Clay Research Awards

The Weinstein Conjecture

Classical mechanics, as formulated by Hamilton, takes place in the context of a configuration space of positions and momenta. Mathematically, this is a manifold M with a symplectic structure and a distinguished function H, the *Hamiltonian*. The symplectic structure is given by a closed 2-form ω such that $\omega^n(x) \neq 0$, for all x in M, where M is of dimension $2n$. Such a manifold carries a natural vector field X_H defined by the condition

$$\omega(X_H, Y) = dH(Y)$$

for all Y. This, the *Hamitonian vectorfield*, defines a flow $\phi_t(x)$ on the manifold. If $x = (q, p)$ give the position and momentum of a particle, its trajectory as time evolves is given by $\phi_t(x)$ for varying t. The flow itself is defined by Hamilton's equations,

$$\dot{p} = \frac{\partial H}{\partial q}, \qquad \dot{q} = -\frac{\partial H}{\partial p}$$

where q and q are Darboux coordinates, giving conjugate positions and momenta.

The flow just defined determines a *dynamical system*. A fundamental problem is whether or not there exist closed orbits for such a system. For example, we hope that the orbit of the earth is both closed and quite stable. Orbits, of course, lie on the level sets of H, which is commonly taken to be the total energy.

In the late 1970s Rabinowitz and Weinstein proved that for $H : R^{2n} \longrightarrow \mathbb{R}$ which has either star-shaped or convex level sets, the corresponding Hamiltonian flow has a periodic orbit on the level sets. In searching for a common generalization, Weinstein observed that a contact structure could be seen as the engine which makes the arguments work.

A contact manifold is an odd-dimensional manifold with a one-form A such that $A \wedge dA^n$ is everywhere nonzero. The kernel of A is a maximally nonintegrable field of hyperplanes in the tangent bundle; the Reeb vector field generates the kernel of dA and pairs to one with A. For a motivating example, consider the unit sphere in $\mathbb{C}^n$, where A is the a standard form which annihilates the maximal complex subspace of the tangent space. If Z is a coordinate vector for $\mathbb{C}^n$, then $A = \sqrt{-1}(\bar{\partial} - \partial) log||Z||^2$ is such a form. In this case the Reeb vector field is the field tangent to the circles in the fibration $S^{2n-1} \longrightarrow \mathbb{CP}^n$ from the sphere to the associated complex projective space.

Cliff Taubes and Claire Voisin delivering their acceptance speeches.

The Weinstein conjecture, stated some thirty years ago, asks whether the Reeb vector field for a contact manifold always has a a closed orbit. By contrast, there exist arbitrary vector fields on the three-sphere not annihilated by dA with no closed orbits. These are the counterexamples to the Seifert Conjecture of Schweitzer, Harrison and Kuperberg.

Hofer proved the Weinstein Conjecture in many special cases in dimension three, for example, the three-sphere and contact structures on any three-dimensional reducible manifold. Taubes' solution to the general conjecture in dimension three is based on a novel application of the Seiberg-Witten equations to the problem. The orbits come from special cycles in the Seiberg-Witten Monopole Floer Homology.

Cliff Taubes receiving the 2008 Clay Research Award that was presented by Landon and Lavinia Clay and President James Carlson.

The Kodaira Conjecture

Geometric structures on a topological manifold often impose restrictions on what kind of manifolds can arise. For example, a symplectic manifold must have nonzero second Betti number, since the symplectic form ω is non-trivial in cohomology. Indeed, if the manifold has dimension $2n$, then ω^n has nonzero integral. Yet more restrictive is the notion of a Kähler manifold – a symplectic manifold for which the form ω has type $(1,1)$ in a compatible complex structure. In that case many topological conditions are satisfied: the odd Betti numbers are even, the cohomology ring is formal, and there are numerous restrictions on the fundamental group. Kähler manifolds abound: any projective algebraic manifold, that is, any submanifold of complex projective space defined by homogeneous polynomial equations, is a Kähler manifold. In complex dimension one, the converse is true: any Kähler manifold (a Riemann surface) is complex projective. In complex dimension two, the converse is false, but just barely: every complex Kähler manifold is the deformation of a projective algebraic manifold. This fact was proved by Kodaira, using his classification theorem for complex surfaces.

The question then arises: is every compact Kähler manifold deformable to projective algebraic one? Although never explicitly stated by Kodaira, this question has become known as the *Kodaira Conjecture*. Alas, the proof in dimension two gives no clue about what happens in higher dimension. The crux of the problem, however, is to show that on the given complex manifold M, can one deform the complex structure so as to obtain a positive $(1,1)$ class in the rational cohomology. That is, one must show that the Hodge structure is *polarizable*. The fundamental theorem here is due to Kodaira: from a closed, rational, positive, $(1,1)$ form, one may construct an imbedding of the underlying manifold into projective space.

There have been various attempts to prove or disprove the conjecture. Since any deformation of M has the same diffeomorphism type as M, a disproof requires a topological invariant defined for Kähler manifolds that distinguishes the projective algebraic ones from those that are not.

The starting point for Voisin's counterexample is the construction of a complex torus T which is not projective algebraic because of the existence of a "wild" endomorphism Φ. This is an endomorphishm whose eigenvalues are non-real and distinct, and such that the Galois group of the field generated by the eigenvalues is as large as possible. An example is given by the companion matrix of the polynomial $x^4 - x + 1$. The second exterior product of a weight one Hodge structure with a wild endomorphism carries no nonzero rational $(1,1)$ classes, so long as the space of elements of type $(1,0)$ has dimension strictly greater than one. Therefore the complex manifold T is not projective algebraic, though it can, of course, be deformed to an algebraic torus. The actual counterexample is a suitable blowup of $T \times T$. Consider the subvarieties $T \times \{0\}$, $\{0\} \times T$, the graph of the diagonal, and the graph of Φ. Blow up the points of intersection of the diagonals of the identity and of Φ and also the intersection of $T \times \{0\}$ with the graph of Φ. Then blow up the proper transforms of the subvarieties to obtain a Kähler manifold V with $H^2(V) \cong \Lambda^2 H^1(T)$. Any deformation of a blowup of a complex torus is obtained first by deforming the torus and then deforming the blowup. From this one sees that the wild endomorphism is preserved. Therefore the Hodge structure $H^2(V)$ contains no rational $(1,1)$ classes, and so V, and indeed any Kähler manifold with the same cohomology ring as V, is not projective algebraic. The same kind of construction yields a disproof of the Kodaira conjecure in dimension four or greater. Voisin also gives simply connected counterexamples in dimension six and greater.

Claire Voisin receiving the 2008 Clay Research Award that was presented by Landon and Lavinia Clay and President James Carlson.