Tamar Ziegler

Tamar Ziegler began by posing the following question.

> (Q1) We are given an infinite abelian group G, a vector space V over some field, and a map $\rho : G \to \mathrm{End}_k(V)$ which is an *almost homomorphism* in the sense that
>
> $$\operatorname{rank}\big(\rho(x+y) - \rho(x) - \rho(y)\big) < r \qquad \forall x, y \in G\,.$$
>
> Does there exist a homomorphism σ close to ρ in the sense that
>
> $$\operatorname{rank}\big(\rho(x) - \sigma(x)\big) < R \qquad \forall x \in G\,?$$

Here R is a constant depending on r, as well as on the choice of the group and the field. The question opens a window on recent developments in ergodic theory and additive combinatorics.

A good starting point for developing techniques to address such problems is Roth's 1952 theorem that a subset $E \subset \mathbb{Z}$ of positive density necessarily contains a three-term arithmetic progression, together with Szemerédi's extension to k-term progressions (1975).

It helpful to explore Roth's result by considering the corresponding statement in a different setting, in which E is a subset of density δ of a vector space V over a finite field k. Here the answer is found by using discrete Fourier analysis: the key point is that either E is 'Fourier uniform' in the sense that the Fourier coefficients of its characteristic function 1_E are small, or else 1_E has a large Fourier coefficient. In the first case E has roughly the same number of three-term arithmetic progressions as a random subset of the same density; in the second, E has increased density on some affine hyperplane.

This idea is not sufficient to establish the analogous result for four-term progressions. One reason is that the count of four-term progressions is biased when the subset is the hypersurface $\{Q(x) = 0\}$ for some high-rank quadratic form Q. Here the Fourier coefficients are small, but the identity

$$Q(x) - 3Q(x+d) + 3Q(x+2d) - Q(x+3d) = 0$$

forces the fourth term of a progression to lie on the hypersurface whenever the first three do, so the hypersurface contains many more four-term progressions than might be expected. Instead it is necessary to adapt the argument by looking at 'quadratic Fourier coefficients'. Either these are all small or E has increased density on a quadratic hypersurface.

$$\begin{tikzcd}
X+u_2+u_3 \arrow[rr] & & X+u_1+u_2+u_3 & \\
& X+u_3 \arrow[ul] \arrow[rr] & & X+u_1+u_3 \arrow[ul] \\
X+u_2 \arrow[uu] \arrow[rr] & & X+u_1+u_2 \arrow[uu] & \\
& X \arrow[ul] \arrow[uu] \arrow[rr] & & X+u_1 \arrow[ul] \arrow[uu]
\end{tikzcd}$$

This raises the general question: if E has a 'biased' number of d-term progressions, does it necessarily have increased density on some hypersurface of degree less than $d-1$? The approach based on *Gowers norms* starts with the observation that, rather than count four-term progressions, it is better to count three-dimensional cubes, as on the left. If there is bias in the count of progressions, then there is also a bias in the count of cubes. This observation is coupled with another, that if $\mathrm{char}(k) > d$, then $P : V \to k$ is a polynomial of degree less that d if and only if it vanishes on every d-dimensional cube in the sense that the alternating sum of its values at the vertices vanishes.

In terms of cubes, the question becomes the following.

> (Q2) If the number of d-dimensional cubes in E is not as in a random set, is it necessarily true that E has increased density on a hypersurface of degree less than d ($\mathrm{char}(k) > d$)?

Before addressing this, Ziegler turned to the ergodic theory approach, which is based on *Furstenberg's correspondence principle*. Under this, the statement that there are k-term progressions in a set $E \subset \mathbb{N}$ is translated to one about multiple return times for a distinguished subset A of a measure space X with a measure-preserving map $T : X \to X$: if

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0 \tag{1}$$

for some n, then E contains a $k+1$-term progression. Furstenbergs's idea was to study such questions about return times by using morphisms to simpler measure-preserving systems.

Ziegler illustrated how this works by outlining an alternative approach to Roth's theorem. The argument has the same general shape as the original, but now the two cases are distinguished by whether or not T has non-trivial eigenfunctions. If not, then

$$\frac{1}{N}\sum_{n\le N}\mu(A\cap T^{-n}A\cap T^{-2n}A)\to\mu(A)^3$$

and we are done; otherwise there are non-trivial eigenfunctions ψ_i, with $\psi(Tx) = \lambda\psi(x)$. When normailzed, one or more of these determine a morphism from X to a much simpler dynamical system,

$$\pi : X \to Z := \prod_i S^1 .$$

This is a *Kronecker system*: Z is an abelian group with Haar measure and multiplication by the eigenvalues λ_i determines a rotation $S : Z \to Z$. In the Kronecker system, the inequality (1) can be established by showing that the sum

$$\frac{1}{N}\sum_{n\le N}\pi_*\mu(\pi_*A\cap S^{-n}\pi_*A\cap S^{-2n}\pi_*A)$$

is positive. By replacing A by its characteristic function, one is then led to consider the asymptotic behaviour of averages the form

$$\frac{1}{N}\sum_{1}^{N}\int \pi_*f(z)\pi_*f(z+n\alpha)\pi_*f(z+2n\alpha)\,\mathrm{d}\pi_*\mu$$

for some positive function f. This is much more straightforward than the original problem.

To extend the picture to four-term and longer progressions, one needs to replace the eigenfunctions ψ_i, on which the action of T is linear, by 'quadratic' and higher degree 'polynomials'. So we now consider a space X with measure μ and a measure-preserving ergodic action $T_u : X \to X$, $u \in G$, of a countable abelian group G. For $f : X \to \mathbb{C}$, put

$$\Delta_u f(x) = f(T_u x)\overline{f(x)}\,.$$

A function P is said to be a *polynomial* of degree less than d if

$$\Delta_{u_d} \ldots \Delta_{u_1} P(x) = 1 \text{ a.e.} \quad \forall u_1, \ldots, u_d \in G\,. \tag{2}$$

A polynomial of degree less than one is constant, assuming ergodicity; and an eigenfunction is a polynomial of degree less than two.

The definition can be re-expressed in terms of values at the vertices of cubes. With $d = 4$, the condition (2) caan be restated as the 'vanishing' of P on cubes, in the multiplicative sense:

$$\prod_{\omega\in\{0,1\}^3} P^{(\omega)}\big(T_{\omega\cdot(u_1,u_2,u_3)}\big)(x) = 1 \text{ a.e.} \tag{3}$$

where $P^{(\omega)}$ is equal to P or $\overline{P}$ as $|\omega|$ is even or odd.

Bergelson, Tao, and Ziegler combined this last idea with the cube construction to reduce the averaging problem for four-term progressions down to simpler *polynomial systems*, in an analogous way to the use of eigenfunctions for the three-term progressions.

By using polynomial systems of higher degree Tao and Ziegler answered question (Q2) positively and constructed bounds extending those found those by Green, Tao, and Gowers in the case $d = 4$ by using additive combinatorics.

Ziegler then explained how these techniques can be used to answer the original question (Q1) in the case $G = W = V$, where W is infinite. The approximating homomorphism is found by making finite approximations, with $k = \mathbb{F}_p$ and $|W| = p^n$. Because ρ is close to linear, the function

$$f(w, x) = \exp\left(2\pi \mathrm{i}\langle\rho(w)x, x\rangle/p\right)$$

is close to being a 'cubic' polynomial in w, x. An exact 'cubic' vanishes on four-dimensional cubes in the sense of (3), while the values of f are biased because ρ is close to being linear. The bias allows one to use the theory above to establish the existence of a cubic $Q(w, z)$ (in the standard sense) such that $\rho(w) - Q(w, \cdot)$ is of bounded rank.

Finally Ziegler outlined a an 'approximate cohomology theory' that she had developed with David Kazhdan that provides a general framework for considering questions of his type.