Constructing the virtual fundamental cycle in the Landau-Ginzburg A model

Huijun Fan
(Joint work with Tyler Jarvis and Yongbin Ruan)

School of Mathematical Sciences
Peking University

January 10, 2009

Superpotential

Definition 0.1

A quasi-homogeneous polynomial $W(x_1, \cdots, x_N)$ satisfies

$$W(\lambda_1^{q_1} x_1, \cdots, \lambda_N^{q_N} x_N) = \lambda W(x_1, \cdots, x_N),$$

where q_i are the weights of x_i.

W is called nondegenerate if the weights of W are uniquely determined and the hypersurface defined by W is non-singular in projective space.

Example 0.2

$$\begin{aligned} &A_n \quad W = x^{n+1},\ n \geq 1; \\ &D_n \quad W = x^{n-1} + xy^2,\ n \geq 4; \\ &E_6 \quad W = x^3 + y^4; \\ &E_7 \quad W = x^3 + xy^3; \\ &E_8 \quad W = x^3 + y^5; \end{aligned}$$

Lemma 0.3

If W is non-degenerate, then the group

$$G_W := \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

of diagonal symmetries of W is finite.

Definition 0.4

We write each element $\gamma \in G_W$ (uniquely) as

$$\gamma = (\exp(2\pi i \Theta_1^\gamma), \ldots, \exp(2\pi i \Theta_N^\gamma)),$$

with $\Theta_i^\gamma \in [0,1) \cap \mathbb{Q}$.

$$J := (\exp(-2\pi i q_1), \ldots, \exp(-2\pi i q_N)),$$

The elements J and J^{-1} will play important roles.

W structure and W curve

Definition 0.5

Orbicurve $(\mathscr{C}, p_1, \cdots, p_k)$: possible orbifold structure at p_i or nodal points; A uniformizing system is given by $z \to z^{m_i}$; Local group $G_{p_i} \cong Z_{m_i}$.
If $\mathscr{L}$ is a orbifold line bundle and (z, s) is the local coordinates of the uniformizing system near p_i, the action of G_{p_i} is $(z, s) \to (\exp(2\pi i/m_i)z, \exp(2\pi iv/m_i)s)$.
We can naturally define the group action at the nodal points.

Definition 0.6

A W-curve $\mathfrak{C} = (\mathscr{C}, p_1, \cdots, p_k, \mathscr{L}_1, \cdots, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s)$ is a genus g orbicurve $\mathscr{C}$, having k marked points and with the W-structure $(\mathscr{L}_1, \cdots, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s)$. A W structure means that the orbifold line bundles $\mathscr{L}_1, \cdots, \mathscr{L}_N$ should satisfy the isomorphisms:

$$\varphi_i : W(\mathscr{L}_1, \cdots, \mathscr{L}_N) \to K_{log} = K_{\mathscr{C}} \otimes (\mathscr{O}(p_1)) \otimes \cdots (\mathscr{O}(p_k)), \forall i = 1, \cdots, s.$$

The action of local group G_p at a marked (or nodal) point p gives an element $\gamma \in G_W$:

$$\gamma = (\exp(2\pi i\Theta_1^\gamma), \ldots, \exp(2\pi i\Theta_N^\gamma))$$

Let $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_k)$ for $\gamma_i \in G_W$. We say a W curve $\mathfrak{C}$ is of type $\boldsymbol{\gamma}$, if at each marked point p_i the group action is given by γ_i.
The automorphism group $\mathrm{Aut}(\mathfrak{C})$ is the extension of the automorphism group $\mathrm{Aut}(\mathscr{C})$ of $\mathscr{C}$ along $\mathrm{Aut}_{\mathscr{C}}(\mathscr{L})$. The later is the fiber multiplication preserving the underlying curve $\mathscr{C}$.

Example 0.7

Consider a W-curve with two irreducible components $\mathscr{C}_1$ and $\mathscr{C}_2$ with marked points $\{p_i | i \in I_1\} \cup \{q_+\} \subset \mathscr{C}_1$ and $\{p_i | i \in I_2\} \cup \{q_-\} \subset \mathscr{C}_2$, such that the components meet at a single node $q = q_+ = q_-$ and such that $I_1 \sqcup I_2 = \{1, \ldots, k\}$. Denote the local group at p_i by $\langle \gamma_i \rangle$. In this case we have

$$\mathrm{Aut}_{\mathscr{C}}(\mathfrak{L}) = G \times_{G/\langle \gamma_+ \rangle} G, \tag{1}$$

where $G \times_{G/\langle \gamma \rangle} G$ denotes the group of pairs (g_1, g_2) such that $g_1 = g_2 \in G/\langle \gamma \rangle$.

Example 0.8

If $\mathscr{C}$ consists of a single (possibly nodal) irreducible component, then we have

$$\mathrm{Aut}_{\mathscr{C}}(\mathfrak{L}) = G. \tag{2}$$

- **Ramond line bundles and Neveu-Schwarz line bundels**: If the orbifold action of a line bundle $\mathscr{L}_i$ at a marked (or nodal) marked point p is trivial, i.e., $\Theta_i^\gamma = 0$, then $\mathscr{L}_i$ is called a Ramond line bundles at p; Otherwise it is called the Neveu-Schwarz line bundle at p.
- If all the line bundles at a marked (or a nodal) point p are Neveu-Schwarz line bundles, then p is called a Neveu-Schwarz point; otherwise it is called a Ramond marked (or nodal) point.
- Let $\gamma \in G$ be the generator of the local group at p, then it has the action to $\mathbb{C}^N$. Define $W_\gamma = W|_{\mathbb{C}^N_\gamma}$.

Moduli space of W curves

- $\overline{\mathscr{M}}_{g,k} := \{(\mathscr{C}, p_1, \cdots, p_k)\}$: moduli space of genus g curves with k marked points.
- $\overline{\mathscr{M}}_{g,k,W}(\gamma) := \{(\mathfrak{C})\}$: moduli space of W-curves $\mathfrak{C}$.It is a stratified space

Natural maps

$$st : \overline{\mathscr{M}}_{g,k,W}(\gamma) \to \overline{\mathscr{M}}_{g,k}\text{ , (forgetting map)}\theta : \overline{\mathscr{M}}_{g,k+1,W}(\gamma, J^{-1}) \to \overline{\mathscr{M}}_{g,k,W}(\gamma)$$

and other cutting-gluing operations.
taughtological ring in $H^*(\overline{\mathscr{M}}_{g,k,W}(\gamma))$. ψ_i, κ_i classes and etc.

Theorem 0.9

For any non-degenerate, quasi-homogeneous polynomial W, the stack $\overline{\mathscr{M}}_{g,k,W}$ is a smooth, compact orbifold (Deligne-Mumford stack) with projective coarse moduli. In particular, the morphism $st : \overline{\mathscr{M}}_{g,k,W} \to \overline{\mathscr{M}}_{g,k}$ is flat, proper and quasi-finite (but not representable).

Rigidified W curve

$\overline{\mathscr{M}}_{g,k,W}(\gamma)$ is not an appropriate working space for analysis, We need consider the moduli space of the rigidified W curves, $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\gamma)$. It is a branched covering space of $\overline{\mathscr{M}}_{g,k,W}(\gamma)$.

Definition 0.10

A rigidification ψ at a marked point p is a local trivialization of the orbifold structure such that it preserves the W-structure, i.e., the diagram commutes

$$\begin{tikzcd} j_p^*\left(\bigoplus_{m=1}^{N} \mathscr{L}_m\right) \arrow[r, "\psi"] \arrow[d, "\phi_\ell \circ W_\ell"'] & {[\mathbb{C}^N/G_p]} \arrow[d, "W_\ell"] \\ j_p^*(K_{log}) \arrow[r, "residue"] & \mathbb{C} \end{tikzcd} \tag{3}$$

where the residue map takes $\frac{dz}{z}$ to 1.

Definition 0.11

A rigidified W-curve is defined as $\mathfrak{C} := \{\mathscr{C}, p_1, \cdots, p_k, \mathscr{L}_1, \mathscr{L}_N, \varphi_1, \cdots, \varphi_s, \psi_1, \cdots, \psi_k\}$, where ψ_i are rigidification at marked point p_i. The moduli space of the rigidified W-curves is the equivalence of those $\mathfrak{C}$, and is denoted by $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\gamma)$.

Metric choices: Choose cylindric metric near a marked or nodal point, i.e., let $|\frac{dz}{z}| = 1$. This metric will induce metrics of line bundles $\mathscr{L}_i$ by W-structure. Let $\mathfrak{C} = (\mathscr{C}, \mathscr{L}, \Psi)$ be a rigidified W-curve with the cylindrical metric. We can define a metric-preserving map:

$$\tilde{I}_1 : \Omega(\tilde{\Sigma}, \bar{\mathscr{L}}_j^{-1} \otimes \Lambda^{0,1}) \to \Omega(\tilde{\Sigma}, \mathscr{L}_j \otimes \Lambda^{0,1}).$$

Now the Witten equation on orbifolds is defined as

$$\widetilde{WM}(\mathfrak{C}, \mathbf{u}) := \bar{\partial} u_i + \tilde{I}_1\left(\overline{\frac{\partial W}{\partial u_i}}\right) = 0, \forall i = 1, \cdots, N.$$

Witten map and its perturbation

Stratified Orbifold Fréchet bundles B^0 and $B^{0,1}$ over $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\gamma)$: The fiber spaces at $\mathfrak{C}$ are:

$$C^\infty(\mathscr{C},\mathscr{L}_j) := \{(u_{j,\nu}) \in \oplus_\nu C^\infty(\mathscr{C}_\nu,\mathscr{L}_j) | u_{j,\nu}(p_\nu) = u_{j,\mu}(p_\mu), \text{ if } \pi_\nu(p_\nu) = \pi_\mu(p_\mu)\}.$$

$$C^\infty(\mathscr{C},\mathscr{L}_j \otimes \Lambda^{0,1}) := \{(u_{j,\nu}) \in \oplus_\nu C^\infty(\mathscr{C}_\nu,\mathscr{L}_j \otimes \Lambda^{0,1})\}.$$

Now the Witten map $\widetilde{WM}$ is viewed as a section from B^0 to $\pi^* B^{0,1}$.
Notice: In uniformizing system, the Witten map is G-equivariant.

Perturbed Witten map: We can modify the Witten map near each Ramond marked or nodal point such that it looks like

$$\bar{\partial} u_i + \tilde{I}_1 \left(\overline{\frac{\partial W + W_{0,\gamma}}{\partial u_i}} \right) = 0, \text{ if } u_i \text{ is Ramond variable.}$$

Where $W_{0,\gamma}$ is a linear perturbation of W_γ such that $W_{0,\gamma} + W_\gamma$ is a holomorphic morse function. The perturbation depends on the parameter $\mathbf{b} = (b_1, \cdots, b_N)$.

Since the perturbation will break the G-equivariance, the perturbed Witten section $WI : B^0 \to B^{1,0}$ is a **multisection**!.

Connection condition at Ramond nodal points The perturbation parameters for the two components connecting at a nodal point p is no independent.

Nonlinear analysis for the perturbed Witten equation

Note: This system is a non-conformal nonlinear Cauchy-Riemann system. So the analysis here is **different** to that for 4d Yang-Mills equation and pseudo-holomorphic curves. It is also different from the Seiberg-Witten equation, whose solutions have C^0-norm estimate.
Near the marked points, it is same as the trajectory equation in Floer's theory, but in the interior it is like the semilinear elliptic equations.

- **Interior estimate**: $||\mathbf{u}||_{C^k} \leq C$.(One of the cruical inequality is given in our first paper, appeared in Comm. Pure Appl. Math.)
- **Convergence at marked or nodal points**: $\mathbf{u} = (u_1, \cdots, u_N) \to \kappa_i$ as $z \to p$, where κ_i is one of the critical points of $W_\gamma + W_{0,\gamma}$.
- **Exponential convergence**

Soliton space

Consider the equation on $(\mathbb{R}^1 \times S^1, -\infty, 0, +\infty, \gamma, *, \gamma^{-1}, \psi)$. (In cylinder coordinates $\zeta = s + i\theta$)

$$\frac{\bar{\partial} u_i}{\partial \bar{\xi}} - 2\overline{\frac{\partial (W + W_{0,\gamma})}{\partial u_i}} = 0.$$

Definition 0.12

The nontrivial solution is called **Soliton solution**, and if the solution is also independent of the angle, then it is called BPS-Soliton.

Notice: The BPS soliton is S^1-invariant solution.

If $\mathbf{u}$ is a solution, then

$$(W_\gamma + W_{0,\gamma})(\kappa^-) - (W_\gamma + W_{0,\gamma})(\kappa^+) = 2\int_{-\infty}^{+\infty}\int_{S^1}\sum_i \left|\frac{\partial (W + W_{0,\gamma})}{\partial u_i}\right|^2.$$

Conclusions:Only if we choose the perturbation parameter $\mathbf{b}$ for γ such that for two different critical points κ^+ and κ^- for $W_\gamma + W_{0,\gamma}$, the following holds

$$\mathrm{Im}(W_\gamma + W_{0,\gamma})(\kappa^i) = \mathrm{Im}(W_\gamma + W_{0,\gamma})(\kappa^j).$$

there exists soliton solutions.

Definition 0.13

If $\mathbf{b}$ is chosen such that for all $\gamma \in G$, $W_\gamma + W_{0,\gamma}$ is a holomorphic morse function, the perturbation is called regular; for regular $\mathbf{b}$, if for any γ the above equality does not hold, the perturbation is called strongly regular.

Theorem 0.14

The regular but not strongly regular parameters $\mathbf{b}$ consists of a generic set in finite real codimension 1 hypersurfaces which separate $\mathbb{C}^N$ into chambers.

Moduli space of stable W sections $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$

Compactness Loss If the moduli space $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ is strongly regular perturbed, then there is no loss of compactness. If it is regular perturbed but not strongly regular perturbed, then there is **compactness loss phenomena** due to the existence of **soliton solutions.** In the latter case, we need add the "**soliton W sections**" into our moduli space. So We need to consider a larger space $\overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$.

- $\overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ is a stratified space stratified by the decorated dual graphs.
- We can define the Gromov-Hausdorff topology such that $\overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ is a compact Hausdorff space.

- If $\overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa}) = \overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ (strongly regular perturbed), then it is a space ”without boundary”.
- If the perturbation is regular but not strongly regular perturbed, then $\overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})$ is a space ”with boundary” and the boundary is related to BPS soliton. Solitons but not BPS solitons appear in the interior of the moduli space(finite automorphism group).

Fredholm theory and the virtual dimension

For any $\mathfrak{C} \in \overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma})$ we have the Witten map:

$$WI_{\mathfrak{C}} : L^p_1(\mathscr{C}, \mathscr{L}_1 \times \cdots \times \mathscr{L}_N) \to L^p(\mathscr{C}, \mathscr{L}_i \otimes \Lambda^{0,1}),$$

which has the following form:

$$WI_{\mathfrak{C}}(\mathbf{u}) = (\bar\partial_{\mathscr{C}} u_1 + \tilde{I}_1\left(\overline{\frac{\partial(W + W_{0,\beta})}{\partial u_1}}\right), \cdots, \bar\partial_{\mathscr{C}} u_N + \tilde{I}_1\left(\overline{\frac{\partial(W + W_{0,\beta})}{\partial u_N}}\right)).$$

Here the perturbation term $W_{0,\beta}$ has the form $\varpi(\zeta)\beta_i W_{0,\gamma}$ which is determined by the combinatorial type of $\mathscr{C}$ and the group element γ and the cut-off section β_i.

We will always set $p > 2$ in our discussion.
We have the linearized operator $D_{\mathfrak{C},\mathbf{u}}WI$ of $WI_{\mathfrak{C}}$ at $\mathbf{u}$:

$$D_{\mathfrak{C},\mathbf{u}}WI(\boldsymbol{\phi}) := D_{\mathfrak{C},\mathbf{u}}WI(\phi_1,\cdots,\phi_N) :=$$
$$\left(\bar{\partial}_{\mathscr{C}}\phi_1 + \sum_j \tilde{I}_1\left(\overline{\frac{\partial^2(W+W_{0,\beta})}{\partial u_1\partial u_j}\phi_j}\right),\cdots,\bar{\partial}_{\mathscr{C}}\phi_N + \sum_j \tilde{I}_1\left(\overline{\frac{\partial^2(W+W_{0,\beta})}{\partial u_N\partial u_j}\phi_j}\right)\right). \tag{4}$$

$D_{\mathfrak{C},\mathbf{u}}WI$ is a map from $L^p_1(\mathscr{C},\mathscr{L}_1\times\cdots\times\mathscr{L}_N)$ to $L^p(\mathscr{C},\mathscr{L}_1\otimes\Lambda^{0,1})\times\cdots L^p(\mathscr{C},\mathscr{L}_N\otimes\Lambda^{0,1})$.

Theorem 0.15

Let $(\mathfrak{C}, \mathbf{u}) \in \overline{\mathscr{M}}^{\mathrm{rig,s}}_{g,k,w}(\boldsymbol{\gamma}, \boldsymbol{\varkappa})$ *and assume that* $\mathfrak{C}$ *is connected. Then its linearized operator* $D_{\mathfrak{C},\mathbf{u}}WI : L^p_1(\mathscr{C}, \mathscr{L}_1 \times \cdots \times \mathscr{L}_N) \to L^p(\mathscr{C}, \mathscr{L}_1 \otimes \Lambda^{0,1}) \times \cdots \times L^p(\mathscr{C}, \mathscr{L}_N \otimes \Lambda^{0,1})$ *is a real linear Fredholm operator of index* $2\hat{c}_W(1-g) - \sum_\tau 2\iota(\gamma_\tau) - \sum_{\tau=1}^k N_{\gamma_\tau}$*, where* $\hat{c}_W = \sum_i (1 - 2q_i)$*,* $\iota(\gamma_\tau) = \sum_i (\Theta_i^{\gamma_\tau} - q_i)$ *and* $N_{\gamma_\tau} = \dim \mathbb{C}^N_{\gamma_\tau}$ *(if* $\mathbb{C}^N_{\gamma_\tau} = \{0\}$*, we set* $N_{\gamma_\tau} = 0$*).*

Corollary 0.16

Let $(\mathbf{u}_{j_1,j_2}, \gamma) \in S_\gamma(\kappa^{j_1}, \kappa^{j_2})$*. Then the linearized operator* $D_{\mathbf{u}_{j_1,j_2}}(WI)$ *is a real linear Fredholm operator of index* 0 *on* $\mathbb{R} \times S^1$*.*

Orientable Kuranishi structure and virtual fundamental cycles

- **Construction of the interior Kuranishi neighborhood**: $(U_\sigma, E_\sigma, s_\sigma, \Psi_\sigma)$ By modified Implicit functional theorem and a priori estimates for the solutions. (No transversality)
- **Construction of the Kuranishi nbhd. on boundary**: Study the BPS soliton carefully including computation of the obstruction bundles (2 cases: Tree and Loop cases).
- **Gluing** Glue the K-nbhd from the lower strata and choose the suitable obstruction bundle on the overlaps.
- **orientation** Show that the Kuranishi structure is orientable and coherent, i.e., the orientation should respect the gluing operation. It is much the same to the treatment of Floer's Hamiltonian trajectory.

Virtual fundamental cycle

- By Fukaya-Ono's machinery, we can get the virtual fundamental cycle $[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa})]^{vir} \in H_*(\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma},\boldsymbol{\varkappa}))$ if the perturbation is strongly regular. Its (real) dimension is $6g-6+2k-2D-\sum_{i=1}^k N_{\gamma_i} = 2((\hat{c}_W-3)(1-g)+k-\sum_{\tau=1}^k \iota(\gamma_\tau)) - \sum_{i=1}^k N_{\gamma_i}$.
  Comment: The real dimension can be odd even for one Ramond marked point.
- We can define the boundary cycles $[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\Gamma)]^{vir}$ w.r.t. the decorated dual graph.
- If two perturbation parameter $\mathbf{b}$ and $\mathbf{b}'$ are in the same chamber of $\mathbb{C}^N$, then the corresponding virtual cycles are the same. Hence the virutal cycles actually depends on the vanishing cycles (or Lefschetz thimbles) of $W_\gamma + W_{0,\gamma}$.

Picard-Lefschetz theory

Before further discussion of the properties of the virtual cycles, We will give a simple description of the vanishing cycles, Lefschetz thimbles and related transformation formulas in the classical Picard-Lefschetz theory.

Now we assume that the perturbation polynomial $W_{0,\tilde{\gamma}}$ is strongly $W_{\tilde{\gamma}}$-regular and sufficiently small such that there are exactly $\mu_{\tilde{\gamma}}$ critical points of $W_{\tilde{\gamma}} + W_{0,\tilde{\gamma}}$ inside a small ball B centered at 0. Let α^i be the critical value of $W_{\tilde{\gamma}} + W_{0,\tilde{\gamma}}$ which lies inside a small neighborhood $T \subset \mathbb{C}$ corresponding to B. Furthermore, we can require the order of these critical values to satisfy $\mathrm{Im}(\alpha_i) < \mathrm{Im}(\alpha_j)$ if $i < j$. Let $\alpha_* \in \partial T$ be a regular value. We take T small enough and define $Y = (W_{\tilde{\gamma}} + W_{0,\tilde{\gamma}})^{-1}(T) \cap B$ and $Y_* = (W_{\tilde{\gamma}} + W_{0,\tilde{\gamma}})^{-1}(\alpha_*)$.

Picard-Lefschetz theory

Take a system of paths $l_i(\tau) : [0,1] \to \mathbb{C}$ connecting α_* and α_i such that

1. the paths l_i have no self-intersections;
2. the paths l_i and l_j intersect only for $\tau = 0$, at the point $\alpha_* = l_i(0) = l_j(0)$;
3. the paths l_i are ordered by the requirement that $\arg l_i'(0) < \arg l_j'(0)$ if $i < j$.

For each path l_i, there exists a corresponding *simple loop* β_i which goes along l_i from α_* to the critical value α_i, then goes anticlockwise around α_i and finally returns to α_* along l_i. The system of these paths l_i's is called distinguished if the corresponding system of simple loops β_i generates the group $\pi_1(T', \alpha_*)$, where $T' = T - \{\alpha_1, \cdots, \alpha_{\mu_{\tilde{\gamma}}}\}$.

Each path l_i induces a unique vanishing cycle $\Delta_i \in H_{N_{\tilde{\gamma}}-1}(Y_*)$ or a Lefschetz thimble $S_i \in H_{N_{\tilde{\gamma}}}(Y, Y_*)$ up to orientation. In singularity theory, the set of these cycles or thimbles forms a basis of the homology group $H_{N_{\tilde{\gamma}}-1}(Y_*) \cong \mathbb{Z}^{\mu_{\tilde{\gamma}}}$, or the relative homology group $H_{N_{\tilde{\gamma}}}(Y, Y_*)$, which is called a distinguished basis of vanishing cycles or thimbles respectively. Let D_* represent the set of all the distinguished bases of vanishing cycles (or thimbles).

Transformation formula of basis

Assume that the boundary of the relative cycle S_i is just Δ_i; then when taking compatible orientations we have the connecting isomorphism:

$$\partial_* : H_{N_{\tilde{\gamma}}}(Y, Y_*) \to H_{N_{\tilde{\gamma}}-1}(Y_*)$$

such that $\partial_*(S_i) = \Delta_i$.
Each simple loop β_i induces the monodromy operators

$$h_{\Delta_i} : H_{N_{\tilde{\gamma}}-1}(Y_*) \to H_{N_{\tilde{\gamma}}-1}(Y_*)$$

and

$$h_{S_i} : H_{N_{\tilde{\gamma}}}(Y, Y_*) \to H_{N_{\tilde{\gamma}}}(Y, Y_*),$$

which have action given by Picard-Lefschetz theory as follows:

$$h_{\Delta_i}(\Delta_j) = \Delta_j + R_{j,i}\Delta_i, \forall j \tag{5}$$

and

$$h_{S_i}(S_j) = S_j + R_{j,i}S_i, \forall j \tag{6}$$

where $R_{j,i} = (-1)^{N_{\tilde{\gamma}}(N_{\tilde{\gamma}}+1)/2}\Delta_j \circ \Delta_i$ and $\Delta_j \circ \Delta_i$ is the intersection number.

Wall-crossing formula

If take a path $\mathbf{b}(\lambda)$ go through the wall of the chamber, then the virtual cycles will transform in the same way as the corresponding vanishing cycles (Lefschetz thimbls) change in the classical Picard-Lefschetz theory.
Let $(b_1(\lambda), \cdots, b_{N_{\tilde{\gamma}}}(\lambda)), \lambda \in [-1, 1]$ be a generic crossing path in $\mathbb{C}^{N_{\tilde{\gamma}}}$.
Let $\{\kappa^1(\pm), \cdots, \kappa^i(\pm), \kappa^{i+1}(\pm), \cdots, \kappa^{\mu_{N_{\tilde{\gamma}}}}(\pm)\}$ be the set of ordered critical points at $\lambda = \pm 1$. We can assume that $\kappa^j(\pm) = \kappa^j$ is fixed for $j \neq i$, $\kappa^i(\pm) = \kappa^i(\lambda = \pm 1)$ and $\mathrm{Im}(\alpha^i(\lambda = 0)) = \mathrm{Im}(\alpha^{i+1})$.
If the perturbation satisfies $\mathrm{Re}\alpha^i(\lambda) < \mathrm{Re}\alpha^{i+1}$, we have the left-transformation:

$$[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(+))]^{vir} = [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(-))]^{vir}, \ \forall j \neq i, i+1 \tag{7}$$

$$\begin{aligned} [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(+))]^{vir} &= [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir} + \\ R_{i,i+1} \cdot [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir} \end{aligned} \tag{8}$$

$$[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(+))]^{vir} = [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir}, \tag{9}$$

where $R_{i,i+1}$ is the intersection number defined as above.

Wall-crossing formula

If the perturbation satisfies Re$\alpha^i(\lambda) >$ Reα^{i+1}, we have the right-transformation:

$$[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(+))]^{vir} = [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^j(-))]^{vir}, \ \forall j \neq i, i+1 \tag{10}$$

$$[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(+))]^{vir} = [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir}, \tag{11}$$

$$[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(+))]^{vir} = [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^i(-))]^{vir} + R_{i,i+1} \cdot [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}', \tilde{\gamma}; \boldsymbol{\varkappa}', \kappa^{i+1}(-))]^{vir} \tag{12}$$

Definition of the virtual cycle $[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma})]^{vir}$

Fix $\boldsymbol{\gamma} = \{\gamma_1, \cdots, \gamma_k\}$ and choose the moduli space $\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\boldsymbol{\gamma}, \boldsymbol{\varkappa})$ to be strongly regular. For each $\gamma \in G$, choose the basis $\{S^-_j(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{-\infty}, \mathbb{Q})$ corresponding to the critical points of $W_\gamma + W_{0,\gamma}$ and the dual basis $\{S_j(\gamma), j = 1, \cdots, \mu_\gamma\}$ in $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma + W_{0,\gamma})^{\infty}, \mathbb{Q})$. Then each combination $\left(S^-_{j_1}(\gamma_1), \cdots, S^-_{j_k}(\gamma_k)\right)$ corresponds to the combination of k critical points, $\boldsymbol{\varkappa}_{j_1 \cdots j_k} := \left(\kappa^-_{j_1}(\gamma_1), \cdots, \kappa^-_{j_k}(\gamma_k)\right)$. We obtain the virtual cycle $[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\Gamma; \boldsymbol{\gamma}, \boldsymbol{\varkappa}_{j_1 \cdots j_k})]^{vir}$ $=: [\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\Gamma; \boldsymbol{\gamma}, S^-_{j_1}(\gamma_1), \cdots, S^-_{j_k}(\gamma_k))]^{vir}$. Now we fix a strongly regular parameter (b^0_i); the Gauss-Manin connection provides the isomorphisms

$$GM_{(b^0_i)} : H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{\pm\infty}, \mathbb{Q}) \to H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma)^{\pm\infty}, \mathbb{Q})$$

Using the isomorphisms we can identify $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W\gamma + W_{0,\gamma})^{\pm\infty}, \mathbb{Q})$ with $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma)^{\pm\infty}, \mathbb{Q})$.

Define

$$\left[\overline{\mathscr{M}}^{\mathrm{rig}}_{W}(\Gamma)\right]^{vir} := \sum_{j_1,\cdots,j_k} \left(\left[\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\Gamma;\boldsymbol{\gamma},\boldsymbol{\varkappa}_{j_1\cdots j_k})\right]^{vir} \otimes \prod_{i=1}^{k} S_{j_i}(\gamma_i) \right) \tag{13}$$

$$\in H_*(\overline{\mathscr{M}}^{\mathrm{rig}}_{g,k,W}(\Gamma)) \otimes \prod_{\tau\in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^N_{\gamma_\tau}, W^{\infty}_{\gamma_\tau}, \mathbb{Q}) \tag{14}$$

By the Wall-crossing formula, We have

Proposition 0.17

The virtual cycle $\left[\overline{\mathscr{M}}^{\mathrm{rig}}_{W}(\Gamma)\right]^{vir}$ *is independent of the choice of the basis* $\{S_{j_i}(\gamma_i)\}$ *of* $H_{N_\gamma}(\mathbb{C}^N_\gamma, (W_\gamma)^{\pm\infty}, \mathbb{Q})$ *at each marked point* p_i*.*

Since the parallel transport induced by the Gauss-Manin connection preserves the inner product of the homology bundle, the above proposition justifies the definition of the virtual cycle $\left[\overline{\mathscr{M}}^{\mathrm{rig}}_{W}(\Gamma)\right]^{vir}$.

Definition of the virtual cycle $[\overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})]^{vir}$

Define

$$[\overline{\mathscr{M}}_W(\Gamma)]^{vir} := \frac{1}{\deg so_\Gamma}(so_\Gamma)_*[\overline{\mathscr{M}}^{\mathrm{rig}}_W(\Gamma)]^{vir},$$

where

$$so_\Gamma : \overline{\mathscr{M}}^{\mathrm{rig}}_W(\Gamma) \to \overline{\mathscr{M}}_W(\Gamma)$$

is the soften map. In particular, one has

$$[\overline{\mathscr{M}}_W(\boldsymbol{\gamma})]^{vir} := \frac{1}{\deg so}(so)_*[\overline{\mathscr{M}}^{\mathrm{rig}}_W(\boldsymbol{\gamma})]^{vir}.$$

Axioms for the virtual cycle $[\overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})]^{vir}$

Because of the Wall-crossing formula, we can tensor the virtual cycle with the dual Lefschetz thimbles to get a G-invariant cycle and then push down to the moduli space $\overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})$ to get a virtual cycle $[\overline{\mathscr{M}}_{W}(\Gamma)]^{vir}$.

We can prove the following axioms for $[\overline{\mathscr{M}}_{W}(\Gamma)]^{vir}$:

- **Dimension:** If D_Γ is not a half-integer (i.e., if $D_\Gamma \notin \frac{1}{2}\mathbb{Z}$), then $\left[\overline{\mathscr{M}}_{W}(\Gamma)\right]^{vir} = 0$. Otherwise, the cycle $\left[\overline{\mathscr{M}}_{W}(\Gamma)\right]^{vir}$ has degree

$$6g - 6 + 2k - 2D_\Gamma = 2\left((\hat{c} - 3)(1 - g) + k - \sum_{\tau \in T(\Gamma)} \iota_\tau\right). \tag{15}$$

So the cycle lies in $H_r(\overline{\mathscr{M}}_{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau\in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^N_{\gamma_\tau}, W^\infty_{\gamma_\tau}, \mathbb{Q})$, where

$$\begin{aligned} r := & \; 6g - 6 + 2k - 2D - \sum_{\tau\in T(\Gamma)} N_{\gamma_\tau} \\ = & \; 2\left((\hat{c} - 3)(1 - g) + k - \sum_{\tau\in T(\Gamma)} \iota(\gamma_\tau) - \sum_{\tau\in T(\Gamma)} \frac{N_{\gamma_\tau}}{2}\right). \end{aligned}$$

- **Symmetric group invariance**: There is a natural S_k-action on $\overline{\mathscr{M}}_{g,k,W}$ obtained by permuting the tails. This action induces an action on homology. That is, for any $\sigma \in S_k$ we have:

$$\sigma_* : H_*(\overline{\mathscr{M}}_{g,k,W}, \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_i}}(\mathbb{C}^N_{\gamma_i}, W^\infty_{\gamma_i}, \mathbb{Q})^G$$

$$\to H_*(\overline{\mathscr{M}}_{g,k,W}, \mathbb{Q}) \otimes \prod_i H_{N_{\gamma_{\sigma(i)}}}(\mathbb{C}^N_{\gamma_{\sigma(i)}}, W^\infty_{\gamma_{\sigma(i)}}, \mathbb{Q})^G.$$

  For any decorated graph Γ, let $\sigma\Gamma$ denote the graph obtained by applying σ to the tails of Γ.

  We have

$$\sigma_* \left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = \left[\overline{\mathscr{M}}_W(\sigma\Gamma)\right]^{vir}. \qquad (16)$$

- **Degenerating connected graphs:** Let Γ be a connected, genus-g, stable, decorated W-graph.
  The cycles $\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir}$ and $\left[\overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})\right]^{vir}$ are related by

$$\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = \tilde{i}^* \left[\overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})\right]^{vir} \tag{17}$$

  where $\tilde{i} : \overline{\mathscr{M}}_W(\Gamma) \to \overline{\mathscr{M}}_{g,k,W}(\boldsymbol{\gamma})$ is the canonical inclusion map.
- **Disconnected graphs:** Let $\Gamma = \coprod_i \Gamma_i$ be a stable, decorated W-graph which is the disjoint union of connected W-graphs Γ_i. The classes $\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir}$ and $\left[\overline{\mathscr{M}}_W(\Gamma_i)\right]^{vir}$ are related by

$$\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = \left[\overline{\mathscr{M}}_W(\Gamma_1)\right]^{vir} \times \cdots \times \left[\overline{\mathscr{M}}_W(\Gamma_d)\right]^{vir}. \tag{18}$$

- **Concavity:**Suppose that all the decorations on tails are *Neveu-Schwarz*, meaning that $\mathbb{C}^N_{\gamma_i} = 0$. In this case we omit the $H_{N_{\gamma_i}}(\mathbb{C}^N_{\gamma_i}, W^\infty_{\gamma_i}, \mathbb{Q})$ from our notation.

  If, furthermore, the universal W-structure $(\mathscr{L}_1, \ldots, \mathscr{L}_N)$ on the universal curve $\pi : \mathscr{C} \to \overline{\mathscr{M}}_W(\Gamma)$ is *concave* (i.e., $\pi_*\left(\bigoplus_{i=1}^t \mathscr{L}_i\right) = 0$), then the virtual cycle is given by capping the top Chern class of the orbifold vector bundle $-R^1\pi_*\left(\bigoplus_{i=1}^t \mathscr{L}_i\right)$ with the usual fundamental cycle of the moduli space:

$$\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = c_{top}\left(-R^1\pi_*\bigoplus_{i=1}^t \mathscr{L}_i\right) \cap \left[\overline{\mathscr{M}}_W(\Gamma)\right]. \tag{19}$$

- **Index zero:** Suppose that $\dim \overline{\mathscr{M}}(\Gamma) = 0$ and all the decorations on tails of Γ and edges are Neveu-Schwarz.
  If the pushforwards $\pi_* \left(\bigoplus \mathscr{L}_i \right)$ and $R^1 \pi_* \left(\bigoplus \mathscr{L}_i \right)$ are both vector spaces of the same dimension, then the virtual cycle is just the degree $\deg(\mathscr{W})$ of the Witten map times the fundamental cycle:

$$\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = \deg(\mathscr{W})[\left[\overline{\mathscr{M}}_W(\Gamma)\right]$$

  where the jth term $\mathscr{W}_j : \pi_* \left(\bigoplus \mathscr{L}_i \right) \longrightarrow R^1 \pi_* \mathscr{L}_j$ of the Witten map is given by $\mathscr{W}_j = \bar{\partial}_{x_i} \overline{W}(x_1, \ldots, x_N)$.

- **Composition law:** Given any genus-g decorated stable W-graph Γ with k tails, and given any edge e of Γ, let Γ_{cut} denote the graph obtained by "cutting" the edge e and replacing it with two unjoined tails τ_+ and τ_- decorated with γ_+ and γ_-, respectively.
  In view of the gluing/cutting commutative diagram:

$$\begin{tikzcd}
 & F \arrow[r, "pr_2"] \arrow[dl, "q"'] \arrow[dd, "pr_1"] & \overline{\mathscr{M}}_W(\Gamma) \arrow[dd, "st_\Gamma"] \\
\overline{\mathscr{M}}_W(\Gamma_{cut}) \arrow[dr, "st_{\Gamma_{cut}}"] & & \\
 & \overline{\mathscr{M}}(\Gamma_{cut}) \arrow[r, "\rho"] & \overline{\mathscr{M}}(\Gamma)
\end{tikzcd} \quad , \tag{20}$$

  the fiber product

$$F := \overline{\mathscr{M}}(\Gamma_{cut}) \times_{\overline{\mathscr{M}}(\Gamma)} \overline{\mathscr{M}}_W(\Gamma),$$

has morphisms

$$\overline{\mathscr{M}}_W(\Gamma_{cut}) \xleftarrow{q} F \xrightarrow{pr_2} \overline{\mathscr{M}}_W(\Gamma).$$

We have

$$\left\langle [\overline{\mathscr{M}}_W(\Gamma_{cut})]^{vir} \right\rangle_{\pm} = \frac{1}{\deg(q)} q_* pr_2^* \left(\left[\overline{\mathscr{M}}_W(\Gamma) \right]^{vir} \right), \tag{21}$$

where

$$\begin{aligned} \langle \quad \rangle_{\pm} : & H_*(\overline{\mathscr{M}}_W(\Gamma_{cut}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^N_{\gamma_\tau}, W^{\infty}_{\gamma_\tau}, \mathbb{Q}) \\ & \otimes H_{N_{\gamma_+}}(\mathbb{C}^N_{\gamma_+}, W^{\infty}_{\gamma_+}, \mathbb{Q}) \otimes H_{N_{\gamma_-}}(\mathbb{C}^N_{\gamma_-}, W^{\infty}_{\gamma_-}, \mathbb{Q}) \longrightarrow \\ & H_*(\overline{\mathscr{M}}_W(\Gamma_{cut}) \otimes \prod_{\tau \in T(\Gamma)} H_{N_{\gamma_\tau}}(\mathbb{C}^N_{\gamma_\tau}, W^{\infty}_{\gamma_\tau}, \mathbb{Q}) \end{aligned}$$

is contraction of the last two factors via the pairing

$$\langle , \rangle : H_{N_{\gamma_+}}(\mathbb{C}^N_{\gamma_+}, W^{\infty}_{\gamma_+}, \mathbb{Q}) \otimes H_{N_{\gamma_-}}(\mathbb{C}^N_{\gamma_-}, W^{\infty}_{\gamma_-}, \mathbb{Q}) \longrightarrow \mathbb{Q}.$$

Forgetting tails:

- Let Γ have its ith tail decorated with J^{-1}, where J is the exponential grading element of G. Further let Γ' be the decorated W-graph obtained from Γ by forgetting the ith tail and its decorations. Assume that Γ' is stable, and denote the forgetting tails morphism by

$$\vartheta : \overline{\mathscr{M}}_W(\Gamma) \to \overline{\mathscr{M}}_W(\Gamma').$$

We have

$$\left[\overline{\mathscr{M}}_W(\Gamma)\right]^{vir} = \vartheta^* \left[\overline{\mathscr{M}}_W(\Gamma')\right]^{vir}. \tag{22}$$

- In the case of $g = 0$ and $k = 3$, then the space $\overline{\mathscr{M}}_W(\gamma_1, \gamma_2, J^{-1})$ is empty if $\gamma_1\gamma_2 \neq 1$ and $\overline{\mathscr{M}}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1}) = \mathscr{B}G$. We omit $H_{N_{J^{-1}}}(\mathbb{C}^N_{J^{-1}}, W^\infty_{J^{-1}}, \mathbb{Q})^G =$ $\mathbb{Q}$ from the notation. In this case, the cycle

  $$\begin{aligned}\left[\overline{\mathscr{M}}_{0,3,W}(\gamma, \gamma^{-1}, J^{-1})\right]^{vir} \in H_*(\mathscr{B}G, \mathbb{Q}) \otimes H_{N_\gamma}(\mathbb{C}^N_\gamma, W^\infty_\gamma, \mathbb{Q})^G \otimes \\ H_{N_{\gamma^{-1}}}(\mathbb{C}^N_{\gamma^{-1}}, W^\infty_{\gamma^{-1}}, \mathbb{Q})^G\end{aligned}$$

  is the fundamental cycle of $\mathscr{B}G$ times the Casimir element. Here the Casimir element is defined as follows. Choose a basis $\{\alpha_i\}$ of $H_{N_\gamma}(\mathbb{C}^N_\gamma, W^\infty_\gamma, \mathbb{Q})^G$, and a basis $\{\beta_j\}$ of $H_{N_{\gamma^{-1}}}(\mathbb{C}^N_{\gamma^{-1}}, W^\infty_{\gamma^{-1}}, \mathbb{Q})^G$. Let $\eta_{ij} = \langle\alpha_1, \beta_j\rangle$ and (η^{ij}) be the inverse matrix of (η_{ij}). The Casimir element is defined as $\sum_{ij} \alpha_i \eta^{ij} \otimes \beta_j$.

Sums of Singularities:
If $W_1 \in \mathbb{C}[z_1, \dots, z_t]$ and $W_2 \in C[z_{t+1}, \dots, z_{t+t'}]$ are two quasi-homogeneous polynomials with diagonal automorphism groups G_1 and G_2, and if we write $W = W_1 + W_2$ then the diagonal automorphism group of W is $G = G_1 \times G_2$. Further, the state space $\mathscr{H}_W$ is naturally isomorphic to the tensor product

$$\mathscr{H}_W = \mathscr{H}_{W_1} \otimes \mathscr{H}_{W_2}, \tag{23}$$

and the space $\overline{\mathscr{M}}_{g,k,W}$ is naturally isomorphic to the fiber product

$$\overline{\mathscr{M}}_{g,k,W} = \overline{\mathscr{M}}_{g,k,W_1} \times_{\overline{\mathscr{M}}_{g,k}} \overline{\mathscr{M}}_{g,k,W_2}.$$

Indeed, since any G-decorated stable graph Γ is equivalent to the choice of a G_1-decorated graph Γ_1 and G_2-decorated graph Γ_2 with the the same underlying graph $\overline{\Gamma}$, we have

$$\overline{\mathscr{M}}_W(\Gamma) = \overline{\mathscr{M}}_{W_1}(\Gamma_1) \times_{\overline{\mathscr{M}}(\overline{\Gamma})} \overline{\mathscr{M}}_{W_2}(\Gamma_2). \tag{24}$$

The natural inclusion

$$\overline{\mathscr{M}}_{g,k,W} = \overline{\mathscr{M}}_{g,k,W_1} \times_{\overline{\mathscr{M}}_{g,k}} \overline{\mathscr{M}}_{g,k,W_2} \xhookrightarrow{\Delta} \overline{\mathscr{M}}_{g,k,W_1} \times \overline{\mathscr{M}}_{g,k,W_2}$$

together with the isomorphism of middle homology induces a homomorphism

$$\begin{aligned}\Delta^* \quad : \quad & \left(H_*(\overline{\mathscr{M}}_{g,k,W_1}, \mathbb{Q}) \otimes \prod_{i=1}^{k} H_{N_{\gamma_{i,1}}}(\mathbb{C}^N_{\gamma_{i,1}}, (W_1)^\infty_{\gamma_{i,1}}, \mathbb{Q})^G \right) \\ & \otimes \left(H_*(\overline{\mathscr{M}}_{g,k,W_2}, \mathbb{Q}) \otimes \prod_{i=1}^{k} H_{N_{\gamma_{i,2}}}(\mathbb{C}^N_{\gamma_{i,2}}, (W_2)^\infty_{\gamma_{i,2}}, \mathbb{Q})^G \right) \\ & \longrightarrow H_*(\overline{\mathscr{M}}_{g,k,W_1+W_2}, \mathbb{Q}) \otimes \prod_{i=1}^{k} H_{N_{(\gamma_{i,1},\gamma_{i,2})}}(\mathbb{C}^N_{(\gamma_{i,1},\gamma_{i,2})}, (W_1 + W_2)^\infty_{(\gamma_{i,1},\gamma_{i,2})}, \mathbb{Q})^{G_1 \times G_2}\end{aligned}$$

The virtual cycle satisfies

$$\Delta^* \left(\left[\overline{\mathscr{M}}_{g,k,W_1} \right]^{vir} \otimes \left[\overline{\mathscr{M}}_{g,k,W_2} \right]^{vir} \right) = \left[\overline{\mathscr{M}}_{g,k,W_1+W_2} \right]^{vir} \qquad (25)$$

Quantum singularity theory and its application

1. State space and pairing
2. Cohomological field theory, quantum ring and potential functions.
3. Landau-Ginzburg B model: Saito's Frobenius manifold and mirror symmetry
4. Mirror symmetry between different singularities
5. Generalized Witten conjecture for ADE-singularities
6. CY/LG correspondence
7. Further questions: singularity of complete intersection.

The End