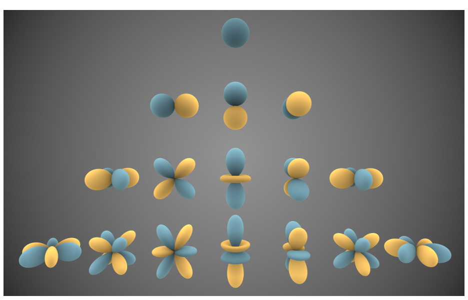
David Ben-Zvi

David Ben-Zvi gave a very elegant and wide-ranging lecture on representation theory as seen through the lens of gauge theory, focusing on four themes:

- Harmonic analysis as the exploitation of symmetry;
- Commutative algebra as a signal of geometry;
- Topology as a source of commutativity;
- Gauge theory as a bridge between topology and representation theory.

These he explored in three settings: representation theory, quantum field theory, and, bridging between them, gauge theory. The first centered on harmonic analysis and various generalizations of the Fourier transform. A basic example is the way in which Fourier series can be understood in terms of the action of the circle group $G = U(1)$ on $L^2(S^1)$. Here one can move back and forth between a representation theory picture and a geometric one in which the central object is a family of vector spaces over the dual $\hat{G}$—the set of irreducible unitary representations of G. The correspondence is through the spectral decomposition of $L^2(S^1)$ under the action of the ring $\mathbb{C}G$ of linear combinations of group elements. In line with the second theme, a geometric picture emerges because G is abelian and $\mathbb{C}G$ is commutative.

By looking at the $U(1)$ representations in this way one has a model of how to understand more complex cases. The extension to non-abelian groups is motivated by studying the action of $SO(3)$ on $L^2(S^2)$. Here the decomposition is the expansion in spherical harmonics, that is, in eigenfunctions of the Laplacian.



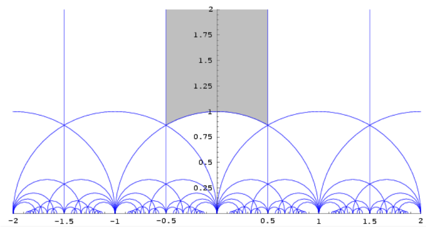
The Laplacian in turn is picked out by the Casimir element $C = (\boldsymbol{i}^2 + \boldsymbol{j}^2 + \boldsymbol{k}^2)$ in the center of the enveloping algebra of $SO(3)$. In this case the unitary dual is $\widehat{SO}(3) = \mathbb{Z}_+$ and C determines a function $\ell \mapsto \ell(\ell + 1)$ on the dual.

The lesson from this example is that to see geometry one should look for commutative algebra. With that in mind, one seeks to access $\hat{G}$ more generally for a nonabelian group by exploiting Schur's lemma, that the center of $\mathbb{C}G$ acts by scalar multiplication in any unitary representation. So a representation determines a function on $\hat{G}$. Conversely functions on $\hat{G}$ give rise to operators that commute with the action of G and so determine symmetries of arbitrary representations of G.

Ben-Zvi moved on to explore this theme within the Langland's program, focusing on automorphic forms and the spectral decomposition of $L^2(X)$ when

$$X = SL_2\mathbb{Z}\backslash SL_2\mathbb{R}/SO_2$$

is the moduli space of elliptic curves. Here in addition to the Laplacian (and higher Casimirs), there are also the actions of the algebras of Hecke operators, one for each prime p. All Hecke operators for almost all p commute.



One can understand the Langlands program in this way: first, study the spectral decomposition of $L^2(X)$; second, identify the joint spectrum with a space of Galois representations.

Turning to the second topic, Ben-Zvi gave a brief outline of quantum theory and quantum field theory, as formalized by Atiyah and Segal, before focusing on topological field theories (TFTs). An n-dimensional TFT is to be thought of as a structure that associates vector spaces $\mathcal{Z}(M)$ of states with $(n-1)$-dimensional manifolds M. More formally a functor satisfying certain axioms, from the cobordism category, which has $(n-1)$-manifolds as objects and cobordisms as morphisms, to the category of vector spaces .

In constructing TFTs, it is simpler to focus on states attached to small patches of space. This leads to a picture in which states attached to a compact n-manifold M are determined by *local boundary conditions*; that is $(n-1)$-dimensional field theories on the $(n-2)$-dimensional boundaries of small regions in M. The boundaries themselves may have interfaces: one also has field theories on the interfaces of boundaries, and on the interfaces of interfaces. Proceeding in this way, one is led into higher category theory, with the interfaces between interfaces giving 2-morphisms and so on.

An important idea here is Lurie's version of the Baez-Dolan cobordism hypothesis, that *an n-dimensional topological field theory $\mathcal{Z}$ is uniquely determined by its higher category of boundary conditions*. This gives a deep geometric perspective on category theory: categories correspond to 2-dimensional quantum field theories and higher categories to higher dimensional theories.

Within this framework, one also has an elegant illustration of the theme 'topology as a source of commutativity'. Measurements at points of space-time are represented by *local operators* in TFT; these correspond to states on small spheres—the boundaries of balls containing the points. Composition of operators is determined by cobordism, with commutativity for $n \geq 2$ following from a topological argument. When $n = 1$ (the quantum mechanics case), the operators do not necessarily commute. The situation mirrors the behavior of homotopy groups: π_1 need not be abelian, while π_n is necessarily abelian for $n \geq 2$.

The final part of the lecture presented the gauge theory of a group G as a bridge between analogous structures that emerge in representation theory and quantum field theory.

Locally, connections are trivial, but they carry pointwise G symmetry. This suggests a focus not on the global picture but on boundary conditions on small pieces of space-time. Motivated by the cobordism hypothesis one is led to consider $(n-1)$-dimensional quantum field theories and representations of G on field theories. In a topological context, these should determine the whole theory.

The approach was illustrated first in the case of two-dimensional topological Yang-Mills theories, in which the boundary conditions are one-dimensional TFTs, that is, quantum theories. The boundary conditions in this case correspond to representations of G.

A more significant realization of the connection is in the geometric Langlands program. Here one studies the moduli space of G-bundles on a Riemann surface C for a complex reductive group G and the action of Hecke operators at points of C. As suggested by Kapustin and Witten, a Hecke operator can be interpreted in terms of the creation of a monopole in the three-dimensional gauge theory on $C \times \mathbb{R}$, and hence, by using the same ideas as those underlying the cobordism hypothesis, as an operator on states on C. Inspired by the topological argument for commutativity of local operators in QFT, Beilinson and Drinfeld used the latter point of view to identify the source of commutativity for Hecke operators, in line with the third theme.

The commutativity of Hecke operators signals the presence of geometry, as in the second theme. The study of their spectrum leads to the identification of Hecke operators with representations of the Langlands dual group $\hat{G}$.