\Omega\frac{\Omega^2}{2}\), & [‘for’ inserted] \(n\Omega\) is substituted \(h\). But by the hypothesis & conditions, \(h\) must be less than \(n\Omega\). Therefore it does not necessarily follow that that which is proved less than \(nC\frac{\Omega^2}{2}\), is also less than \(Ch\frac{\Omega^2}{2}\). You see my objection.

2ndly. See Note to page 102 : If the “completion of the [‘first’ inserted] Series” [109v] in this page is unnecessary, surely it is equally unnecessarily in the first Series of page 100 ; for the same observation applies to the latter as to the former, viz : that the additional term is comminuent with \(w\).

3dly. See page 99 (line 8 from the bottom) :

\[\int_a^x \varphi x \, dx = (x - a)a + \frac{(x-a)^2}{2} = \frac{x^2-a^2}{2}\]

This is another form of \(\int_a^{a+h} xdx = ha + \frac{h^2}{2}\) 8 lines above, & of the limit of the summation for \(\varphi x = x\) in the previous page. And therefore it appears to me that it ought to be

\[\int_a^x x \, dx = (x - a)a + \frac{(x-a)^2}{2} = \frac{x^2-a^2}{2}\]

I do not see what business \(\varphi x\) has.
Now at last, I have done troubling you. __
I am very anxious on all these points. ____
With many apologies, believe me
Yours very truly
A. A. Lovelace
Dear Mr De Morgan. Since dispatching my letter yesterday, I remember that I have not even quite fully & correctly stated the whole points of difference ['between' inserted] \( \int \sqrt{a^2 - x^2} x^{n-2} \, dx \) and \( \int \sqrt{v^2} \, du \). I think I stated that \( \int \sqrt{a^2 - x^2} x^{n-2} \, dx = \int \sqrt{v^2} \, du \, \frac{1}{x} \); that in other words the 1st side differs from \( \int \sqrt{v^2} \, du \) in containing a factor \( (-\frac{1}{x}) \). But it differs also in containing \( dx \) as well, which in writing yesterday I omitted I believe to notice. So that \( \int \sqrt{a^2 - x^2} x^{n-2} \, dx = \int \sqrt{v^2} \, du \, \frac{1}{x} \, dx \) or the 1st side differs from \( \int \sqrt{v^2} \, du \) in containing \( -\frac{1}{x} \, dx \). Is not this what I ought to have stated? Or is there still any confusion?

I also wish to observe upon what I wrote on Friday as to the application of the Differential & Integral Calculus to \( \frac{2a^2}{x^2} \), [110v] that I am aware this formula \( 'e = \frac{gt^2}{2} \) inserted can be derived from \( V = gt \), by the simple Theory of algebraical proportion; but that I was anxious to know how it is derived in the other way.

I will with your leave ['(which I do not wait for)'] inserted, send you my paper making it out on the doctrine of Proportions.

You must tell me if I presume too much on your kindness to me. I am so engaged at present with my mathematical & scientific plans & pursuits that I can think of little else; & perhaps may be a plague & bore to my friends about [something crossed out] these subjects; for after my interruption from Paris & London pursuits & occupations, my whole heart is with my renewed studies; & every minuitia even is a matter of the greatest interest.

Believe me
Yours most truly
A. A. Lovelace
You will receive two papers on $e = \frac{gt^2}{2}$ tomorrow evening, or Wed$^{dy}$. One of them is to show the absurdity of the supposition that the spaces might be as the velocities; on merely abstract grounds it could not be.
Dear Mr De Morgan. I enclose you a paper (marked No 1) from which I think you will see that I now quite understand the real relationship between $\int \frac{x^n \, dx}{\sqrt{a^2 - x^2}}$ and $\int \frac{\sqrt{a^2 - x^2} \, x^{n-2} \, dx}{x}$; & that I [something crossed out] am now aware I wanted to apply to the latter what is not intended to be directly applied to it at all; & that ['my' inserted] getting both $du$ and $dx$ in, was a complete puzzle & blunder. For where a few lines previously $(-n - T) \int \sqrt{a^2 - x^2} x^{n-2} \, dx$ is substituted for $\int \sqrt{a^2 - x^2} \times (-n - T) x^{n-2} \, dx$, $du$ ceases of course to enter under the integrated quantity, [something crossed out] since it has been decomposed & otherwise distributed.

I am still occupied on pages 108, 109, 110, & hope to complete to page 112 during this week. I find this part requires studying with great care. I think you anticipated this.

I must now thank you very much for your two letters; & will proceed to notice one or two points [112v] in your replies to my enquiries. I see that in objecting to what I called the division of $\frac{dV}{dx}$, when $dV$ is substituted for $\frac{dV}{dx} \, dx$, I took a completely wrong view of the matter. It does so happen that the expression (derived from a separate & distinct Theorem) which we may substitute for $\frac{dV}{dx} \, dx$ coincides in form with what we may call the numerator $dV$ of the difference. But the $dV$ that is substituted is not therefore derived from $\frac{dV}{dx}$, at least ['not directly or' inserted] from the decomposition of that which is indecomposable [sic].

I return again my former paper (marked No 2) with a clearer explanation of what I intended to convey by the term equivalent; a term which it seems I had no business to use in the application which I ['there' inserted] meant to make of it.

I enclose (marked No 3) my answer to your "Try to prove the following. It is only when $y = ax$ " $(a$ being constant) that $\frac{dy}{dx} = \frac{y}{x}$. I do not feel quite sure that my proof is a proof. But I think
it is too.

Now about $v = gt$ and $s = \frac{1}{2}gt^2$; a subject which troubles me not a little.

Is the following a correct development of the note in Useful Knowledge Mechanics? I re-copy the notes first:

[113r] $V = \frac{dS}{dT} = gT$. Hence $dS = gTdT$, which being "integrated gives $S = \frac{1}{2}gT^2$"

[something crossed out on two lines]
The Integral of $\frac{dS}{dT}$ or of $gTdT$ will obviously give us $S$; & we know that $\int gTdT = \frac{1}{2}gT^2 + C$, (by formula of page 104 of the Calculus).

But it appears to me that the statement above "Hence $dS = gTdT$" is an unnecessary intermediate step:

It is true that $\int \frac{dS}{dT}dT = \int dS$,

that is providing we extend the theorem

$$\int fx\, dT = \int fx\, dx$$
to the case when $fx = 1$, which I conclude it is allowable to do, since 1 may be considered a function of anything, I imagine; thro' the formula

$$\frac{fT}{fx} = 1.$$ But tho' true, yet the above ['clause' inserted] appears to me ['an' inserted] unnecessary introduction.

I am not sure that I have explained myself well.

With respect to this formula

$\frac{1}{2}gt^2$, & it's [sic] derivation & application; I have referred as you desired to pages 27, 28, & have [113v] fully refreshed my memory upon them. But I do not feel this helps me much. In this first place the process is the converse of that I enquired upon. $S$ is there give, & $V$ is to be derived from $S$. My position was; $V$ given, & $S$ to be derived from $V$.

I understand the process of pages 27, 28, considered as a distinct & separate thing. But I do not identify it with Differentiation or Integration.

I, (knowing by abstract rules & theorems) that $2x$ is the diff. of $x^2$, see that the limit $2t$ which comes out, might be perfectly well expressed by $\frac{dt^2}{dt}$. And that we may put the result of the Differentiation of $t^2$, and the result of all the reasoning of pages 27, 28, indifferently one for the other. But I only see it as I see that
in the processes $12 \div 4 = 3 \quad 1 + 2 = 3$ we
might indifferently put the results (3, in both cases)
one for the other. There may, for anything I yet
see or understand, be as little connection between
the abstract process of Differentiation and the
Stone-falling process, as between the above processes
of Division & Addition, which latter tho' their results
agree, cannot be identified, or one made to represent
[114r] the other.
I apprehend [something crossed out] you will perhaps answer me
here, that I must wait patiently for Chapter 8,
in which (page 143) I see something very like an
explanation of all I want. __ At the same time
I think it better to express fully my difficulties.

I am very anxious to see your Comments on
my two papers ['sent the other day' inserted] upon $\frac{1}{2}gt^2$. For I do not see
where the flaw in them can be; & yet I suppose
there is one. It is some comfort in the confusions
& puzzles one makes, that they are always
exceedingly amusing to me, after they are cleared
away. And this is at least some compensation
for the plague of them before. __
With many thanks,

Yours most truly

A. A. Lovelace
Dear Mr De Morgan. You must be beginning to think me lost. I have been however hard at work, with the exception of 10 days complete interruption from company. I have now many things to enquire. First of all; can I spend an evening with Mrs De Morgan & yourself on Tuesday the 24th? On that day we go to Town to remain till Friday, when we move down to Ashley for 2 months at least. I would endeavour to be early in Gower St; before eight or not later than eight. And I feel as if I should have many mathematical things to discuss.

Now to my business: 

1stly: I send you a paper marked 1, containing my development of two Integrals in page 116, 
\[
\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1} \left( \frac{x-a}{a} \right)
\]
And 
\[
\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x + a + \sqrt{2ax + x^2}) + \log 2
\]
The former one I think is plain enough, & I and the book are quite agreed upon it. Not so with the latter one, & I begin to suspect the book. I cannot make it anything but 
\[
\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x + a + \sqrt{2ax + x^2})
\]
or else 
\[
= \log\left(\frac{x}{2} + a + \sqrt{2ax + x^2}\right) + \log 2
\]
I have tried various methods; but the only one which I find hold good at all, is that applied in page 115 to 
\[
\int \frac{dx}{\sqrt{a^2+x^2}}
\]
& which seems clearly to bring out my result above. By the bye I have a remark to make on the Integration of 
\[
\int \frac{dx}{\sqrt{a^2+x^2}}
\]
developed [sic] in page 115. 

Line 10 (from the bottom), you have 
\[
xdx = ydy: 
\]
This is obvious, & similarly I deduce in my paper No 1, 
\[
(2a + x)dx = ydy.
\]
But I see no use in what follows, “and 
\[
ydx + xdx = ydx + ydy.
\]
It is equally obvious with the former equation, but seems to me to have no purpose in bringing out the results, which I deduce as follows:
Since \( xdx = ydy \), we have \( \frac{dx}{y} = \frac{du}{x} \).

Therefore by the Theorem of page 48, or at least ['by' inserted] a Corollary of it, we have \( \frac{dx+du}{x+y} = \frac{dx}{y} \), whence &c, &c.

And this is the method also which I have used [sic] in developing \( \int \frac{dx}{\sqrt{2ax+x^2}} \).

2ndly: Page 113, lines 16 ['&c 17' inserted] from the bottom, you say "The first form becomes impossible when \( x \) is greater than \( \sqrt{c} \), for in that case the Integral becomes the "Logarithm of a Negative Quantity". Now there are surely certain cases in which negative quantities may be powers, & therefore may have Logarithms.

All the odd whole numbers may surely be the Logarithms of Negative Quantities. \( (-a) \times (-a) = a^2 \) But \( (-a) \times (-a) \times (-a) = -a^3 \) or \( (+a) \times (+a) = a^2 \) \( (+a) \times (+a) \times (+a) = a^3 \)

3 is here surely the Logarithm of a Negative Quantity. Similarly a negative quantity multiplied into itself any odd number of times will give a negative result.

3rdly: In the Paper marked 3, which I return again ['for reference' inserted] ; I perfectly understand the proof by means of the Logarithms (added by you), why \( \frac{dy}{dx} \) can only = \( \frac{y}{x} \)
when \( y \) is either = \( x \), or = \( ax \) (\( a \) being Constant)

Your proof is perfect, but still I do not see that mine was not sufficient, tho' derived from much more general grounds.

My argument was as follows : Given us \( \frac{dy}{dx} = \frac{y}{x} \), what conditions must be fulfilled in order to make this equation possible? \( \) Firstly : I see that [sic] since \( \frac{dy}{dx} \) means a Differential Co-efficient, which from it’s [sic] nature (being a Limit) is a constant & fixed thing, \( \frac{y}{x} \) must also be a constant & fixed quantity. That is \( y \) must have to \( x \) a constant Ratio which we may call \( a \). \( \) This seems to me perfectly valid. And surely a Differential Co-efficient is as fixed & invariable in it’s [sic] nature as anything under the sun can be.

To be sure you may say that there is a different Differential Co-efficient for every different initial value of \( x \) taken to start from, thus :
\[
\frac{d(x^2)}{dx} = 2x \quad \text{if} \quad x = a, \quad \frac{d(x^2)}{dx} = 2a
\]
if \( x = b \), \( \frac{d(x^2)}{dx} = 2b \)

And this is perhaps what invalidates my argument above.

4\textsuperscript{thly}: In the two papers folded together & marked 2, which I also again return for reference, I perfectly see that tho’ mathematically correct. I was completely wrong in my application. But my proofs do apply to any two different & independent velocities, whatever of two different bodies, or of the same body moving at two different uniform ratio \([sic]\) at different epochs. 

Thus my paper (marked upon it 1\textsuperscript{st} Paper) proves [117r] the following: that the Spaces moved over at two different times, in virtue of the Velocity acquired at the end of each of those times, (the impelling cause being supposed to cease at the end respectively of each time fixed on), would be to each other as the squares of the times fixed on. But I perfectly see that this is quite a different & independent consideration from that of the Space actually moved over by a body impelled by an accelerating force, & how wholly inapplicable my ‘former’ inserted view of it was.

I have been especially studying this subject of \([something crossed out]\) Accelerating Force, & believe that I now understand it very completely. I found I could not rest upon it at all, until I made the whole of the subject out entirely to my satisfaction: I enclose you (marked 4) the first of a Series of papers I am making out in the different parts \([of ‘inserted]\) it. This one is the more general development of the particular case of Gravitation in pages 27, 28; & my more especial object in it has been the identification of the results arrived at in this real application, with the Mathematical Differential Co-efficient. 

I have worked most earnestly & incessantly at the Application of the Differential & Integral Calculus to the [117v] subject of Accelerating Force, & Accelerated Motion, during the last 2 or 3 weeks. It has interested me beyond everything. After making out (according to my own notions) the two papers on \( v = \frac{ds}{dt} \), and \( s = \int vdt \), (the first of which I now send, & the Second you will have in a day or two), I attacked your Chapter 8, pages 144, 145, worked out all the
Formulae there; & had excessive trouble with my third paper on $t = \int \frac{ds}{v}$, (now successfully terminated); and I am now on $f = \frac{dv}{dt}$, page 146. You will perhaps not approve my having thus run a little riot, & anticipated. But I think it has done me great good. And I am anxious to know if I may read the rest of this Chapter 8, before reading Chapter 7 on Trigonometrical Analysis; & if I am likely to understand it all without having read Chapter 7.

I shall probably write again tomorrow; or if not, certainly I shall on Tues\textsuperscript{dy}.

We are very anxious to know if there is no time between the 1\textsuperscript{st} Nov\textsuperscript{r} & the middle of Feb\textsuperscript{r}, when you & M\textsuperscript{rs} De Morgan (& family) would come & stay here for as long as you can & would like. We should be delighted if you would remain 2 or 3 weeks.

[118r] And if this should be impossible for you, perhaps still you would bring M\textsuperscript{rs} De Morgan & the children here, & remain a few days; having them to stay longer.

We both of us assure you that it would be no inconvenience whatever to us; but rather contrary the greatest pleasure. And I am certain it would do M\textsuperscript{rs} De Morgan good to be here for a time. Pray consider my proposal; at any rate for her & the children, if your own avocations should make it impossible for you even.

Believe me Yours most truly A. A. Lovelace
Dear M’ De Morgan. I send you today, 1stly:
a paper in which I have proved the Theorem
\[ \int f x \frac{dx}{dt} dt = \int f x \, dx \] (of pages 102, 103), backwards.
That is I have assumed \[ \int f x \, dx \], & deduced
from it that when \( x = \psi t \), then \[ \int f x \, dx = \int f x \frac{dx}{dt} dt \];
whereas in the book \( \int f x \frac{dx}{dt} dt \) was assumed,
& the process was exactly reversed. __
I do not send this as having any advantage
over the other proof. Merely because it happened
accidentally to strike me, & I wrote it down ;
& I now ‘may’ crossed out] enclose it for inspection, to see that I
have correctly deduced each step. __

2ndly: I enclose you my Second Paper on ‘the’ inserted] Accelerating
Force subject. This one is the explanation of
\[ S = \int v \, dt \]
Tomorrow I hope to send you the one on \( t = \int \frac{ds}{\pi} \).
[119v] I hope I am not plaguing you very much.
I am anxious to read up to a certain point
before moving to Devonshire. __
Believe me

Yours most truly
A. A. Lovelace
Dear Mr De Morgan. You have received safely I hope my packet of yesterday, & my packet sent on Tuesday. I now re-enclose you the paper marked 1. There is another Integral added at the bottom. Also I have altered one or two little minutiae in the development of \( \int \frac{dx}{\sqrt{2ax + x^2}} \) above, which you had omitted to correct.

I quite understand your observations upon it, & see the mistake I had made; & which related to the Differential \( dy \), and \( d(\varphi x) \)

If \( y = \varphi x \), then \( dy = d(\varphi x) = \frac{d(\varphi x)}{dx} dx = \left\{ x \cdot \frac{d(\varphi x)}{dx} + \varphi x \cdot \frac{dx}{dx} \right\} dx = x \cdot \frac{d(\varphi x)}{dx} dx + \varphi x dx = x \cdot d\varphi x + \varphi x dx \)

Or if \( y^2 = \varphi x \), then \( dy^2 = x \cdot d\varphi x + \varphi x dx \)

or \( \frac{d(y^2)}{dy} dy = x \cdot d\varphi x + \varphi x dx \)

or \( 2ydy = x \cdot d\varphi x + \varphi x dx \), and \( ydy = \frac{1}{2}x \cdot d\varphi x + \frac{1}{2}x \varphi x dx \)

This is all now right in my head.

In \( \int \frac{dx}{\sqrt{2ax + x^2}} \) we arrive then in my corrected paper, at

\[ \int \frac{dx}{\sqrt{2ax + x^2}} = \log(x + a + \sqrt{2ax + x^2}) \]

\[ \int \frac{dx}{\sqrt{2ax + x^2}} = \log \left( \frac{x}{2} + \frac{a}{2} + \frac{\sqrt{4ax + x^2}}{2} \right) + \log 2 \]

which, as you observe “again with the book all but the log 2, which being a Constant, matters nothing.”

Very true; but why did you then insist the log 2 in page 116? It seems as if put in on purpose to be effaced in the parenthesis (Omit the Constant).

And it might just as well have been log 3, log 4, log (anything in the world).

As to my two papers marked 2 (& which I again return, merely for the convenience of reference), I see that in order to make them valid, as applying each to two separate & different velocities, they should be re-written (which is not worth while), & the terms of the enumerations altered as follows:

“If two quantities \( V, V' \) be respectively equal to the “Ratios \( \frac{S}{T}, \frac{S'}{T'} \), and if \( V : V' = T : T' \), then the values “\( S, S' \) must be to each other as the squares of \( T, T' \)
“are to one another” &c, &c

At last I believe I have it quite correctly. __

As for \( \frac{dy}{dx} = \frac{y}{x} \), I see my fallacy about \( \frac{y}{x} \) being a fixed quantity.

About page 113, “The first form becomes impossible when \( x \) is greater than \( \sqrt{c} \), for &c”, I fancy I had a little misunderstood the mathematical meaning of the words impossible quantity. I have loosely interpreted it as being equivalent to “an absurdity”, or at least to “an absurdity, unless an extension be made in the ordinary meaning of words”. And in this instance I perceived that if the Logarithm be an odd number, there would be no absurdity even without extension in the meaning of terms; because that it would then merely imply a negative Base; which negative Base, would I think be admitted theoretically (tho’ inconvenient practically) on the common beginner’s instruction on the Theory of Logarithms. __ Am I right? __

By the bye this subject reminds me that I think I find a mistake in page 117, line 13 (from the top)

“\((n \text{ an integer}) \int_{-a}^{+a} x^n dx = 0 \text{ when } n \text{ is odd, } = \frac{2a^{n+1}}{n+1} \text{ when } n \text{ is even}”

It seems to me just the reverse, thus:

\[ \int_{-a}^{+a} x^n dx = \frac{a^{n+1}}{n+1} - \left(\frac{(-a)^{n+1}}{n+1}\right) = \frac{a^{n+1} - (-a)^{n+1}}{n+1} = \]

or 0 if \( n + 1 \) be even

(I now see it while working; for if \( n + 1 \) be even, \( n \) must be odd.)

and vice versa.

So I need not trouble you upon this; as I have solved my difficulty whilst stating it. I had only looked at this Integral in [‘a’ inserted] great hurry, this morning.

I hope on Sunday to send you two remaining papers I have to make out, on the Accelerating Force subject;

upon \( f = \frac{dv}{dt} \), and \( v = \int f dt \)

I think I have been encouraged by your great kindness, so as to give you really no Sinecure just at this moment.
Yours most truly

A. A. Lovelace
Dear Mr. De Morgan. I have rather a large batch now for you altogether:

1stly: I am in the middle of the article on Negative & Impossible Quantities; & I have a question to put on page 134, (Second Column, lines 1, 2, 3, 4, 5 from the bottom)

\[(a + bk)^{m+nk} = \varepsilon^A \cos B + k \varepsilon^A \sin B &c\]

I have tried a little to demonstrate this Formula; but before I proceed further in spending more time upon it, I think I may as well ask if it is intended to be demonstrable by the Student. For you know I sometimes try to do more than anyone means me to attempt. I have as yet only got thus far [something crossed] with the above formula: If in \((a + bk)^{m+nk}\),

\[r \text{ is given } = \sqrt{a^2 + b^2}, \text{ ['and' inserted]} \tan \theta = \frac{b}{a}; \text{ then } \sin \theta = b \cos \theta = a\]

and \((a + bk)^{m+nk} = (\cos \theta + k \sin \theta)^{m+nk} = \varepsilon^{k(m\theta)} \times \{\varepsilon^{k(n\theta)}\}^k\]

or \((\cos .m\theta + k \sin .n\theta) \times (\cos .n\theta + k \sin .m\theta)^k\), and

[123v] I dare say that from some of these transformations, the Second Side of the given equation, with the determination of \(A\) and \(B\), may be deduced. But it appears to me [‘it must be’ inserted] a very complicated process; & therefore I should like to know before I undertook it, that I was not wasting time [‘in’ deleted] doing so.

2ndly: I am plagued over page 135 of the Calculus. It is not that there is any one thing in it which I do not clearly see. But it is the depth of the whole argument which I cannot manage to discover. I should say that whole argument from “We now know &c” page 134, to “We can therefore take a function, “which, for a particular value of \(x\), &c, &c” page 135.

It seems to me all to be much ado about nothing: and I do not see what is arrived at by means of it [something crossed out]. A very complicated process appears to be used in the 1st Paragraph of page 135, to prove that when \(h\) is small then the Increment in \(\varphi x\) is very nearly represented by \(\varphi \prime a + h\), which was
already shown in page 134. And then suddenly in the Second Paragraph the Formula $\varphi a + \varphi' a(x - a) + \frac{\varphi'' a(x - a)^2}{2}$ is introduced, & I do not understand a quoi bon the closing conclusion drawn from it.

3rdly: I am not sure that I agree to what you say in preference (for ascertaining Maxima & Minima) of the direct ascertainment of the value of $\varphi' x$, over the ordinary method. Because it seems to me in many cases impossible after you have determined 0 or $\alpha$ values of $\varphi' x$, to determine further that the sign does change at them & how it changes, unless by means of the ordinary rule. I have written out and enclosed an example from Peacock, in which unless I had used the ordinary rule, after I had determined 0 values for $\varphi' x$, I should have been at my wits’ end how to bring out the conclusion.

4thly: I send you a little Maxima & Minima Theorem of my own, which occurred to me by accident; It is for $\varphi x = x^2 - mx$. After proving it by the Differential Calculus, I have given a direct proof of another sort. I merely wrote this [‘direct proof’ inserted], because it [something crossed out] occurred to me; but it gave me a great deal of trouble, & I think was rather a work of supererogation; but I believe it is quite correct. You will find enclosed in the same sheet the demonstration of “What is the number whose excess above it’s [sic] Square “Root is the least possible?” (see page 133 of the Calculus); and on the reverse side of this latter [something crossed out] is the “verification round the 4 Right Angles” for the continual increase together of $x$ and it’s [sic] tangent (See page 132). But here I have something further to add. In this Chapter VIII, we hear of Differential Co-efficient which become $= 0$, or $= \alpha$. In this very [124v] instance, $1 + \tan^2 x$ is alternately $= 1$, and $= \alpha$. Now according to my previous ideas, the terms Differential Co-efficient was only applied to some finite quantity; and referring to pages 47, 48, where one acquired one’s first ideas of a Differential Co-efficient, I think it is there clearly explained that the term is only used with reference to a finite limit. But in this Chapter VIII, there seems to be a considerable extension of meaning on
the subject.

5thly. In page 132, it is very clearly deduced that
the Ratio of a [something crossed out] Logarithm to it’s [sic] number is increasing
as long as \( x < \varepsilon \), and afterwards decreases.

The proof is most obvious. But, unluckily, the
conclusion seems to me to be contrary to the fact; at
least the first half of the conclusion, not the latter half.

On this principle: from the very nature of a
Logarithm, it is obvious that (\( x \) being \( \varepsilon \)), for
equal increments to the log, \( x \), there will be
larger & larger Increments to \( x \). The one being in
arithmetical, the other in geometrical progression.
Therefore clearly the Ratio of the Logarithm to the
number, is a diminishing one. But then the
same thing seems to me to apply [something crossed out] when
\( x < \varepsilon \). Surely there is then just the same
[125r] arithmetical & geometrical progression for equal
Increments of the Logarithms. I suppose there is
some link that I have over-looked.

I send you two Integrations worked out. They
are from Peacock. I in vain spent hours over the
one marked 2, of which I could make nothing by
any method that I devised; until in despair, I
looked thro’ your Chapter XIII to see if I could there
find any hints; & accordingly at page 277, I
found a general formula which included this
case. But I do not believe I should ever have
hit upon it by myself. The Integral marked 1,
might of course be proved also in the same way;
tho’ [‘my’ crossed out] the method [‘I have used’ inserted] is sufficient in this instance.

I have written out no more papers on
Forces. In fact there is only one more that is
left for me, viz: \( f = v \frac{dv}{ds} \). And for this I see
no occasion; for I am sure that I must thor’ly
understand it, after all I have written.

I quite see [‘the truth’ inserted] your remarks on my having treated
Acceleration of Velocity as being identical with
Force; whereas, (as I now understand it), it is
simply the measure of Force, & our only way of
getting at expressions for this latter. On the subject
[125v] of \( v^2 = 2 \int fds + C \); I have considered it a great
deal; and any direct demonstration of it, after the
manner of my other papers, seems to me to be quite impracticable. Neither \( \int vdv \), nor \( \int f.dS \)
[something crossed out] now appear to me to have any actual proto-types in the real motion. Here then suggests itself to me the question: “then are there certain truths & conclusions which can be arrived at by pure analysis, & in “no other way?” And also, how far abstract analytical expressions must express & mean something real, or not. In short, it has suggested to me a good deal of enquiry, which I am desirous of being put in the way of satisfying.

By the bye, I fear that one little paper of mine dropped out of the last packet. It was a little pencil memorandum on [‘the meaning of’ inserted] \( \int f.dS \); & there were remarks upon it, (if you remember) in my accompanying letter. It bears upon the above question. I could write it out again, if it has been lost.

Is not this a budget indeed?

Yours most truly

A. A. Lovelace
Dear Mr De Morgan. I have more to say to you than ever; (beginning with many thanks for your bountiful replies to my last packet).

I will begin with the Article on Negative & Impossible Quantities, on which I have a good deal to remark. I have finished it; & I think with on the whole great success. I need scarcely say that I like it parenthetically. I enclose you the demonstration of the formula \((a + bk)^{m+nk} = \varepsilon^A \cos B + k.\varepsilon^A \sin B\), which I found exceedingly easy, after your observations.

I should tell you that the allusions to the Irreducible Case of Cubic Equations in this Article, has so excited my curiosity on the subject, that I have attacked the chapter on Cubic Equations in [page 47 inserted] of R. Murphy’s treatise on the Theory of Equations (Library Useful Knowledge), hoping there to gain some light on the subject. For I know not to what exactly this alluded, (my Algebra wits, as you say, not having been quite proportionally stretched with some of my other wits). I have got thro’ the first two pages; and [127v] shall have to write you some remarks upon these, either in this letter, or in one as soon as possible.

But as yet I meet with nothing about \(\sqrt{(a + b\sqrt{-1})} + \sqrt{(a - b\sqrt{-1})}\)

I hope I shall be able to understand the rest of the Chapter.

At the bottom of my demonstration of \((a + bk)^{m+nk}\), you will find a memorandum (simple as to the working out) of the formula \(\cos(a + bk)\), see page 137 of the Cyclopedia. You there say that such a formula may be interpreted by it’s [sic] identical expression on the Second Side. That is to say I imagine that the meaning of \(\cos(a + b\sqrt{-1})\), which (as before pointed out in the case of the line \(h\)) is a misapplication of symbols, may be got at thro’ an examination of the results arrived at by [‘the application of’ inserted] symbolical rules to this unmeaning or mis-meaning expression. That if in a calculation, such an answer as \(\cos(a + b\sqrt{-1})\) were worked out, the answer means in fact
the remaining side of a parallelogram in which
\[ \cos \alpha \frac{\sqrt{b^2 + \epsilon^2} - \sqrt{b^2 - \epsilon^2}}{2} \] is a diagonal, and
\[ \sin \alpha \frac{\sqrt{b^2 - \epsilon^2} - \sqrt{b^2 + \epsilon^2}}{2} \] the other side: the diagonal itself being a 4th Proportional
to 1, \[ \cos \alpha \frac{\sqrt{b^2 + \epsilon^2} + \sqrt{b^2 - \epsilon^2}}{2} \text{ inclined to 1 [that is to the Unit-Line] as the \cos \alpha \text{ is}}; \]
& the remaining side being a 4th Proportional to 1, \[ \sin \alpha \frac{\sqrt{b^2 - \epsilon^2} + \sqrt{b^2 + \epsilon^2}}{2} \text{ inclined to 1 at an angle equal to the sum of a} \]
[128r] Right-Angle and the angle made by \sin \alpha \text{ with the Unit-Line.}

I enclose you an explanation I have written out (according to the Definition of this Geometrical Algebra), of the two formulae for the Sine and Cosine. I am at work now on the Trigonometrical Chapters of the Differential Calculus.

I do not agree to what is said in page 119 [of the Calculus] that results would be the same whether we worked [something crossed out] algebraically with forms expressive of quantities or not. It is true that [in' inserted] the form \( a + \sqrt{m} - \sqrt{n} \), if \(-1\) be substituted for \(m\) and \(n\), the results come out the same as if we work with \(a\) only, but were the form \( a + \sqrt{m}, a - \sqrt{m}, a \times \sqrt{m} \), or fifty others one can thin of, surely the substitution of \(-1\) for \(m\) will not bring out results the same as if we worked with \(a\) only; and in fact can only do so when the impossible expression is so introduced as to neutralize itself, if I may so speak. I think I have explained myself clearly. __

It cannot help striking me that this extension of Algebra ought to lead to a further extension similar in nature, to Geometry in Three-Dimensions; & that again perhaps to a further extension into some unknown region, & so on ad-infinitum possibly. And that it is especially the consideration of an angle = \( \sqrt{-1} \), which should lead to this; a symbol, which when it appears, sees to me in no way more [128v] satisfactorily accounted for & explained than was formerly the appearances [which’ inserted] Bombelli in some degree cleared up by showing that at any rate they (tho’ in themselves unintelligible) led to intelligible & true results. You do hint in parts of page 136 at the possibility of something of this sort. __
I enclose you also a paper I have written explaining a difficulty of mine in the Definitions of this Geometrical Algebra.

It appears to me that there is no getting on at all without this Algebra. In the 3rd Chapter of your Trigonometry (which I have just been going thro’), though there are no impossible quantities introduced; yet how unintelligible are such formulae as $2ac \cos B$, $a \sin B$, or any in short where lines are multiplied into lines, if one only takes the common notion of a line into a line being a Rectangle.

I cannot send more today; but I have many other matters to write on; especially the Logarithmic Theory at the end of the Article. I am considering it very carefully; & studying at the same time the Article on Logarithms in the Cyclopedia. And I believe I shall have much to say on it all.

The passage I wanted to ask you about in Lamé’s 1st Vol, is pages 54, 55, 56, on the Resultant of the pressures of a liquid on a vase. I want to know if I ought to understand these three pages, or if they entail some knowledge of mathematical (especially of trigonometrical) application to Mechanics, which I do not yet possess.

I hope you receive game regularly.

Yours most truly

A. A. Lovelace

P.S. Did you ever hear of a Science called Descriptive Geometry? I think Monge is the originator of it.
[130r] Dear M' de Morgan

I have for the last fortnight ['been' inserted] daily intending to write to you on mathematical matters ; & now I do not think it worth while because I have an idea of being in Town on Tuesd'y next for a day or two. And if could see you on Wedd'y, either by your kindly coming to me [130v] at 12 o’clock that day, or by my going to spend Wedd'y Evg in Gower St, it would answer the purpose much better. Besides having a list of ['particular,' inserted] little things to ask you about ; I am now anxious to consult you again as to my general progress & way of going on. I have one or two little difficulties just now.

I believe we shall remain on here for some weeks longer. My intended journey to Town is only on particular business. And by the bye it is not to be known that I am going. My mother [131r] even has no idea of it ; & I do not wish that she should. So I will thank you & Mrs De Morgan to mention nothing about it to any one.

How is she? And when to be confined? ___

Yours most truly

A. A. Lovelace

Ashley-Combe
Porlock
Somerset
Wedd'y 27th Octr ['1841' added by later reader]
Dear Mr De Morgan. As I find my journey to Town is extremely uncertain, & may possibly even not take place at all, I will trouble you without further delay on the more important of my present points of difficulty.

I will begin with those relating to Chapter 9th of the Calculus, which I am now studying. I have arrived at page 156.

page 132 : (at the bottom). I make \[ u = \cos^{-1} \left( \frac{1 - \varepsilon^2(C-x)}{\varepsilon^2(C-x) + 1} \right) \]

instead of \[ u = \cos^{-1} \left( \frac{\varepsilon^2(C-x)-1}{\varepsilon^2(C-x) + 1} \right) \]

I enclose a paper with my version of it.

page 153 : “For instance, we should not recommend “the student to write the preceding thus, \[ d^2u + d^2x.du = 0, \]

“tho’ is it certainly true that upon the implicit “suppositions with regard to the successive Increments, “\[ \Delta^2 u.\Delta x + \Delta^2 x.\Delta u \]
diminishes without limit as compared “with (\(\Delta x\))^3.” Why this comparison with (\(\Delta x\))^3?

[132v] Had the expression been \[ \frac{\Delta^2 u.\Delta x + \Delta^2 x.\Delta u}{(\Delta x)^3} \] instead of \[ \Delta^2 u.\Delta x + \Delta^2 x.\Delta u \], it would then be clear that if the Numerator diminished without limit with respect to the Denominator, the fraction itself would approach without limit to \(0\). But as it is, I see no purpose answered by a comparison with (\(\Delta x\))^3.

Also, I not only do not see the object of this comparison, but I do not perceive the fact itself either.

Where is the proof that \[ \frac{\Delta^2 u.\Delta x + \Delta^2 x.\Delta u}{(\Delta x)^3} \] does diminish without limit with respect to (\(\Delta x\))^3?

Page 135 : (at the top) : There is a slight misprint \[ C = K^2 + K^{12} \] instead of \[ C = K^2 + K^2 \]

Page 156 : (line 9 from the top) : \[ u = C.\sin \theta + C'.\cos \theta + \frac{1}{8}\theta.\sin \theta \]

(Explain this step?)

Now I cannot “explain this step”.

In the previous line, we have :

(1) ... \[ u = C'\sin \theta + C'' \cos \theta + \frac{1}{8}\theta.\sin \theta + \frac{1}{4} \cos \theta \] (quite clear)

(2) ... And \[ u = \cos \theta - \frac{d^2u}{d\theta^2} \] (by hypothesis)
= \frac{1}{4} \cos \theta + \left( \frac{3}{4} \cos \theta - \frac{d^2 u}{dx^2} \right)

whence one may conclude that

C. \sin \theta + C' \cos \theta + \frac{1}{2} \sin \theta = \frac{3}{4} \cos \theta - \frac{d^u}{dx^2}

But how $u = C \sin \theta + C' \cos \theta + \frac{1}{2} \theta$ is to be deduced

[133r] I do not discover: By subtracting $\frac{1}{4} \cos \theta$ from both sides of (1), we get

$u - \frac{1}{4} \cos \theta = C \sin \theta + C' \cos \theta + \frac{1}{2} \theta$

But unless $\frac{1}{4} \cos \theta = 0$, (which would only be the case I conceive if $\theta = \frac{\pi}{4}$), I do not see how to derive the equation in line 9 of the book.

Page 156: Show that $\frac{d^2 u}{dx^2} - u = X$ (a function of $x$) gives $u = C \varepsilon^x + C' \varepsilon^{-x} + \frac{1}{2} \varepsilon^x \int \varepsilon^{-x} X. du - \frac{1}{2} \varepsilon^{-x} \int \varepsilon^x X. dx$

I have,

$\frac{d^2 u}{dx^2} - u = X$  
$u = K \varepsilon^x + K' \varepsilon^{-x}$

$\frac{du}{dx} = K \varepsilon^x + K' \varepsilon^{-x} + \frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x}$

Assume $\frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x} = 0$

Then $\frac{du}{dx} = K \varepsilon^x + K' \varepsilon^{-x}$, and $\frac{d^2 u}{dx^2} = K \varepsilon^x + K' \varepsilon^{-x} + \frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x}$

$X = \frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x}$

$0 = \frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x}$

which tell nothing at all as to the values of $\frac{dK}{dx}, \frac{dK'}{dx}$ of $K, K'$

If we had

$X = \frac{dK}{dx} \varepsilon^x - \frac{dK'}{dx} \varepsilon^{-x}$

$0 = \frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x}$

the expression in the book will be then at once deduced.

[133v] But I do not see how to get these two latter equations co-existent.

I enclose an attempt of mine, making the assumed to be $\frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x} = x^2$ instead of $= 0$;

and also [one inserted] making this relation to be $K + K' = x^3$, but which latter I found led to such very complicated results that I proceeded but a little way, thinking it a probable loss of time to go on.

With the relationship $\frac{dK}{dx} \varepsilon^x + \frac{dK'}{dx} \varepsilon^{-x} = x^2$, I am as unsuccessful as with $= 0$.

I defer to another letter some other difficulties of mine not relating to this Chapter, but partly to some remaining points in the 8th Chapter, & partly to miscellaneous matters.
I hope Mrs De Morgan & the “large boy” continue to flourish. So Mrs De M has beat the Queen in the race, out & out! 

Yours most truly

A. A. L.
Dear Mr. De Morgan. I hope you intend to christen the “large boy” by the name of Podge, with which I am particularly pleased.

I am much obliged by your letter. I send a corrected version (now I believe quite right) of \( \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - u = X \); on my assumed supposition \( \frac{\mathrm{d}K}{\mathrm{d}x} e^x + \frac{\mathrm{d}K'}{\mathrm{d}x} e^{-x} = 0 \). As for my other assumption \( K + K' = x^3 \), it is so complicated a one that I have not thought it worth while to pursue it’s [sic] development.

I cannot think how I could be so negligent as to forget that \( e^{-x} \) is a function of \((-x)\) which is itself a function of \( x \). A complete oversight; as indeed most of the enquiries in my last letter seem to have been.

I should perhaps mention that lately [something crossed out] I have had my mind a good deal distracted by some circumstances of considerable annoyance & anxiety to me; & I have certainly studied much less well & more negligently in consequence. Indeed the last few weeks I have not at all got on as I wished and intended; & I find that to force myself, (when disinclined & distraite), beyond a certain point is very disadvantageous. So on these occasions I just keep gently going, without however attempting very much. I am hoping now to get a good lift again before long; as I think I am returning to a more settled & concentrated state of mind. I mention all this as an excuse for some errors & over-sights which I conceive are more likely just at present to creep into my performances than would usually be the case.

Now to business: Chapter VIII:

1. I send you two Problems on hypotheses of my own, intended as being worked out on the model of those in page 150. There are three different Hypotheses. In the one where I obtain \( t = \frac{1}{\sqrt{2}} \int \frac{ds}{\sqrt{x^{-1} - a^{-1}}} \) I have not attempted to develop this Integral further. Perhaps I ought to have done so; but it was only my object to get quite a general expression.

2. Page 141; (lines 9, 10, from the bottom): Series in page 116 (of Chapter VI), it was shown that \( \int \frac{ds}{\sqrt{2k^2 - x^2}} = \sin^{-1} \frac{2 - k}{x} = \)
\[ (v \sin^{-1} \frac{x}{2}) + \left( \frac{w}{2} \right), \] I do not see how it can be said (page 141) that the Constant may have any value \( P \).

3. I have never succeeded in properly understanding the Paragraph beginning page 134, ending page 135, on which I before applied to you; & the paragraph of page 148 – (marked 2) – has only added to my mistiness on the subject. There is something or other which I cannot get at in the argument & it's [sic] objects. That of page 135 seems very like another way of arriving at Taylor’s Theorem. The expression taken in line 25 from the top, I conclude to be arrived at as follows:

Having obtained \( \varphi a + \varphi' a.(x - a) \); a function agreeing in value and diff-co with \( \varphi x \) when \( x = a \), let us now find a function agreeing not only in these two points but also in second diff-co with \( \varphi x \), when \( x = a \); (the same conditions being continued of \( \varphi' a, \varphi'' a \));

We see therefore that \( \varphi' a \) must be of the form \( \varphi'' a.x + m \) where \( m = \varphi' a - \varphi'' a.a \)

Substituting this in \( \varphi a + \varphi' a.(x - a) \) we have

\[
\varphi a + (\varphi'' a.x + m)(x - a) = \varphi a + (\varphi'' a.x - a + \varphi' a)(x - a) = \varphi a + \varphi' a.(x - a) + \varphi'' a(x - a)^2
\]

Similarly we may obtain \( \varphi a + \varphi' a.(x - a) + \varphi'' a.(x - a)^2 + \varphi''' a.(x - a)^3 \)

(By the bye I don’t see how you get \( (x-a)^2 \) and \( (x-a)^3 \) as I make it).

But I cannot perceive what all this is for; & (as I mentioned below), paragraph 2 of page 148 has added to my blindness. I am sorry to plague you again about it. On receiving your former reply, I felt none the wiser; but determined to wait, thinking I might see it as I went on, which is often the case with difficulties.

I now proceed to some miscellaneous matters.


2. Article “Negative & Impossible Quantities” P. Cyclopaedia – page 137 “If the logarithm of two Units inclined at angles \( \theta \) and \( \theta' \) be added, (the bases being inclined at angles \( \varphi \) and \( \varphi' \)); the result is the logarithm of a Unit inclined &c, &c” I cannot develop this; but I enclose some remarks upon it.

3. In the treatise you sent me on the “Foundation of Algebra”,
I cannot make out ['in' inserted] the least [something crossed out] (page 5), about the general solution of $\varphi^2x = ax$. I suspect I do not understand the notation $f^{-1}x$. I quite understand $f^2x$ or $\varphi^2x$, $f^n x$ or $\varphi^n x$. Judging by analogy, from page 82 of the Differential & Integral Calculus, (where $\Delta^{-1}x$ is explained), I conceive $f^{-1}x$ or $\varphi^{-1}x$ may mean “the quantity which having had an operation $f$ or $\varphi$ performed “with & upon it, is $= x$.” But I have considered much over the last half of this page 5, & I can’t understand it. __

I have one or two other matters still to write about; but they do not press; & this is plenty I think for today. __ Pray congratulate Mrs De Morgan on the arrival & prosperity of Podge. __

Yours most truly

A. A. L.
Dear Mr. De Morgan,

In consequence of a sudden decision that I must go to Town tomorrow, I write without delay to beg for a line in St. James’ Sq., if you will kindly fix any hour on Sat’day between nine & three o’clock, that may be most convenient to you to call on me there. I am very anxious to see you; & hope it may not be very inconvenient to you.

I had not the least idea till an hour ago, that I should go to Town this week. It is in consequence of letters unexpectedly received.

I will just again mention that I do not wish my journey to be known; & I have told no one except those necessarily concerned.

In much haste

Yours most truly

A. A. Lovelace
Dear Mr. De Morgan. What will you say when I write that my intended journey is again delayed? & cannot now take place before Sunday or Monday week. I fear you will never trust me again; but I must beg to assure you that this uncertainty has been quite unavoidable.

Everything was arranged last night for our setting out at six this morning; when unexpected circumstances obliged us to delay once more [‘fulfilling’ inserted] our intentions. I shall really feel quite ashamed next time, to write to you on a similar appointment; altho’ I do not think it is possible that another put-off should occur.

May I hope meanwhile to receive a reply to my last letter of Monday. I am going on very well indeed I believe now, with my studies; next week I expect a few interruptions however from some engagements I have made.

With many apologies, pray [139r] believe me

Yours most truly

A. A. Lovelace
My Dear Mr De Morgan.

I am to be in Town (for as certain as any human affairs can be), next week. May I hope to see you on Tuesday; any hour you like ['to fix' inserted] between 12 & 4 o’clock?

Thanks for yesterday’s packet. I shall probably send you a Budget in a day or two again; as it will save time when I see [140v] you. For I have so many Memoranda down to ask about, that it will be best all the principal things should just be noted to you first. Then we shall know at once where & what to begin upon. 

I am in the middle of Operation; indeed nearly thro’ it; & find no difficulties scarcely; at least none of any moment.

I see I have made a bother of $u = C\varepsilon^x + C'\varepsilon^{-x} + \frac{1}{2}\varepsilon^x \int \varepsilon^{-x} X \, dx - \frac{1}{2}\varepsilon^{-x} \int \varepsilon^x X \, dx$ [141r]; not from any mis-comprehension of it; but from too much haste in writing out. As I will explain in my promised packet. 

Yours always
Most truly
A. A. L

Ashley-Combe
Wed
Mr Dear M’r De Morgan. [something crossed out] I said \textit{Wed}dy. At least I meant to do so. On Tuesday I have already an engagement in the morning. Perhaps you have written Tuesday by mistake. But of you cannot come on \textit{Wed}dy, then I must put off my Tuesday’s engagement, that I may see you then. If it is the same to you however, I should much prefer \textit{Wed}dy. Can you kindly give me one line tomorrow to say which it is to be. I shall get [‘it’ inserted] in the evening in St James’ Sq. Now I proceed to business:\n
1\textsuperscript{stly}: You have mistaken my intentions I think about the formulae of pages 155, 156. My enclosures 1 & 2 will explain. ___

2\textsuperscript{ndly}. Enclosure 3 contains the demonstration of “Exercise” page 159

3\textsuperscript{rdly}. Enclosure 4 “Exercise” page 158

4\textsuperscript{thly}: About the Constant in page 141 : I still am unsatisfied. I perfectly understand that “any value” consists with everything in page 141. The principle is I conceive exactly the same as that by which in page 149, \( y = a + \sin x \) instead of \( y = \sin x \).

I only mean that this result seems inconsistent with page 116 when it is shown that the Constant must = \( \frac{m}{2} \). ___

5\textsuperscript{thly}: page 161, (line 14 from the top):
\[
\varphi''(x + \theta h, y + k) - \varphi''(x + \theta h, y) = \varphi''_1(x + \theta h, y + vk).k
\]

\( v < 1 \)

Why is \( v \) introduced at all? ___

I have as follows:
\[
\frac{\varphi''(x + \theta h, y + k) - \varphi''(x + \theta h, y)}{k} = \varphi''_1(x + \theta h, y)
\]

if \( k \) diminishes without limit ; \( (k \) being = \( \Delta y \))
or \( \varphi''(x + \theta h, y + k) - \varphi''(x + \theta h, y) = \varphi''_1(x + \theta h, y)k \)

But I do not see how \( v \) comes in. ___

6\textsuperscript{thly}: I have several remarks to make altogether on the Article Operation. I will only now subjoin two. I believe on the whole that I understand the
Article very well.
See page 443, at the top, (2nd Column):

\[ \varphi^2 + 2\varphi\psi + \psi^2, \text{ or } (x^2)^2 + 2(x^3)^2 + (x^3)^3 \]

should be it appears to me \( \varphi^2 + 2\varphi\psi + \psi^2, \text{ or } (x^2)^2 + 2x^3x^3 + (x^3)^2 \)

or \( (x^2)^2 + 2(x^3)^2 + (x^3)^2 \)

\[ = (x^2)^3 + 3(x^3)^2 \]

[143r] I only allude to \( (x^3)^3 \), instead of \( (x^3)^2 \) as I make it.

See page 444, at the bottom, (2nd column):

“Where \( B_0, B_1, \& c \) are the values of \( fy \) and its
“successive diff-co’s [sic] when \( y = 0, \& c, \& c \)”
Surely it should be when \( y = 1 \).

The same as when immediately afterwards, (see page
445, 1st column, at the top), in developping [sic] \( (2 + \Delta)^{-1} \varphi x; \)
\( B_0, B_1 \& c \) are the values of \( fy \& its Co-efficients \)
when \( y = 2, \& c, \& c \).

I have referred to Numbers of Bernoulli
& to Differences of Nothing; in consequence of
reading this Article Operation. And find that
I must read that on Series also.

I left off at page 165 of the Calculus; &
suppose that I may now resume it; (when I return
here that is).

I will not trouble you further in this letter.
But I have a formidable list of small matters
down, against I see you.

Yours most sincerely
A. A. Lovelace
Dear Mr De Morgan, I am going on well; [‘quite’ inserted] as I could wish. I have done much since I saw you; & you will have all the results of the last few days in good time. I enclose you now two papers; one on $f = \frac{dv}{dt}$, the other on $\int_a^b f \, dt$.

You will have next those on $v \frac{dv}{dt} = f$, and $v^2 = 2 \int f \, ds + C$. This latter I think I have succeeded in analysing to my mind.

I have [‘now’ inserted] two observations to make: [something crossed out]

1stly: I think I have detected a slight error in one of my former papers, that on $t = \int \frac{ds}{v}$. I return it for reference. In order in the [something crossed out] Summation

\[
\left\{ \frac{1}{\varphi s} + \frac{1}{\varphi (s+ds)} + \cdots + \frac{1}{\varphi (2s)} \right\} \, ds
\]

to end with $\frac{1}{\varphi (2s)}$, I should have begun with $\frac{1}{\varphi (s+ds)}$ not with $\frac{1}{\varphi s}$.

If the time elapsed during the first fraction of Space [144v] (starting from $s$) were [‘made’ inserted] $= \frac{1}{\varphi s}$, then the time for the last of the Fractions necessary to complete up to $2s$, would be $\frac{1}{\varphi (2s)}$ and not $\frac{1}{\varphi (2s)}$ which it ought to be.

I don’t know that this affects the correctness of the ultimate limit of the Summation. But here, where the Summation itself is made to represent a hypothetical movement, it is clearly wrong.

The error is avoided in the former paper I had written on $s = \int v \, dt$, which I likewise return to refer to this Point.

2ndly: In considering a priori the Integral $\int f \, ds$,

I am inclined still to adhere to my original opinion (expressed in the pencil Memorandum I showed you & [‘which I’ inserted] now return). I should premise that I now mention this merely as a curious subject of investigation, not because it is concerned in the [something crossed out] papers I am making out upon $v^2 = 2 \int f \, ds + C$, in which I have avoided the direct consideration of $\int_a^b f \, ds$. ___
I am disposed to contend that tho’ \( ds \) does here represent Space, that still the \( ds \) fraction of any one of the terms of the Summation, say \( \varphi(a + n.ds)ds \) means the same fraction of \( \varphi(a + n.ds) \) which \( ds \) is of [145r] a Unit of Space; & therefore that since \( \varphi(a + n.ds) \) represents Force, (or ['uniform' inserted] Acceleration of Velocity for 1 Second in operation during the performance of the length \( ds \), the \( ds \) fraction of this expression must represent the ['ds part of this Force or the' inserted] actual Acceleration for \( \frac{1}{ds} \) of a Second. I treat \( ds \) as an abstract quantity. And so I conceive [something crossed out] \( dt \) must be treated in \( s = \int v.dt \), ['ds’ inserted] in \( t = \int \frac{ds}{v} \), \( dt \) in \( \int f.dt \), &c, &c. 

I should tell you that I am much pleased with the observation you added to my inverse demonstration of \( \int fx.dx \), \( \frac{dx}{dt} \), and that I quite understand ['why’ inserted] my proof can only be admissible on the Infinitesimal Leibnitzian Theory. But this theory is to my mind the only intelligible or satisfactory one. In fact, (notwithstanding it’s ['sic error], I should call it the only true one. 

By and bye, you will have some observations of mine upon Differential Co-efficients & Integrals, abstractly considered. I have been thinking much upon them. I am going on with Chapter VIII. 

By the bye, I believe you will receive somehow tomorrow [145v] a book (the 1st Vol of Lamé’s Cours de Physique) in which there is a passage which I will write to you about as soon as I find time. 

I forgot to mention it to you on Thursday ; & so have ordered the Book to be sent to you, that I might write about it sometime. 

Believe me

Yours very truly

A. A. Lovelace
Dear Mr De Morgan. I am very much obliged for your long letter. The Formula in Peacock comes out quite correct now that I have written diff. co of \((b - x)^4 = 4(b - x)^3 \times (-1) = -4(b - x)^3\).

It is odd that notwithstanding the caution you gave me in Town on this very point, I should have fallen into the trap. There is nothing like one’s own blunders after all for instruction. I do not however understand why example (19) page 4, has not come out wrong also in my working out. I enclose a copy of my solution, and it appears to me it ought to be wrong, because I surely should have had diff. co of \((1 - x)^4 = 4(1 - x)^3 \times (-1) = -4(1 - x)^3\), whereas I have diff. co of \((1 - x)^4 = 4(1 - x)^3\).

On looking over my development again very carefully, I am inclined to think that my solution \([146v] \frac{(1+x)^2}{(1-x)^3} \times (7 + x)\), comes out right only because I have managed to make another blunder of a sign in the course of the proofs, which has corrected the first blunder. I therefore now write on the other side of the paper, what I think it should be.

The note in page 2 I do not imagine to be of any consequence. It is on “rendering the Differentiation of complicated Functions sometimes much easier” by means of three Theorems from Maclaurin’s Fluxions.

Certainly had I thought a little more upon what I read some weeks ago, before I wrote my last letter to you, I should not have sent the question about \(du = \varphi(x) \times dx\) [flourishes at tops of stems of ‘d’s, here and after]. I [‘must have’ inserted] forgot exactly what a Differential Co-efficient means, when I did so. But how is it then that in your 1st Chapter of the Differential Calculus there is no mention of the multiplication by \(dx\)?
I conclude that the real Differential Co-efficient is \( \frac{du}{dx} = \varphi(x) \), and that Peacock’s solutions are” not strictly speaking Differential Co-efficients. ”

I think pages 13 to 15 of your Elementary [147r] Illustrations bear considerably upon the observations in your letter, do they not? ___

Your explanation of Euler’s proof of the Binomial Theorem is perfectly satisfactory to me. Unluckily I have not any Book here which contains the Theory of Combinations. I wanted to refer to this when reading page 215, as I have forgotten it in it’s [sic] particulars. However this can very well wait a short time, & I have only to take the Formula for Combinations for granted meanwhile.

The necessity of the truth of \((1 + x)^n \times (1 + x)^m = (1 + x)^{n+m}\) for all values of \(n\) and \(m\), since it is true when they are whole numbers, I shall probably see more clearly at some further time. ___

I can explain exactly what my difficulty is in Chapter X. _ “For instance, if we “know that \( \varphi(xy) = x \times \varphi y \), supposing this always “true, it is true when \( y = 1 \), which gives \( \varphi(x) = “= x \times \varphi(1) \). But \( \varphi(1) \) is an independent quantity, “made by writing 1 instead of \( y \) in \( \varphi(y) \). Let us “call it \( c \ & c. \)” _

It is this substitution of 1 and of \( c \), and consequent ascertainment of the form which will [147v] satisfy the equation, which is all dark to me. It is ditto in lines 12, 13, & 14 from the top. ___

I understand quite well I believe from “We have seen that if \( \varphi x = c^2 \ & c” \), all through the next page.

That I do not comprehend at all the means of deducing from a Functional Equation the form which will satisfy it, is I think clear from my being quite unable to solve the example at the end of the Chapter “Shew that the equation “\( \varphi(x + y) + \varphi(x - y) = 2\varphi x \times \varphi y \) is satisfied “by \( \varphi x = \frac{1}{2}(a^x + a^{-x}) \). “I have tried several times, substituting first 1 for \( x \), then 1 for \( y \). but I can make nothing whatever of it, and I think it is evident there is
something that has preceded, which I have
not understood. The 2\textsuperscript{nd} example given for
practice “Shew that \( \varphi(x + y) = \varphi x + \varphi y \) can
“have no other solution than \( \varphi x = ax \)”, I
have not attempted.

I have a question to ask upon page 229.

“By extracting a sufficiently high root of \( z \), we
[148r] “can bring \( z^m \) as near to 1 as we please, or
“make \( z^m - 1 \) as small as we please ; that is
“(page 187) \( z^m - 1 \) may be made as nearly equal
“to the sum of the whole series as we please”. 

I cannot find what it is that is referred
to in page 187 ; and Secondly, it appears to me
somewhat of a contradiction that a quantity
\( z^m - 1 \) which can certainly be made as small
as we please by the diminution of \( m \), should
become as near as we please to a fixed
limit or sum (the \( \log z \) I conclude is the
sum of the series, referred to), since by continued
diminution the quantity \( z^m - 1 \) may become a
great deal less than the sum of the Series, &
keep receding from it.

To return to Chapter \( X \), there is one other
thing in it that I do not understand. Page
205, lines 5, 6, 7 from the bottom. It seems
to me fallacious to substitute first one value
0, for a letter ; & then another value, let \( y = -x \),
[148v] in the same equation & in a manner at the
same time. How can the two suppositions
consist together at all.

I go on well with the Trigonometry, &
have nearly finished the Number & Magnitude.
I think there is another Erratum in page
34 of the Trigonometry, line 13 from the bottom

\[ \frac{OM}{ON} \cdot \frac{ON}{OP} \cdot \frac{NR}{NP} \cdot \frac{NP}{NO} \text{ &c} \]

should be

\[ \frac{NR}{NP} \cdot \frac{NP}{OP} \text{ &c} \]

I am really ashamed to send you such
troublesome letters.

Believe me

Yours most truly

A. A. Lovelace
Dear Mr De Morgan. I have I believe made some little progress towards the comprehension of the Chapter on Notation of Functions, & I enclose you my Demonstration of one of the Exercises at the end of it: “Show that the equation $\varphi(x + y) = \varphi x + \varphi y$ can be satisfied by no other solution than $\varphi x = ax$.” At the same time I am by no means satisfied that I do understand these Functional Equations perfectly well, because I am completely baffled by the other Exercise: “Shew that the equation $\varphi(x + y) + \varphi(x - y) = 2\varphi x \times \varphi y$ is satisfied by $\varphi x = \frac{1}{2}(a^x + a^{-x})$ “for every value of $a$”. I do not know when I have been so tantalized by anything, & should be ashamed to say how much time I have spent upon it, in vain.

These Functional Equations are complete Will-o’-the-Wisps to me. The moment I fancy I have really at last got hold of something tangible & substantial, it all recedes further & further & vanishes again into thin air. But now for this perplexing $\varphi x = \frac{1}{2}(a^x + a^{-x})$. I believe I have left no method untried; but I cannot get further than as below, with any certainty:

$$\varphi(x + y) + \varphi(x - y) = 2\varphi x \times \varphi y$$

$$\therefore 2\varphi x = \frac{\varphi(x + y) + \varphi(x - y)}{\varphi y}$$

Since $x$ and $y$ may have any values whatever, (at least such I conclude is of course intended), let $y = 0$. We have then

$$2\varphi x = \frac{\varphi(x) + \varphi(x)}{\varphi(0)}$$

$$\therefore 2\varphi x \times \varphi(0) = \varphi(x) + \varphi(x)$$

or $2\varphi x \times \varphi(0) = 2\varphi x$

$\varphi(0)$ must = 1, or = $a^0$, since $a^0$ is the only function of 0 which can = 1

I think so far is correct in itself, but whether [150r] it be the [‘right’ inserted] road to the rest is another question.

At any rate, I have not succeeded in proving
it such. To assume that since \( \varphi(0) = a^0 \),
\( \varphi(x + y) = a^{x+y}, \varphi(x - y) = a^{x-y}, \varphi y = a^y \)
appears to me scarcely warrantable; and besides in that case it must be equally
assumed that \( \varphi x = a^x \), (there being the same
ground for the one assumption as for the others),
and we should then have,
\[
\begin{align*}
\varphi x &= \frac{a^x + a^{-y}}{a^y} \\
\therefore a^x &= a^{x+y} + \frac{1}{a^y} \\
\therefore a^x &= a^x + a^{x-2y}
\end{align*}
\]
most clearly absurd, independent of it’s [sic] being
discordant with the book.

Once I thought I had hit on something very
clever indeed, and wrote as follows :
\[
\varphi(x + y) = \varphi(x + y.1) = \{\varphi(1)\}^{x+y} \text{ by equation } (\varphi a)^n = \varphi(na)
\]
page 205, entirely forgetting that the \( \varphi \) of that
equation had nothing whatever to do with the
\( \varphi \) of any other equations; (a disagreeable truth
[150v] which did not occur to me until 24 hours
later). I then had \( \varphi(x - y) = \varphi(x - y.1) = \{\varphi(1)\}^{x-y} \)
\( \varphi(y) = \varphi(y.1) = \{\varphi(1)\}^y, \) and
\[
2\varphi x = \frac{(\varphi(1))^{x+y} + (\varphi(1))^{x-y}}{(\varphi(1))^y} = \frac{(\varphi(1))^{x+y}}{(\varphi(1))^y} + \frac{1}{(\varphi(1))^y}
= \{\varphi(1)\}^x + \{\varphi(1)\}^{x-2y}, \text{ and supposing}
\]
x = y, then \( 2\varphi x = \{\varphi(1)\}^x + \frac{1}{2}\{\varphi(1)\}^{-x} \)
or calling \( \varphi(1) = a, \varphi x = \frac{1}{2}(a^x + a^{-x}) \)
But besides my unwarrantable assumption of
\( \varphi(x + y.1) = \{\varphi(1)\}^{x+y} \), there was this in the
result which was unsatisfactory, that it was
necessary to assume \( x = y \), and the result seemed
to hold good in that case alone. Also, when to
verify, I tried \( x = 1, \therefore \varphi(1) = \frac{1}{2}(a^1 + a^{-1}), \)
which ought to have come out \( a = a \), I could
make neither head or tail of it. _Well, I
abandoned this, & tried all sorts of other resources.
[151r] I understand to work out something by means
similar to those in page 205 and in the
Problem I send; but equally unsuccessfully.
I also in equation (A) page 204, changed
\( \varphi(x + y) \) into \( \varphi(x - y) \) and investigated this,
thinking I might derive a hint possibly from it. In short, many & various are the experiments I have made, but I will not detail any more. Indeed I think you may be possibly heartily sick of what I have detailed. But I wished to show you that I have not failed from want of trying, at least; & also to give you the chance of smiling at my expense [sic].

I shall have to trouble you with another letter shortly, on other knotty points. Really I do not give you a Sinecure. Your letters are however well bestowed, in as far as the use they are of to me, can make them so, and the great encouragement that such assistance is to me to continue my Studies with zeal & spirit. We are to return to Surrey very [151v] soon. I expect to have occasion to trouble you again before we go, & after that I shall hope to see you & M* De Morgan in Town, where I intend to be for two or three days in about a fortnight.

Yours very truly

A. A. L
Dear Mr De Morgan. I am indeed extremely obliged to you for all your late communications. In two or three days more, I shall have several observations & to send you in reply to some of them.

My object in writing today, is to make another enquiry concerning the substitution of \( \varphi_1 (a + h) - \varphi_1 a \) for \( \varphi(a + h) - \varphi a \) in page 100, for I perceive on carefully examining the passage, that I do not quite understand it

\[ \varphi_1 x = \varphi x + C, \text{ which } \{ \text{last side' inserted} \} \text{ means the Primitive Function } \{ \text{of } \varphi' x \text{ inserted} \} \]

and the Primitive Function means the Function which differentiated gives \( \varphi' x \)

Therefore \( \varphi_1 (a + h) = \varphi(a + h) + C \)

And \( \varphi_1 a = \varphi a + C \)

Consequently \( \varphi_1 (a + h) - \varphi_1 a = \varphi(a + h) + C - (\varphi a + C) = \varphi(a + h) - \varphi a \)

[152v] This is my version of it. But you tell me,

\[ \varphi_1 (a + h) = \varphi(a + h) + C \]

\[ \varphi_1 a = \varphi a, \text{ (which ought to be I say} \]

\[ \varphi_1 a = \varphi a + C) \]

From which we should have,

\[ \varphi_1 (a + h) - \varphi_1 a = \varphi(a + h) + C - \varphi a \]

Consequently \( \varphi_1 (a + h) - \varphi_1 a \) is not equal to \( \varphi(a + h) - \varphi a \) as is required to be proved, but is \( = \varphi(a + h) - \varphi a + C \).

I cannot unravel this at all.

Secondly: I do not see why the Indefinite Integral only is \( \varphi x + C \) = Primitive Function of \( \varphi' x \)

The argument at the top of page 101 seems to apply equally to the Definite Integral.

As follows : It is proved that
\[ \varphi_1(a + h) - \varphi_1a = \int_a^{a+h} \varphi'x \, dx \]

\[ \varphi_1a \] is just as much here an arbitrary Constant as it is in \( \varphi_1x - \varphi_1a = \int_a^x \varphi'x \, dx \)

Therefore \( \int_a^{a+h} \varphi'x \, dx = \varphi_1(a + h) + C_1 \)

\[ = \varphi(a + h) + C + C_1 \]

\[ = \varphi(a + h) + \text{an arbitrary Constant} \]

[153r] just as with \( \varphi_1x - \varphi_1a = \int_a^x \varphi'x \, dx \)

Thirdly: With respect [‘to’ inserted] the assumption that when \( a \) is arbitrary, then any function of \( a \), say \( \varphi a \), is also arbitrary or may be anything we please, seems to me not always valid.

For instance if \( \varphi a = a^0 \), it must be always = 1. We may assume \( a = \) anything we like, but \( \varphi a \) will not in this case be arbitrary.

It is curious how many little things I [something crossed out] discover in this Chapter, which in looking back upon them, I find I have only half-understood.

I shall be exceedingly obliged, if you can answer these points soon; I think a word almost may explain them, & they rather annoy me.

Believe me, with many thanks

Yours very truly

A. A. Lovelace
Dear M’r De Morgan

It would give me great pleasure to see you again, & to have a chat with you. If therefore I am in Town on Tuesday week, (which is likely), do you think you could call on me at 11 o’clock. I should propose going to [154v] Gower St to see M’ns De Morgan, but on this occasion I should not have [something crossed out] time to go so far.

If therefore you answer me affirmatively, I shall not write again unless I should be prevented from going.

I have been doing little of late in Mathematics; just amusing myself with Murphy’s Theory of Equations (in the Useful Knowledge); & looking into one or two other little matters, rather than [155r] studying them. So I have no very diligent reckoning to make you.

My kind remembrances to Mns De Morgan.

Yours very truly

A. A. Lovelace
P. S. Excuse my materials.

Dear Mr De Morgan. In much haste, I write to tell you that I must adjourn the proposed engagement for Monday next, until Monday week; owing to an unforeseen obstacle for next Monday afternoon. You have very probably already written to me; & if so, I shall (to save you trouble) take it for granted if I do not hear, [156v] that the hour fixed will be for the Monday week instead.

I was obliged to stay in Town last night, unexpectedly; but am just setting out...

I was very glad to find Mrs De Morgan so well & cheerful, & moreover that she approved of a proposition I made her; (not a proposition of our particular kind however).

I remain

Yours very truly

A. A. L
Dear M’ De Morgan

In consequence of your kind reply to my former note, Lord L think I had better send you the paper, (which has been put into type, tho’ not published). Should you be at leisure & disposed to look at it, you will at once see how much stuff there is in it, & of what a solid quality.

Some few corrections are wanting, before it finally goes to proofs.

There is also one single sentence, (I think in page 22), about Irish Priests & Douai, which must be altered or cut out, for a Roman Catholic Editor.

Much of the Paper, tho’ on so dry a subject, is amusing enough. If you look at it, & should discover my inaccuracies or anything which might be made clearer, pray be kind enough to mention it.

I will not now detain you further.
Believe me very faithfully yrs
A. A. Lovelace
Dear Mr De Morgan. I am most particularly vexed & annoyed at finding that you have just been here, & are gone without ['my' inserted] seeing you. Owing to the message received by the carriage the other day, I had reserved this morning particularly [159v] for our affairs, & I cannot but think that either my footman must have blundered the matter & sent you away, or that there has been some misunderstanding or other. Your time is so valuable that it is without measure vexatious you should have come all the way here for nothing, & when I was especially waiting for you too. 

Now it does so happen quite by chance that I have this evening at liberty. May I come to Gower St between eight & nine therefore, & so repair the mistake of this morning? I am quite in a fuss about my mathematics, for I am in much want of a little lift at this moment; & you know how I have my progress [160v] at heart.
I write in a great hurry; & I only hope that such a contrariety will never happen again.

You must forgive my writing in such a fuss; but I cannot imagine how the thing has happened. Believe me

Yours very truly

A. A. Lovelace
Dear M' De Morgan

Yes — tomorrow

morning between \( \frac{1}{2} \) past
twelve & one o’clock —
will do. (Remember I
write Tuesday Night).

I am much provoked

at my stupid servants’
stupid blunder, but I

[161v] believe the poor man

meant to be highly
discerning & well-judging.

I was at the moment

treating for an Opera

Box for [‘the night of’ inserted] Rachel’s Benefit,

which I conclude the

footman thought a

much more important

matter than any

spectacles in the world

could [something crossed out] betoken. He is

a new man, & does not

know my people yet.

[162r] In haste

Yours most truly obliged

A. Ada Lovelace

Tuesday Night

St James’ Sqre

[162v] Augustus De Morgan Esqre

69. Gower St
Ockham

Thursday

Many thanks for the packet of yesterday morning. I shall write in reply to it tomorrow. I send today some Integral Developments (see pages 116, 117); & another of the Accelerating Force papers.

Any hour you may appoint on Thursday Morning [something torn away] I will be ready. Shall I say 10 o’clock?
When an equation involves only two variables, it is easy enough to write all differential equations so as to contain nothing but differential coefficients; thus
\[ y = \log x \quad \frac{dy}{dx} = \frac{1}{x} \]

If we prefer to write \( dy = \frac{dx}{x} \), it must be under a new understanding.

By \( \frac{dy}{dx} \), we mean the limit of \( \frac{\Delta y}{\Delta x} \), not any value which \( \frac{\Delta y}{\Delta x} \) ever can have, but that which it constantly tends towards, as \( \Delta x \) is diminished without limit.

But in \( dy = \frac{dx}{x} \), we cannot by \( dy \) and \( dx \) mean limits, for the limits are zeros; and \( 0 = \frac{0}{x} \), though very true, is unmeaning.

What then do we mean by this

When \( y = \log x \), \( dy = \frac{dx}{x} \)

we mean that \( \Delta y = \frac{\Delta x}{x} \), as \( \Delta x \) diminishes without limit, not only diminishes without limit, but diminishes without limit as compared with \( \Delta x \) or \( \Delta y \). So that, if we call it \( a \), or if

\[ \Delta y - \frac{\Delta x}{x} = a \]

Then \( a \) is useless, and we might as well write 0. For since the processes of the differential calculus always terminate in taking limits of ratios and since

\[ \frac{\Delta y}{\Delta x} - \frac{1}{x} = \frac{a}{\Delta x} \]

(or some transformation of this sort) must come at last, our limiting equation must be

\[ \frac{dy}{dx} - \frac{1}{x} = \text{Limit of} \quad \frac{a}{\Delta x} = 0 \]

The truth of every equation differentially written, as \( dy = p \, dx \), is always absolutely speaking, only approximate: but the approximation is relatively closer and closer. Understand it as if it were

\[ dy = (p + \lambda) \, dx \]

where \( \lambda \) diminishes with \( dx \), so that the error made in \( dy \) by writing \( dy = p \, dx \) is not only diminishes
with $dx$, but becomes a smaller and smaller fraction of $dx$: because $\lambda$ diminishes without limit

[169r] All this is conveniently signified in the language of Leibnitz, namely, that when $dx$ is infinitely small, $dy - p\,dx$ is as nothing (or infinitely small) when compared with $dx$, or $dy$ is (relatively to its own value) infinitely near to $p\,dx$.

The differential might easily be avoided when there are only two variables, and even when there are more, provided we only want to use one independent variable at a time. Thus

$$u = \varphi(x, y, z)$$

may give the equations

$$\frac{du}{dx} = P, \quad \frac{du}{dy} = Q, \quad \frac{du}{dz} = R$$

But when we want to make $x, y,$ and $z$ all vary together, we have no notion of a differential coefficient attached to this simultaneous variation, unless we suppose some one new variable on which $x, y,$ and $z$ all depend, and the variation of which sets them all varying together.

[169v] If this new variable be $t$, and if $x, y,$ and $z$ be severally functions of $t$, we have then

\[ \frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt} + \frac{du}{dz} \cdot \frac{dz}{dt} \]

Thus if $u = xy^2 \varepsilon^2$

\[ \frac{du}{dt} = y^2 \varepsilon^2 \frac{dx}{dt} + 2xy \varepsilon^2 \frac{dy}{dt} + xy^2 \varepsilon^2 \frac{dz}{dt} \]

But observe that this makes $x, y,$ and $z$, (which we want to be independent of one another) really functions of one another: thus if $x = t^2$, $y = \log t$, we must have $y = \log \sqrt{x}$. We might it is true avoid this by the following supposition. Let $x, y,$ and $z$, instead of being given functions of $t$, be unassigned and
arbitrary functions, which we can always make whatever functions we please. We can then really hold $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ to be independent of one another, for it is always in our power to assign them any values we like. But this method would be awkward, and would put continual impediments in our way. It is better therefore to avoid that notation which while it makes the first step by supposing relations to exist between $x$, $y$, and $z$, immediately contradict that supposing by making these relations mean any relations.

If in $\varphi(x, y, z)$ we suppose $x$, $y$, and $z$ to be simultaneously altered into $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, then $\varphi(x, y, z)$ takes the value $\varphi(x + \Delta x, y + \Delta y, z + \Delta z)$ which may be expounded as follows

$$\varphi + \frac{d\varphi}{dx}\Delta x + \frac{d\varphi}{dy}\Delta y + \frac{d\varphi}{dz}\Delta z + A\Delta x\Delta y + B\Delta y\Delta z + C\Delta z\Delta x + D\Delta x^2 + E\Delta y^2 + F\Delta z^2 + &c + &c$$

say

$$\varphi + \frac{d\varphi}{dx}\Delta x + \frac{d\varphi}{dy}\Delta y + \frac{d\varphi}{dz}\Delta z + M = 0$$

If it be required that $\varphi = \text{constant}$, or $\varphi = c$ we must have

$$\frac{d\varphi}{dx}\Delta x + \frac{d\varphi}{dy}\Delta y + \frac{d\varphi}{dz}\Delta z + M = 0$$

Now if we were to leave out $M$, and say

$$\frac{d\varphi}{dx}\Delta x + \frac{d\varphi}{dy}\Delta y + \frac{d\varphi}{dz}\Delta z = 0$$

we should of course commit an error: but it is one the magnitude of which relatively to $\Delta x$, for instance, diminishes without limit as the increments $\Delta x$, $\Delta y$, $\Delta z$, are diminished without limit. The considerations already given apply here again: because all the terms contain $[sic]$ in $M$, diminish without limit as compared with those which are not [something crossed out] contained in $M$. This rejection of all terms except those of the first order is always accompanied and
I should, if asked whether this equation is absolutely true, answer no. If then asked why I write it, I should answer that it leads to truth, and for this reason that it is more and more nearly true as \( dx \) &c are diminished: not because \( \frac{d\varphi}{dx} \frac{dx}{dx} + \&c \) diminishes in that case, though undoubtedly it does so; but because it diminishes as compared with \( dx \), &c. Hence, when we form ratios and take their limits, it matters nothing, as to the results we obtain, whether we write

1. \( \frac{d\varphi}{dx} dx + \&c = -M \)
2. \( \frac{d\varphi}{dx} dx + \&c = 0 \)

marked by writing \( dx \) for \( \Delta x \), \( dy \) for \( \Delta y \), &c.
\[
\begin{align*}
\sin 90 & = \frac{1}{4} \sqrt{3 + \sqrt{5}} + \frac{1}{4} \sqrt{3 - \sqrt{5}} + \frac{1}{4} \sqrt{3 + \sqrt{5}} + \frac{1}{4} \sqrt{3 - \sqrt{5}} = \\
\sin 30 & = \frac{1}{4} \sqrt{5 + \sqrt{5}} - \frac{1}{4} \sqrt{5 - \sqrt{5}} + \frac{1}{4} \sqrt{5 + \sqrt{5}} - \frac{1}{4} \sqrt{5 - \sqrt{5}} = \\
\sin 10 & = \frac{1}{2} \sqrt{3 + \sqrt{5}} - \frac{1}{2} \sqrt{3 - \sqrt{5}} + \frac{1}{2} \sqrt{3 + \sqrt{5}} - \frac{1}{2} \sqrt{3 - \sqrt{5}} = \\
\sin 50 & = \frac{1}{4} \sqrt{2} + \frac{1}{4} \sqrt{7 - \sqrt{5}} = \frac{1}{2} \sqrt{3 + \sqrt{5}} - \frac{1}{2} \sqrt{3 - \sqrt{5}} = \\
\sin 70 & = \frac{1}{4} \sqrt{2} + \frac{1}{4} \sqrt{7 - \sqrt{5}} = \frac{1}{2} \sqrt{3 + \sqrt{5}} + \frac{1}{2} \sqrt{3 - \sqrt{5}} = \\
\sin 60 & = \frac{1}{4} (1 + \sqrt{5}) \\
\sin 20 & = \frac{1}{4} (-1 + \sqrt{5}) \\
\frac{1}{4} (1 + \sqrt{5}) = \frac{1}{4} (-1 + \sqrt{5}) + \frac{1}{2} \\
1 + \sqrt{5} = -1 + \sqrt{5} + 2 = 1 + \sqrt{5} \\
\sin a = \pm \sqrt{\frac{1}{2} R^2 - \frac{1}{2} \cos a} = \sqrt{\frac{1}{2} R^2 + \frac{1}{2} R \sqrt{1 - \sin^2 a}} = \\
= \pm \frac{1}{2} \sqrt{R^2 + R \sin a} + \frac{1}{2} \sqrt{R^2 - R \sin a} \\
\frac{1}{2} R^2 \pm \frac{R}{2} \sqrt{1 - \sin^2 a} = \frac{1}{4} (R^2 + R \sin a) + \frac{1}{4} (R^2 - R \sin a)
\end{align*}
\]
\[-2 \times \frac{1}{2} \times \frac{1}{3}\sqrt{R^2 + R \sin a} \times \sqrt{R^2 - R \sin a}\]

\[\frac{1}{2} R^2 \mp \frac{R}{2} \sqrt{1 - \sin^2 a} = \frac{1}{3} R^2 + \frac{1}{3} R \sin a + \frac{1}{4} R^2 - \frac{1}{2} R \sin a\]

\[-\frac{1}{2} \sqrt{R^2 + R \sin a} \times \sqrt{R^2 - R \sin a}\]

\[-\frac{1}{2} \sqrt{R^2 + R \sin a} \times \sqrt{R^2 - R \sin a}\]

\[\frac{R}{2} \sqrt{1 - \sin^2 a} = -\frac{1}{2} \sqrt{R^1 - R^2 \sin^2 a} =\]

\[\mp \frac{R}{2} \sqrt{R^2 - \sin^2 a}\]
Theorem. Page 199.

If \( N \) be a function of \( x \) and \( y \), giving \( \frac{dN}{dx} = p + q \frac{dy}{dx} \)
then the equation \( \frac{du}{dx,dy} = V \frac{dN}{dx,dy} \) is incongruous & self-contradictory, except upon the assumption that \( u \) is, as to \( x \) and \( y \), a function of \( N \); or contains \( x \) and \( y \) only thro’ \( N \).

Let \( N = \psi(x,y) \) give \( y = \chi(N,x) \), and suppose, if possible, that the substitution of this value of \( y \) in \( u \) gives \( u = \beta(N,x) \), \( x \) not disappearing with \( y \). Then \( x \) and \( y \)

\[
\frac{du}{dx,dy} = \frac{d\beta}{dN} \frac{dN}{dx,dy} + \frac{d\beta}{dx} = \frac{d\beta}{dN} \cdot (\frac{dN}{dx} + \frac{dN}{dy}) + \frac{d\beta}{dx} = \frac{d\beta}{dN} \cdot \frac{dN}{dx,dy} + \frac{d\beta}{dx}
\]

[universal, is true on the supposition that \( x \) does not vary, or that \( \frac{d\beta}{dx} = 0 \). This gives \( \frac{d\beta}{dN} = V \); or \( \frac{du}{dx} = V \frac{dN}{dx,dy} + \frac{d\beta}{dx} \) being independent of the variations &c, &c. Hence \( \frac{d\beta}{dN} = 0 \) always; or \( \beta \) does not contain \( x \) directly, &c.

I think the above is correct. I cannot see the use (page 200) of introducing \( t \) in the proof there given. Is it possible that I have committed an error in my original understanding of the enunciation [sic] of the Theorem; & that the \( du \) [of the equation'] crossed out and the \( dN \) of the equation \( du = V.dN \), do not mean the \( du \) and \( dN \) derived from differentiating with respect to the quantities \( x \) and \( y \), already introduced; but with respect to [‘some’ crossed out] other given quantity?

I suspect so.

[the following appears underneath in pencil — still in Ada’s hand]

\[
\begin{align*}
    u &= \beta(N,x) \\
    \frac{du}{dx} &= \frac{d\beta}{dN} \frac{dN}{dx} + \frac{d\beta}{dx} \\
    \frac{du}{dx,dy} &= \\
\end{align*}
\]
This complete differential of \( \varphi \), as it is called namely
\[
\frac{d\varphi}{dx}.dx + \frac{d\varphi}{dy}.dy + \frac{d\varphi}{dz}.dz
\]
is a perfectly distinct thing from
\[
\frac{d\varphi}{dx} + \frac{d\varphi}{dy} + \frac{d\varphi}{dz}
\]
and also from \( \frac{d^3\varphi}{dx\,dy\,dz} \).

Read again page 86 when \( x \) is changed to \( \end{87} \)

page 198–199 & the references

\[
\begin{align*}
\frac{d\varphi}{dx} \text{ is } & d\varphi \\
\frac{d\varphi}{dy} \text{ is } & d\varphi
\end{align*}
\]

But the first means the \( d\varphi \) which is caused by variation of \( x \), and the second has the same reference to \( y \).
[176r] [diagram] \((a - x)x = ax - x^2\)

[further diagrams: one Pythagoras-related]

[176v] [Königsberg bridges diagrams, some with labels in Babbage’s hand]

[Pythagoras-related diagrams]

[177r] [further Königsberg bridges diagrams and related jotting]

[in Babbage’s hand] 1 No return

3

5

If there are an odd \(N^o\) of Bridges 2 No ending except

if you do not begin you must end 4 in island

in it

[written at 90°, spanning bottom of 176v and 177r]

\[
\begin{array}{ccc}
2 & 9 & 4 \\
7 & 5 & 3 \\
6 & 1 & 8
\end{array}
\]

\[
(a \pm b)^2 = a^2 + b^2 \pm 2ab
\]

\[
(a - b) \times (a + b) = a^2 - b^2
\]

\[
\sin \frac{1}{2}a = \sqrt{\frac{1}{2}R^2 - \frac{1}{2}R \cos a}
\]

\[
\text{for } \cos a \text{ put } \pm \sqrt{R^2 - \sin^2 a}
\]

\[
\sin \frac{1}{2}a = \sqrt{\frac{1}{2}R^2 \mp \frac{1}{2}R \sqrt{R^2 - \sin^2 a}}
\]

Let \(\sin \frac{1}{2}a = \frac{1}{2} \sqrt{R^2 + R \sin a \mp \frac{1}{2} \sqrt{R^2 - R \sin a}}
\]

\[
\frac{1}{2}R^2 \pm \frac{1}{2}R \sqrt{R^2 - \sin^2 a} = \frac{1}{4}R^2 + \frac{1}{4}R \sin a \pm \frac{1}{4}R^2 - \frac{1}{4}R \sin a \mp 2 \times \frac{1}{2} \times \frac{1}{2} \sqrt{R^2 + R \sin a \times \sqrt{R^2 - R \sin a}}
\]

\[
\mp \frac{1}{2}R \sqrt{R^2 - \sin^2 a} = \frac{1}{2} \sqrt{R^4 - R^2 \sin^2 a} = \frac{1}{2} \sqrt{R^2 - \sin^2 a}
\]

\[
\mp \frac{1}{2}R \sqrt{R^2 - \sin^2 a} = \mp \frac{1}{2}R \sqrt{R^2 - \sin^2 a}
\]
25th August 1843

\[ u = R + X \]
\[ P = \frac{dR}{dx} + \frac{dx}{dx} \]
\[ \frac{dX}{dx} = P - \frac{dR}{dx} \]
\[ X = \int \left( P - \frac{dR}{dx} \right) dx \]
\[ P = -\frac{y^2}{x^2\sqrt{x^2+y^2}} \]
\[ R = \frac{\sqrt{x^2+y^2}}{x} \]

\[
\frac{1}{e^{x+1}-e^x} - \frac{1}{(e^{x+1})^2} \frac{\left(-e^{x}+e^{x+2}e^{2[x\leqslant -1]}\right)}{(e^{x+1})^3} \frac{e^{2x}-e^{-x}}{(e^{x+1})^3} \]
25th Augst 1843

\[ \frac{x}{1 + \frac{x^2}{2} + \frac{x^3}{3} + \cdots} \]

\[ \begin{array}{ccc}
0 & \frac{\phi x}{\psi x} & \frac{\phi' x}{\psi' x} \\
0 & 0 & 0 \\
\frac{x}{e^x - 1} & 1 & e^x \\
\end{array} \]

Co. of \( x^{2n} \) in \( \frac{\frac{1}{2} x}{e^x + 1} \)
\[ = \frac{1}{2^{2n}} \text{ co. of } x^{2n} \text{ in } \frac{x}{e^x + 1} \]
\[ = \frac{1}{2^{2n}} \text{ co. of } x^{2n-1} \text{ in } \frac{1}{e^x + 1} \]
\[ = \frac{1}{2^{2n}} \frac{1}{1.2.3\ldots2n-1} \frac{d}{dx} [e^{-x}]^{2n-1} \frac{1}{e^x + 1} \]

when \( x = 0 \)