PREPARTORY NOTES ON p-ADIC HODGE THEORY

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- *Everyone should learn the basic formalism of Witt vectors before arriving in Hawaii.* A nice succinct development of this circle of ideas is given in notes of Benji Fisher that are provided in a separate file, along with the first half of §4.2 below (up through and including Remark 4.2.4).
- *Please try to at least skim over §1–§2 and §5 before the summer school begins.* The course will go much further into the theory than do these preparatory notes (e.g., (ϕ, Γ)-modules, overconvergence, integral methods, etc.). You are not expected to have mastered §1, §2, and §5 beforehand, but the course will begin with a quick overview of that stuff, so prior awareness with the style of thinking in those sections will be very helpful.
- Don't worry – *you are NOT expected to have read all of these notes before Hawaii* (but feel free to do so if time permits). The material in §3–§4 and §6 is where things really get off the ground, but that stuff is harder to digest. The hardest part for a beginner is probably §4 after Remark 4.2.4, especially the main constructions there (the ring R and the ring B_{dR}).

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Date: April 15, 2009.

B.C. was partially supported by NSF grants DMS-0600919 and DMS-0917686, and we are grateful to B. Cais, I. Ikeda, R. Schoof, and S. Unver for comments on an earlier version of these notes.

1. Motivation

1.1. **Tate modules.** Let E be an elliptic curve over a number field F, and fix an algebraic closure $\overline{F}/F$ and a prime number p. A fundamental arithmetic invariant of E is the $\mathbf{Z}$-rank of its finitely generated Mordell-Weil group $E(F)$ of rational points over F. This is conjecturally encoded in (and most fruitfully studied via) the p-adic representation of $G_F := \mathrm{Gal}(\overline{F}/F)$ associated to E. Let us review where this representation comes from, as well as some of its interesting properties.

For each $n \geq 1$ we can choose an isomorphism of abelian groups

$$\iota_{E,n} : E(\overline{F})[p^n] \simeq (\mathbf{Z}/p^n\mathbf{Z})^2$$

in which G_F acts on the left side through the finite Galois group quotient $\mathrm{Gal}(F(E[p^n])/F)$ associated to the field generated by coordinates of p^n-torsion points of E. By means of $\iota_{E,n}$ we get a representation of this finite Galois group (and hence of G_F) in $\mathrm{GL}_2(\mathbf{Z}/p^n\mathbf{Z})$. As n grows, the open kernel of this representation shrinks in G_F. It is best to package this collection of representations into a single object: we can choose the $\iota_{E,n}$'s to be compatible with respect to reduction modulo p-powers on the target and the multiplication map $E[p^{n+1}] \to E[p^n]$ by p on the source to get an isomorphism of $\mathbf{Z}_p$-modules

$$\mathrm{T}_p(E) := \varprojlim E(\overline{F})[p^n] \simeq \mathbf{Z}_p^2$$

on which G_F acts through a *continuous* representation

$$\rho : G_F \to \mathrm{GL}_2(\mathbf{Z}_p);$$

passing to the quotient modulo p^n recovers the representations on torsion points as considered above.

For any prime $\wp$ of F we choose an embedding of algebraic closures $\overline{F} \hookrightarrow \overline{F_\wp}$ (i.e., we lift the $\wp$-adic place of F to one of $\overline{F}$) to get a decomposition subgroup $G_{F_\wp} \subseteq G_F$, so we may restrict ρ to this subgroup to get a continuous representation $\rho_\wp : G_{F_\wp} \to \mathrm{GL}_2(\mathbf{Z}_p)$ that encodes local information about E at $\wp$. More specifically, if $I_\wp \subseteq G_{F_\wp}$ denotes the inertia subgroup and we identify the quotient $G_{F_\wp}/I_\wp$ with the Galois group $G_{k(\wp)}$ of the finite residue field $k(\wp)$ at $\wp$ then we say that $\rho_\wp$ (or ρ) is *unramified* at $\wp$ if it is trivial on $I_\wp$, in which case it factors through a continuous representation $G_{k(\wp)} \to \mathrm{GL}_2(\mathbf{Z}_p)$. In such cases it is natural to ask about the image of the (arithmetic) Frobenius element $\mathrm{Frob}_\wp \in G_{k(\wp)}$ that acts on $\overline{k(\wp)}$ by $x \mapsto x^{q_\wp}$, where $q_\wp := \#k(\wp)$.

Theorem 1.1.1. *If $\wp \nmid p$ then E has good reduction at $\wp$ (with associated reduction over $k(\wp)$ denoted as $\overline{E}$) if and only if $\rho_\wp$ is unramified at $\wp$. In such cases, $\rho_\wp(\mathrm{Frob}_\wp)$ acts on $\mathrm{T}_p(E)$ with characteristic polynomial $X^2 - a_{E,\wp}X + q_\wp$, where $a_{E,\wp} = q_\wp + 1 - \#\overline{E}(k(\wp)) \in \mathbf{Z} \subseteq \mathbf{Z}_p$.*

Remark 1.1.2. Observe that $a_{E,\wp}$ is a rational integer that is independent of the choice of p (away from $\wp$). By Hasse's theorem, $|a_{E,\wp}| \leq 2\sqrt{q_\wp}$. If we had only worked with the representation $\rho \bmod p^n$ on p^n-torsion points rather than with the representation ρ that encodes all p-power torsion levels at once then we would only obtain $a_{E,\wp} \bmod p^n$ rather than $a_{E,\wp} \in \mathbf{Z}$. By the Hasse bound, this sufficies to recover $a_{E,\wp}$ when $q_\wp$ is "small" relative to p^n (i.e., $4\sqrt{q_\wp} < p^n$).

It was conjectured by Birch and Swinnerton-Dyer that $\mathrm{rank}_{\mathbf{Z}}(E(F))$ is encoded in the behavior at $s = 1$ of the Euler product

$$L_{\mathrm{good}}(s, E/F) = \prod_{\mathrm{good}\,\wp} (1 - a_{E,\wp} q_\wp^{-s} + q_\wp^{1-2s})^{-1};$$

this product is only known to make sense for $\mathrm{Re}(s) > 3/2$ in general, but it has been meromorphically continued to the entire complex plane in many special cases (by work of Taylor-Wiles and its generalizations). For each p, the G_F-representation on $\mathrm{T}_p(E)$ encodes all Euler factors at primes $\wp$ of good reduction away from p by Theorem 1.1.1. For this reason, the theory of p-adic representations of Galois groups turns out to be a very convenient framework for studying the arithmetic of L-functions.

Question 1.1.3. Since the notion of good reduction makes sense at $\wp$ without any reference to p, it is natural to ask if there is an analogue of Theorem 1.1.1 when $\wp | p$.

This question was first answered by Grothendieck using p-divisible groups, and his answer can be put in a more useful form by means of some deep results in p-adic Hodge theory: the property of being unramified at $\wp$ (for $\wp \nmid p$) winds up being replaced with the property of being a *crystalline* representation at $\wp$ (when $\wp | p$). This latter notion will be defined much later, but for now we wish to indicate why unramifiedness cannot be the right criterion when $\wp | p$. The point is that the determinant character $\det \rho_\wp : G_{F_\wp} \to \mathbf{Z}_p^\times$ is infinitely ramified when $\wp | p$. In fact, this character is equal to the p-adic cyclotomic character of $F_\wp$, a character that will be ubiquitous in all that follows. We therefore now recall its definition in general (and by Example 1.1.5 below this character is infinitely ramified on $G_{F_\wp}$).

Let F be a field with a fixed separable closure F_s/F and let p be a prime distinct from $\mathrm{char}(F)$. Let $\mu_{p^n} = \mu_{p^n}(F_s)$ denote the group of p^nth roots of unity in $F_s^\times$, and let μ_{p^∞} denote the rising union of these subgroups. The action of G_F on μ_{p^∞} is given by $g(\zeta) = \zeta^{\chi(g)}$ for a unique $\chi(g) \in \mathbf{Z}_p^\times$: for $\zeta \in \mu_{p^n}$ the exponent $\chi(g)$ only matters modulo p^n, and $\chi(g) \bmod p^n \in (\mathbf{Z}/p^n\mathbf{Z})^\times$ describes the action of g on the finite cyclic group μ_{p^n} of order p^n. Thus, $\chi \bmod p^n$ has open kernel (corresponding to the finite extension $F(\mu_{p^n})/F$) and χ is continuous. We call χ the *p-adic cyclotomic character* of F.

Remark 1.1.4. Strictly speaking we should denote the character χ as $\chi_{F,p}$, but it is permissible to just write χ because p is always understood from context and if F'/F is an extension (equipped with a compatible embedding $F_s \to F'_s$ of separable closures) then $\chi_{F,p}|_{G_{F'}} = \chi_{F',p}$.

Example 1.1.5. Let F be the fraction field of a complete discrete valuation ring R with characteristic 0 and residue characteristic p. Hence, $\mathbf{Z}_p \subseteq R$, so we may view $\overline{\mathbf{Q}}_p \subseteq \overline{F}$. In this case $F(\mu_{p^\infty})/F$ is infinitely ramified, or in other words $\chi : G_F \to \mathbf{Z}_p^\times$ has infinite image on the inertia subgroup $I_F \subseteq G_F$. Indeed, since $e := \mathrm{ord}_F(p)$ is finite $F(\mu_{p^n})$ has ramification degree e_n over F satisfying $e_n \cdot e \geq \mathrm{ord}_{\mathbf{Q}_p(\mu_{p^n})}(p) = p^{n-1}(p-1)$, so $e_n \to \infty$.

1.2. **Galois lattices and Galois deformations.**

Moving away from elliptic curves, we now consider a wider class of examples of p-adic representations arising from algebraic geometry, and we shall formulate a variant on Question 1.1.3 in this setting.

Let X be an algebraic scheme over a field F; the case of smooth projective X is already very interesting. For a prime $p \neq \mathrm{char}(F)$, the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X_{F_s}, \mathbf{Z}_p)$ are finitely generated $\mathbf{Z}_p$-modules that admit a natural action by $G_F = \mathrm{Gal}(F_s/F)$ (via pullback-functoriality of cohomology and the natural G_F-action on $X_{F_s} = X \otimes_F F_s$), and these modules not be torsion-free. Hence, the G_F-action on them is not described via matrices in general, but satisfies a continuity condition in the sense of the following definition.

Definition 1.2.1. Let Γ be a profinite group. A *continuous representation of* Γ on a finitely generated $\mathbf{Z}_p$-module Λ is a $\mathbf{Z}_p[\Gamma]$-module structure on Λ such that the action map $\Gamma \times \Lambda \to \Lambda$ is continuous (or, equivalently, such that the Γ-action on the finite set $\Lambda/p^n\Lambda$ has open kernel for all $n \geq 1$). These form a category denoted $\mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$, and $\mathrm{Rep}_{\mathbf{F}_p}(\Gamma)$ is defined similarly.

Example 1.2.2. If a $\mathbf{Z}_p[\Gamma]$-module Λ is finite free as a $\mathbf{Z}_p$-module then $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$ if and only if the matrix representation $\Gamma \to \mathrm{GL}_n(\mathbf{Z}_p)$ defined by a choice of $\mathbf{Z}_p$-basis of Λ is a continuous map.

Example 1.2.3. Let F be a number field and consider the action by G_F on $\mathrm{H}^i_{\text{ét}}(X_{F_s}, \mathbf{Z}_p)$ for a smooth proper scheme X over F. This is unramified at all but finitely many primes $\wp$ of F (i.e., $I_\wp \subseteq G_F$ acts trivially) due to "good reduction" properties for X at all but finitely many primes (and some general base change theorems for étale cohomology). However, if X has good reduction (appropriately defined) at a prime $\wp$ over p then this p-adic representation is rarely unramified at $\wp$. Is there a nice property satisfied by this p-adic representation at primes $\wp|p$ of good reduction for X, replacing unramifiedness? Such a replacement will be provided by p-adic Hodge theory.

Galois representations as in Example 1.2.3 are the source of many interesting representations, such as those associated to modular forms, and Wiles developed techniques to prove that various continuous representations $\rho : G_F \to \mathrm{GL}_n(\mathbf{Z}_p)$ not initially related to modular forms in fact arise from them in a specific manner. His technique rests on deforming ρ; the simplest instance of a deformation is a continuous representation

$$\widetilde{\rho} : G_F \to \mathrm{GL}_n(\mathbf{Z}_p[\![x]\!])$$

that recovers ρ at $x = 0$ and is unramified at all but finitely many primes of F. A crucial part of Wiles' method is to understand deformations of $\rho|_{G_{F_\wp}}$ when $\wp|p$, and some of the most important recent improvements on Wiles' method (e.g., in work of Kisin [4], [5]) focus on precisely such $\wp$. For these purposes it is essential to work with Galois representations having coefficients in $\mathbf{Z}_p$ or $\mathbf{F}_p$, a necessary prelude to many such considerations is a solid understanding of the case of $\mathbf{Q}_p$-coefficients, and much of p-adic Hodge theory is focused on this latter case. This leads us to make the following definition.

Definition 1.2.4. A *p-adic representation* of a profinite group Γ is a representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{Q}_p}(V)$ of Γ on a finite-dimensional $\mathbf{Q}_p$-vector space V such that ρ is continuous (viewing $\mathrm{Aut}_{\mathbf{Q}_p}(V)$ as $\mathrm{GL}_n(\mathbf{Q}_p)$ upon choosing a basis of V). The category of such representations is denoted $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$.

Exercise 1.2.5. For $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(\Gamma)$, prove that the scalar extension $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$ lies in $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$.

The example in Exercise 1.2.5 is essentially the universal example, due to the next lemma.

Lemma 1.2.6. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$, there exists a Γ-stable $\mathbf{Z}_p$-lattice $\Lambda \subseteq V$ (i.e., Λ is a finite free $\mathbf{Z}_p$-submodule of V and $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda \simeq V$).*

Proof. Let $\rho : \Gamma \to \mathrm{Aut}_{\mathbf{Q}_p}(V)$ be the continuous action map. Choose a $\mathbf{Z}_p$-lattice $\Lambda_0 \subseteq V$. Since $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_0$, we naturally have $\mathrm{Aut}_{\mathbf{Z}_p}(\Lambda_0) \subseteq \mathrm{Aut}_{\mathbf{Q}_p}(V)$ and this is an open subgroup. Hence, the preimage $\Gamma_0 = \rho^{-1}(\mathrm{Aut}_{\mathbf{Z}_p}(\Lambda))$ of this subgroup in Γ is open in Γ. Such an open subgroup has finite index since Γ is compact, so Γ/Γ_0 has a finite set of coset representatives $\{\gamma_i\}$. Thus, the finite sum $\Lambda = \sum_i \rho(\gamma_i)\Lambda_0$ is a $\mathbf{Z}_p$-lattice in V, and it is Γ-stable since Λ_0 is Γ_0-stable and $\Gamma = \coprod \gamma_i \Gamma_0$. ∎

1.3. **Aims of p-adic Hodge theory.** In the study of p-adic representations of $G_F = \mathrm{Gal}(\overline{F}/F)$ for F of finite degree over $\mathbf{Q}_p$, it is very convenient in many proofs if we can pass to the case of an algebraically closed residue field. In practice this amounts to replacing F with the completion $\widehat{F^{\mathrm{un}}}$ of its maximal unramified extension inside of $\overline{F}$ (and replacing G_F with its inertia subgroup I_F; see Exercise 1.3.2(1) below). Hence, it is convenient to permit the residue field k to be either finite or algebraically closed, and so allowing perfect residue fields provides a good degree of generality.

Definition 1.3.1. A *p-adic field* is a field K of characteristic 0 that is complete with respect to a fixed discrete valuation that has a perfect residue field k of characteristic $p > 0$.

Exercise 1.3.2. Let K be a p-adic field with residue field k.

(1) Explain why the valuation ring of K is naturally a local extension of $\mathbf{Z}_p$, and prove that $[K : \mathbf{Q}_p]$ is finite if and only if k is finite.
(2) Let $K^{\mathrm{un}} \subseteq \overline{K}$ denote the maximal unramified extension of K inside of a fixed algebraic closure (i.e., it is the compositum of all finite unramified subextensions over K). Prove that the completion $\widehat{K^{\mathrm{un}}}$ is naturally a p-adic field with residue field $\overline{k}$ that is an algebraic closure of k, and use Krasner's Lemma to prove that $I_K := G_{K^{\mathrm{un}}}$ is naturally isomorphic to $G_{\widehat{K^{\mathrm{un}}}}$ as profinite groups. More specifically, prove that $L \rightsquigarrow L \otimes_{K^{\mathrm{un}}} \widehat{K^{\mathrm{un}}}$ is an equivalence of categories from finite extensions of K^{un} to finite extensions of $\widehat{K^{\mathrm{un}}}$.

Most good properties of p-adic representations of G_K for a p-adic field K will turn out to be detected on I_K, so replacing K with $\widehat{K^{\mathrm{un}}}$ is a ubiquitious device in the theory (since $I_K := G_{K^{\mathrm{un}}} = G_{\widehat{K^{\mathrm{un}}}}$ via Exercise 1.3.2(2); note that K^{un} is not complete if $k \neq \overline{k}$). The goal of p-adic Hodge theory is to identify and study various "good" classes of p-adic representations of G_K for p-adic fields K, especially motivated by properties of p-adic representations arising from algebraic geometry over p-adic fields.

The form that this study often takes in practice is the construction of a dictionary that relates good categories of p-adic representations of G_K to various categories of semilinear algebraic objects "over K". By working in terms of semilinear algebra it is often easier to deform, compute, construct families, etc., than is possible by working solely with Galois representations. There are two toy examples of this philosophy that are instructive before we take up the development of the general theory (largely due to Fontaine and his coworkers), and we now explain both of these toy examples (which are in fact substantial theories in their own right).

Example 1.3.3. The theory of *Hodge–Tate representations* was inspired by Tate's study of $\mathrm{T}_p(A)$ for abelian varieties A with good reduction over p-adic fields, and especially by Tate's question as to how the p-adic representation $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ arising from a smooth proper K-scheme X is related to the Hodge cohomology $\oplus_{p+q=n} \mathrm{H}^p(X, \Omega^q_{X/K})$. This question concerns finding a p-adic analogue of the classical Hodge decomposition

$$\mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}^n_{\mathrm{top}}(Z(\mathbf{C}), \mathbf{Q}) \simeq \bigoplus_{p+q=n} \mathrm{H}^p(Z, \Omega^q_Z)$$

for smooth proper $\mathbf{C}$-schemes Z.

In §2 we will define the notion of a Hodge–Tate representation of G_K, and the linear algebra category over K that turns out to be related to Hodge–Tate representations of G_K is the category $\mathrm{Gr}_{K,f}$ of finite-dimensional graded K-vector spaces (i.e., finite-dimensional K-vector spaces V equipped with a direct sum decomposition $V = \oplus_q V_q$, and maps $T : V' \to V$ that are K-linear and satisfy $T(V'_q) \subseteq V_q$ for all q).

Example 1.3.4. A more subtle class of representations arises from the Fontaine–Wintenberger theory of norm fields, and gives rise to the notion of an *étale φ-module* that will arise repeatedly (in various guises) throughout p-adic Hodge theory. The basic setup goes as follows. Fix a p-adic field K and let K_∞/K be an infinitely ramified algebraic extension such that the Galois closure K'_∞/K has Galois group $\mathrm{Gal}(K'_\infty/K)$ that is a p-adic Lie group. The simplest such example is $K_\infty = K'_\infty = K(\mu_{p^\infty})$, in which case K_∞/K is infinitely ramified by Example 1.1.5 and the infinite subgroup $\mathrm{Gal}(K_\infty/K) \subseteq \mathbf{Z}_p^\times$ that is the image of the continuous p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ is closed and hence open. (Indeed, the p-adic logarithm identifies $1 + p\mathbf{Z}_p$ with $p\mathbf{Z}_p$ for odd p and identifies $1 + 4\mathbf{Z}_2$ with $4\mathbf{Z}_2$ for $p = 2$, and every nontrivial closed subgroup of $\mathbf{Z}_p$ is open.) Another interesting example that arose in work of Breuil and Kisin is the non-Galois extension $K_\infty = K(\pi^{1/p^\infty})$ generated by compatible p-power roots of a fixed uniformizer π of K, in which case $\mathrm{Gal}(K'_\infty/K)$ is an open subgroup of $\mathbf{Z}_p^\times \ltimes \mathbf{Z}_p$.

For any K_∞/K as above, a theorem of Sen ensures that the closed ramification subgroups of $\mathrm{Gal}(K'_\infty/K)$ in the upper numbering are of finite index, so in particular K_∞ with its natural absolute value has residue field k' that is a finite extension of k. The Fontaine–Wintenberger theory of norm fields [9] provides a remarkable functorial equivalence between the category of separable algebraic extensions of K_∞ and the category of separable algebraic extensions of an associated local field E of equicharacteristic p (the "field of norms" associated to K_∞/K). The residue field of E is naturally identified with k', so non-canonically we have $E \simeq k'(\!(u)\!)$.

Upon choosing a separable closure of K_∞, the Fontaine–Wintenberge equivalence yields a separable closure for E and an associated canonical topological isomorphism of the associated absolute Galois groups

$$G_{K_\infty} \simeq G_E. \tag{1.3.1}$$

Because E has equicharacteristic p, we will see in §3 that the category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ is equivalent to a category of semilinear algebra objects (over a certain coefficient ring depending on E) called étale φ-modules. This equivalence will provide a concrete illustratration of many elementary features of the general formalism of p-adic Hodge theory.

If K_∞/K is a Galois extension with Galois group Γ then G_K-representations can be viewed as G_{K_∞}-representations equipped with an additional "Γ-descent structure" that encodes the descent to a G_K-representation. In this way, (1.3.1) identifies $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ with the category of (φ, Γ)-*modules* that consists of étale φ-modules endowed with a suitable Γ-action encoding the descent of an object in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E) = \mathrm{Rep}_{\mathbf{Z}_p}(G_{K_\infty})$ to an object in $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$. The category of (φ, Γ)-modules gives a remarkable and very useful alternative description of the entire category $\mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ in terms of objects of semilinear algebra.

2. Hodge–Tate representations

From now on, K will always denote a p-adic field (for a fixed prime p) in the sense of Definition 1.3.1, and we fix a choice of algebraic closure $\overline{K}/K$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ is denoted G_K, and we write $\mathbf{C}_K$ to denote the completion $\widehat{\overline{K}}$ of $\overline{K}$ endowed with its unique absolute value extending the given absolute value $|\cdot|$ on K. It is a standard fact that $\mathbf{C}_K$ is algebraically closed. Sometimes we will normalize the absolute value by the requirement that $\mathrm{ord}_K := \log|\cdot|$ on $K^\times$ satisfies $\mathrm{ord}_K(p) = 1$, and we also write $|\cdot|$ and ord_K to denote the unique continuous extensions to $\mathbf{C}_K$ and $\mathbf{C}_K^\times$ respectively.

The p-adic Tate module $\varprojlim \mu_{p^n}(\overline{K})$ of the group GL_1 over K is a free $\mathbf{Z}_p$-module of rank 1 and we shall denote it as $\mathbf{Z}_p(1)$. This does not have a canonical basis, and a choice of basis amounts to a choice of compatible system $(\zeta_{p^n})_{n\geq 1}$ of primitive p-power roots of unity (satisfying $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$). The natural action of G_K on $\mathbf{Z}_p(1)$ is given by the $\mathbf{Z}_p^\times$-valued p-adic cyclotomic character $\chi = \chi_{K,p}$ from §1.1, and sometimes it will be convenient to fix a choice of basis of $\mathbf{Z}_p(1)$ and to thereby view $\mathbf{Z}_p(1)$ as $\mathbf{Z}_p$ endowed with a G_K-action by χ.

For any $r \geq 0$ define $\mathbf{Z}_p(r) = \mathbf{Z}_p(1)^{\otimes r}$ and $\mathbf{Z}_p(-r) = \mathbf{Z}_p(r)^\vee$ (linear dual: $M^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Z}_p)$ for any finite free $\mathbf{Z}_p$-module M) with the naturally associated G_K-actions (from functoriality of tensor powers and duality), so upon fixing a basis of $\mathbf{Z}_p(1)$ we identify $\mathbf{Z}_p(r)$ with the $\mathbf{Z}_p$-module $\mathbf{Z}_p$ endowed with the G_K-action χ^r for all $r \in \mathbf{Z}$. If M is an arbitrary $\mathbf{Z}_p[G_K]$-module, we let $M(r) = \mathbf{Z}_p(r) \otimes_{\mathbf{Z}_p} M$ with its natural G_K-action, so upon fixing a basis of $\mathbf{Z}_p(1)$ this is simply M with the modified G_K-action $g.m = \chi(g)^r g(m)$ for $g \in G_K$ and $m \in M$. Elementary isomorphisms such as $(M(r))(r') \simeq M(r+r')$ (with evident transitivity behavior) for $r, r' \in \mathbf{Z}$ and $(M(r))^\vee \simeq M^\vee(-r)$ for $r \in \mathbf{Z}$ and M finite free over $\mathbf{Z}_p$ or over a p-adic field will be used without comment.

2.1. **Theorems of Tate–Sen and Faltings.** Let X be a smooth proper scheme over a p-adic field K. Tate discovered in special cases (abelian varieties with good reduction) that although the p-adic representation spaces $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ for G_K are mysterious, they become *much simpler* after we apply the drastic operation

$$V \rightsquigarrow \mathbf{C}_K \otimes_{\mathbf{Q}_p} V,$$

with the G_K-action on $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ defined by $g(c \otimes v) = g(c) \otimes g(v)$ for $c \in \mathbf{C}_K$ and $v \in V$. Before we examine this operation in detail, we introduce the category in which its output lives.

Definition 2.1.1. A $\mathbf{C}_K$*-representation of* G_K is a finite-dimensional $\mathbf{C}_K$-vector space W equipped with a continuous G_K-action map $G_K \times W \to W$ that is semilinear (i.e., $g(cw) = g(c)g(w)$ for all $c \in \mathbf{C}_K$ and $w \in W$). The category of such objects (using $\mathbf{C}_K$-linear G_K-equivariant morphisms) is denoted $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

This is a p-adic analogue of the notion of a complex vector space endowed with a conjugate-linear automorphism. In concrete terms, if we choose a $\mathbf{C}_K$-basis $\{w_1, \dots, w_n\}$ of W then we may uniquely write $g(w_j) = \sum_i a_{ij}(g) w_i$ for all j, and $\mu : G_K \to \mathrm{Mat}_{n\times n}(\mathbf{C}_K)$ defined by $g \mapsto (a_{ij}(g))$ is a continuous map that satisfies $\mu(1) = \mathrm{id}$ and $\mu(gh) = \mu(g) \cdot g(\mu(h))$ for all $g, h \in G_K$. In particular, μ takes its values in $\mathrm{GL}_n(\mathbf{C}_K)$ (with $g(\mu(g^{-1}))$ as inverse to $\mu(g)$) but beware that μ is *not* a homomorphism in general (due to the semilinearity of the G_K-action).

Example 2.1.2. If $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ then $W := \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is an object in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. We will be most interested in W that arise in this way, but it clarifies matters at the outset to work with general W as above.

The category $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is an abelian category with evident notions of tensor product, direct sum, and exact sequence. If we are attentive to the semilinearity then we can also define a reasonable notion of duality: for any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the *dual* $W^\vee$ is the usual $\mathbf{C}_K$-linear dual on which G_K acts according to the formula $(g.\ell)(w) = g(\ell(g^{-1}(w)))$ for all $w \in W$, $\ell \in W^\vee$, and $g \in G_K$. This formula is rigged to ensure that $g.\ell : W \to \mathbf{C}_K$ is $\mathbf{C}_K$-linear (even though the action of g^{-1} on W is generally not $\mathbf{C}_K$-linear). Since G_K acts continuously on W and on $\mathbf{C}_K$, it is easy to check that this action on $W^\vee$ is continuous. In concrete terms, if we choose a basis $\{w_i\}$ of W and describe the G_K-action on W via a continuous function $\mu : G_K \to \mathrm{GL}_n(\mathbf{C}_K)$ as above Example 2.1.2 then $W^\vee$ endowed with the dual basis is described by the function $g \mapsto g(\mu(g^{-1})^t)$ that is visibly continuous. Habitual constructions from linear algebra such as the isomorphisms $W \simeq W^{\vee\vee}$ and $W^\vee \otimes W'^\vee \simeq (W \otimes W')^\vee$ as well as the evaluation morphism $W \otimes W^\vee \to \mathbf{C}_K$ are easily seen to be morphisms in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

The following deep result of Faltings answers a question of Tate.

Theorem 2.1.3 (Faltings)**.** *Let K be a p-adic field. For smooth proper K-schemes X, there is a canonical isomorphism*

$$\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \bigoplus_q (\mathbf{C}_K(-q) \otimes_K \mathrm{H}^{n-q}(X, \Omega^q_{X/K})) \tag{2.1.1}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$*, where the G_K-action on the right side is defined through the action on each* $\mathbf{C}_K(-q) = \mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(-q)$*. In particular, non-canonically*

$$\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq \bigoplus_q \mathbf{C}_K(-q)^{h^{n-q,q}}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$*, with* $h^{p,q} = \dim_K \mathrm{H}^p(X, \Omega^q_{X/K})$*.*

This is a remarkable theorem for two reasons: it says that $\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ as a $\mathbf{C}_K$-representation space of G_K is a direct sum of extremely simple pieces (the $\mathbf{C}_K(-q)$'s with suitable multiplicity), and we will see that this isomorphism enables us to recover the K-vector spaces $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ from $\mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ by means of operations that make sense on all objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. This is a basic example of a *comparison isomorphism* that relates one p-adic cohomology theory to another. However, of the greatest significance is that (as we shall soon see) we *cannot* recover the p-adic representation space $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ from the Hodge cohomologies $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ in (2.1.1). In general, $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ loses a lot of information about V. This fact is very fundamental in motivating many of the basic constructions in p-adic Hodge theory, and it is best illustrated by the following example.

Example 2.1.4. Let E be an elliptic curve over K with split multiplicative reduction, and consider the representation space $\mathrm{V}_p(E) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathrm{T}_p(E) \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. The theory of Tate curves provides an exact sequence

$$0 \to \mathbf{Q}_p(1) \to \mathrm{V}_p(E) \to \mathbf{Q}_p \to 0 \tag{2.1.2}$$

that is non-split in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K'})$ for all finite extensions K'/K inside of $\overline{K}$.

If we apply $\overline{K} \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.1.2) then we get an exact sequence

$$0 \to \overline{K}(1) \to \overline{K} \otimes_{\mathbf{Q}_p} \mathrm{V}_p(E) \to \overline{K} \to 0$$

in the category $\mathrm{Rep}_{\overline{K}}(G_K)$ of semilinear representations of G_K on $\overline{K}$-vector spaces. We claim that this sequence cannot be split in $\mathrm{Rep}_{\overline{K}}(G_K)$. Assume it is split. Since $\overline{K}$ is the directed union of finite subextensions K'/K, there would then exist such a K' over which the splitting occurs. That is, applying $K' \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.1.2) would give an exact sequence admitting a G_K-equivariant K'-linear splitting. Viewing this as a split sequence of $K'[G_{K'}]$-modules, we could apply a $\mathbf{Q}_p$-linear projection $K' \to \mathbf{Q}_p$ that restricts to the identity on $\mathbf{Q}_p \subseteq K'$ so as to recover (2.1.2) equipped with a $\mathbf{Q}_p[G_{K'}]$-linear splitting. But (2.1.2) has no splitting in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{K'})$, so we have a contradiction. Hence, applying $\overline{K} \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.1.2) gives a non-split sequence in $\mathrm{Rep}_{\overline{K}}(G_K)$, as claimed.

This non-splitting over $\overline{K}$ makes it all the more remarkable that if we instead apply $\mathbf{C}_K \otimes_{\mathbf{Q}_p} (\cdot)$ to (2.1.2) then the resulting sequence in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ does (uniquely) split! This is a special case of the second part of the following fundamental result that pervades all that follows. It rests on a deep study of the ramification theory of local fields.

Theorem 2.1.5 (Tate–Sen)**.** *For any p-adic field K we have $K = \mathbf{C}_K^{G_K}$ (i.e., there are no transcendental invariants) and $\mathbf{C}_K(r)^{G_K} = 0$ for $r \neq 0$ (i.e., if $x \in \mathbf{C}_K$ and $g(x) = \chi(g)^{-r}x$ for all $g \in G_K$ and some $r \neq 0$ then $x = 0$). Also, $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(r)) = 0$ if $r \neq 0$ and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K)$ is 1-dimensional over K.*

More generally, if $\eta : G_K \to \mathbf{Z}_p^\times$ is a continuous character and $\mathbf{C}_K(\eta)$ denotes $\mathbf{C}_K$ with the twisted G_K-action $g.c = \eta(g)g(c)$ then $\mathbf{C}_K(\eta)^{G_K} = 0$ if $\eta(I_K)$ is infinite and $\mathbf{C}_K(\eta)^{G_K}$ is 1-dimensional over K if $\eta(I_K)$ is finite (i.e., if the splitting field of η over K is finitely ramified). Also, $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta)) = 0$ if $\eta(I_K)$ is infinite.

In this theorem, we define $\mathrm{H}^1_{\mathrm{cont}}(G_K, \cdot)$ using continuous 1-cocycles.

Example 2.1.6. Let $\eta : G_K \to \mathbf{Z}_p^\times$ be a continuous character. We identify $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta))$ with the set of isomorphism classes of extensions

$$0 \to \mathbf{C}_K(\eta) \to W \to \mathbf{C}_K \to 0 \tag{2.1.3}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ as follows: using the matrix description

$$\begin{pmatrix} \eta & * \\ 0 & 1 \end{pmatrix}$$

of such a W, the homomorphism property for the G_K-action on W says that the upper right entry function is a 1-cocycle on G_K with values in $\mathbf{C}_K(\eta)$, and changing the choice of $\mathbf{C}_K$-linear splitting changes this function by a 1-coboundary. The continuity of the 1-cocycle says exactly that the G_K-action on W is continuous. Changing the choice of $\mathbf{C}_K$-basis of W that is compatible with the filtration in (2.1.3) changes the 1-cocycle by a 1-coboundary. In this way we get a well-defined continuous cohomology class, and the procedure can be reversed (up to isomorphism of the extension structure (2.1.3) in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$).

Theorem 2.1.5 says that all exact sequences (2.1.3) are split when $\eta(I_K)$ is infinite. Moreover, in such cases the splitting is *unique*. Indeed, any two splittings $\mathbf{C}_K \rightrightarrows W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ differ by an element of $\mathrm{Hom}_{\mathrm{Rep}_{\mathbf{C}_K}(G_K)}(\mathbf{C}_K, \mathbf{C}_K(\eta))$, and by chasing the image of $1 \in \mathbf{C}_K$ this Hom-set is identified with $\mathbf{C}_K(\eta)^{G_K}$. But by the Tate–Sen theorem this vanishes when $\eta(I_K)$ is infinite.

The real importance of Theorem 2.1.5 is revealed when we consider an arbitrary $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ admitting an isomorphism as in Faltings' Theorem 2.1.3:

$$W \simeq \bigoplus_q \mathbf{C}_K(-q)^{h_q}. \tag{2.1.4}$$

Although such a direct sum decomposition is non-canonical in general (in the sense that the individual lines $\mathbf{C}_K(-q)$ appearing in the direct sum decomposition are generally not uniquely determined within W when $h_q > 1$), we shall see that for any such W there is a *canonical* decomposition $W \simeq \oplus_q(\mathbf{C}_K(-q) \otimes_K W\{q\})$ for a canonically associated K-vector space $W\{q\}$ with dimension h_q.

Example 2.1.7. In (2.1.4) we have $W^{G_K} \simeq \oplus_q(\mathbf{C}_K(-q)^{G_K})^{h_q} \simeq K^{h_0}$ by the Tate–Sen theorem, so $h_0 = \dim_K W^{G_K}$. A priori it is not clear that $\dim_K W^{G_K}$ should be finite for typical $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Such finiteness holds in much greater generality, as we shall see, and the W that arise as in (2.1.4) will be intrinsically characterized in terms of such finiteness properties.

2.2. **Hodge–Tate decomposition.** The companion to Theorem 2.1.5 that gets p-adic Hodge theory off the ground is a certain lemma of Serre and Tate that we now state. For $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and $q \in \mathbf{Z}$, consider the K-vector space

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W \mid g(w) = \chi(g)^{-q}w \text{ for all } g \in G_K\}, \tag{2.2.1}$$

where the isomorphism rests on a choice of basis of $\mathbf{Z}_p(1)$. In particular, this isomorphism is not canonical when $q \neq 0$ and $W\{q\} \neq 0$, so $W\{q\}$ is canonically a K-subspace of $W(q)$ but it is only non-canonically a K-subspace of W when $q \neq 0$ and $W\{q\} \neq 0$. More importantly, $W\{q\}$ is not a $\mathbf{C}_K$-subspace of $W(q)$ when it is nonzero. In fact, $W\{q\}$ contains no $\mathbf{C}_K$-lines, for if $x \in W\{q\}$ is nonzero and cx lies in $W\{q\}$ for all $c \in \mathbf{C}_K$ then $g(c) = c$ for all $c \in \mathbf{C}_K$ and all $g \in G_K$, which is absurd since $\overline{K} \subseteq \mathbf{C}_K$.

We have a natural G_K-equivariant K-linear multiplication map

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \simeq W,$$

so extending scalars defines maps

$$\mathbf{C}_K(-q) \otimes_K W\{q\} \to W$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for all $q \in \mathbf{Z}$.

Lemma 2.2.1 (Serre–Tate)**.** *For $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the natural $\mathbf{C}_K$-linear G_K-equivariant map*

$$\xi_W : \bigoplus_q (\mathbf{C}_K(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $W\{q\} = 0$ *for all but finitely many* q *and* $\dim_K W\{q\} < \infty$ *for all* q*, with* $\sum_q \dim_K W\{q\} \le \dim_{\mathbf{C}_K} W$*; equality holds here if and only if* ξ_W *is an isomorphism.*

Proof. The idea is to consider a hypothetical nonzero element in $\ker \xi_W$ with "shortest length" in terms of elementary tensors and to use that $\ker \xi_W$ is a $\mathbf{C}_K$-subspace yet each $W\{q\}$ contains no $\mathbf{C}_K$-lines. To carry out this strategy, consider a nonzero $v = (v_q)_q \in \ker \xi_W$. We choose such v with minimal length, where the length $\ell(x)$ for

$$x = (x_q) \in \oplus_q(\mathbf{C}_K(-q) \otimes_K W\{q\})$$

is defined as follows. For an element x_q of $\mathbf{C}_K \otimes_K W\{q\}$ we define $\ell(x_q)$ to be the least integer $n_q \ge 0$ such that x_q is a sum of n_q elementary tensors, and for a general $x = (x_q)$ we define $\ell(x) = \sum \ell(x_q)$ (which makes sense since $\ell(x_q) = 0$ for all but finitely many q). Observe that $\mathbf{C}_K^\times$-scaling preserves length.

There is some q_0 such that v_{q_0} is nonzero, so if we rename $W(q_0)$ as W then we can arrange that $v_0 \ne 0$. By applying a $\mathbf{C}_K^\times$-scaling we can also arrange that v_0 has a minimal-length expression $v_0 = \sum_j c_{j,0} \otimes y_{j,0}$ with $c_{j,0} \in \mathbf{C}_K^\times$, $y_{j,0} \in W\{0\} = W^{G_K}$, and some $c_{j_0,0} = 1$.

Clearly $g(v) - v \in \ker \xi_W$, and for each $q \in \mathbf{Z}$ its qth component is $g(v_q) - v_q$. If $\sum c_{j,q} \otimes y_{j,q}$ is a minimal-length expression for v_q then since $g(v_q) - v_q = \sum(\chi(g)^{-q} g(c_{j,q}) - c_{j,q}) \otimes y_{j,q}$ we see that $\ell(g(v_q) - v_q) \le \ell(v_q)$, so $g(v_q) - v_q$ has length at most $\ell(v_q)$. But $g(v_0) - v_0 = \sum_j (g(c_{j,0}) - c_{j,0}) \otimes y_{j,0}$ has strictly smaller length than v_0 because $c_{j_0,0} = 1$. Hence, $g(v) - v \in \ker \xi_W$ has strictly smaller length than v, so it vanishes. Thus, $v = g(v)$ for all $g \in G_K$, so $v_q \in (\mathbf{C}_K(-q) \otimes_K W\{q\})^{G_K} = \mathbf{C}_K(-q)^{G_K} \otimes_K W\{q\} = 0$ if $q \ne 0$ and $v_0 \in W\{0\}$ by the Tate–Sen theorem. That is, $\ker \xi_W \subseteq W\{0\}$. Thus, $\ker \xi_W$ is a $\mathbf{C}_K$-subspace of $W\{0\}$ inside of $\mathbf{C}_K \otimes_K W$, yet $W\{0\}$ contains no $\mathbf{C}_K$-lines, so $\ker \xi_W = 0$. ∎

Remark 2.2.2. An alternative formulation of the Serre–Tate lemma can be given in terms of the K-subspaces

$$W[q] := \{w \in W \mid g(w) = \chi(g)^{-q} w \text{ for all } g \in G_K\} \subseteq W$$

instead of the K-subspaces $W\{q\} \subseteq W(q)$ from (2.2.1) for all $q \in \mathbf{Z}$. Indeed, since non-canonically we have $W[q] \simeq W\{q\}$, the Serre–Tate lemma says exactly that the $W[q]$'s are finite-dimensional over K and vanish for all but finitely many q, and that these are mutually $\mathbf{C}_K$-linearly independent within W in the sense that the natural map $\oplus(\mathbf{C}_K \otimes_K W[q]) \to W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is injective.

In the special case $W = \mathbf{C}_K \otimes_{\mathbf{Q}_p} \mathrm{H}^n(X_{\overline{K}}, \mathbf{Q}_p)$ for a smooth proper scheme X over K, Faltings' Theorem 2.1.3 says that ξ_W is an isomorphism and $W\{q\}$ (rather than $W[q]$!) is canonically K-isomorphic to $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ for all $q \in \mathbf{Z}$.

Example 2.2.3. Let $W = \mathbf{C}_K(\eta)$ for a continuous character $\eta : G_K \to \mathbf{Z}_p^\times$. By the Tate–Sen theorem, $W\{q\} = \mathbf{C}_K(\eta\chi^{-q})^{G_K}$ is 1-dimensional over K if $\eta\chi^{-q}|_{I_K}$ has finite order (equivalently, if $\eta = \chi^q\psi$ for a finitely ramified character $\psi : G_K \to \mathbf{Z}_p^\times$) and $W\{q\}$ vanishes otherwise. In particular, there is at most one q for which $W\{q\}$ can be nonzero, since if $W\{q\}, W\{q'\} \ne 0$ with $q \ne q'$ then $\eta = \chi^q\psi$ and $\eta = \chi^{q'}\psi'$ with finitely ramified $\psi, \psi' : G_K \rightrightarrows \mathbf{Z}_p^\times$, so $\chi^r|_{I_K}$ has finite image for $r = q - q' \ne 0$, which is absurd (use Example 1.1.5).

An interesting special case of Example 2.2.3 is when K contains $\mathbf{Q}_p(\mu_p)$, so $\chi(G_K)$ is contained in the pro-p group $1+p\mathbf{Z}_p$. Hence, $\eta=\chi^s$ makes sense for all $s\in\mathbf{Z}_p$, and for $W=\mathbf{C}_K(\eta)$ with such η the space $W\{q\}$ vanishes for all q when $s\notin\mathbf{Z}$ whereas $W\{-s\}$ is 1-dimensional over K if $s\in\mathbf{Z}$. Thus for $s\in\mathbf{Z}_p$ the map $\xi_{\mathbf{C}_K(\chi^s)}$ vanishes if $s\notin\mathbf{Z}$ and it is an isomorphism if $s\in\mathbf{Z}$. The case $s\notin\mathbf{Z}$ is of "non-algebraic" nature, and this property situation is detected by the map $\xi_{\mathbf{C}_K(\chi^s)}$.

Definition 2.2.4. A representation W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is *Hodge–Tate* if ξ_W is an isomorphism. We say that V in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is *Hodge–Tate* if $\mathbf{C}_K\otimes_{\mathbf{Q}_p}V\in\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is Hodge–Tate.

Example 2.2.5. If W is Hodge–Tate then by virtue of ξ_W being an isomorphism we have a non-canonical isomorphism $W\simeq\oplus\mathbf{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ with $h_q=\dim_K W\{q\}$. Conversely, consider an object $W\in\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ admitting a finite direct sum decomposition $W\simeq\oplus\mathbf{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ with $h_q\geq 0$ for all q and $h_q=0$ for all but finitely many q. The Tate–Sen theorem gives that $W\{q\}$ has dimension h_q for all q, so $\sum_q\dim_K W\{q\}=\sum_q h_q=\dim_{\mathbf{C}_K}W$ and hence W is Hodge–Tate. In other words, the intrinsic property of being Hodge–Tate is equivalent to the concrete property of being isomorphic to a finite direct sum of various objects $\mathbf{C}_K(r_i)$ (with multiplicity permitted).

For any Hodge–Tate object W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ we define the *Hodge–Tate weights* of W to be those $q\in\mathbf{Z}$ such that $W\{q\}:=(\mathbf{C}_K(q)\otimes_{\mathbf{C}_K}W)^{G_K}$ is nonzero, and then we call $h_q:=\dim_K W\{q\}\geq 1$ the *multiplicity* of q as a Hodge–Tate weight of W. Beware that, according to this definition, $q\in\mathbf{Z}$ is a Hodge–Tate weight of W precisely when there is an injection $\mathbf{C}_K(-q)\hookrightarrow W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, as opposed to when there is an injection $\mathbf{C}_K(q)\hookrightarrow W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. For example, $\mathbf{C}_K(q)$ has $-q$ as its unique Hodge–Tate weight.

Obviously (by Example 2.2.5) if W is Hodge–Tate then so is $W^\vee$, with negated Hodge–Tate weights (compatibly with multiplicities), so it is harmless to change the definition of "Hodge–Tate weight" by a sign. In terms of p-adic Hodge theory, this confusion about signs comes down to later choosing to use covariant or contravariant functors when passing between p-adic representations and semilinear algebra objects (as replacing a representation space with its dual will be the mechanism by which we pass between covariant and contravariant versions of various functors on categories of representations).

2.3. **Formalism of Hodge–Tate representations.** We saw via Example 2.2.5 that for any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, W is Hodge–Tate if and only if its dual $W^\vee$ is Hodge–Tate. By the same reasoning, since

$$\left(\oplus_q\mathbf{C}_K(-q)^{h_q}\right)\otimes_{\mathbf{C}_K}\left(\oplus_{q'}\mathbf{C}_K(-q')^{h'_{q'}}\right)\simeq\oplus_r\mathbf{C}_K(-r)^{\sum_i h_i h'_{r-i}}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ we see that if W and W' are Hodge–Tate then so is $W\otimes W'$ (with Hodge–Tate weights that are suitable sums of products of those of W and W'); it is clear that $W\oplus W'$ is also Hodge–Tate. To most elegantly express how the Hodge–Tate property interacts with tensorial and other operations, it is useful to introduce some terminology.

Definition 2.3.1. A ($\mathbf{Z}$-)*graded vector space* over a field F is an F-vector space D equipped with direct sum decomposition $\oplus_{q\in\mathbf{Z}}D_q$ for F-subspaces $D_q\subseteq D$ (and we define the *qth* *graded piece* of D to be $\mathrm{gr}^q(D) := D_q$). Morphisms $T : D' \to D$ between graded F-vector spaces are F-linear maps that respect the grading (i.e., $T(D'_q) \subseteq D_q$ for all q). The category of these is denoted Gr_F; we let $\mathrm{Gr}_{F,f}$ denote the full subcategory of D for which $\dim_F D$ is finite.

For any field F, Gr_F is an abelian category with the evident notions of kernel, cokernel, and exact sequence (working in separate degrees). We write $F\langle r\rangle$ for $r \in \mathbf{Z}$ to denote the F-vector space F endowed with the grading for which the unique non-vanishing graded piece is in degree r. For $D, D' \in \mathrm{Gr}_F$ we define the *tensor product* $D \otimes D'$ to have underlying F-vector space $D \otimes_F D'$ and to have qth graded piece $\oplus_{i+j=q}(D_i \otimes_F D'_j)$. Likewise, if $D \in \mathrm{Gr}_{F,f}$ then the *dual* $D^\vee$ has underlying F-vector space given by the F-linear dual and its qth graded piece is $D^\vee_{-q}$.

It is easy to check that with these definitions, $F\langle r\rangle \otimes F\langle r'\rangle = F\langle r+r'\rangle$, $F\langle r\rangle^\vee = F\langle -r\rangle$, and the natural evaluation mapping $D \otimes D^\vee \to F\langle 0\rangle$ and double duality isomorphism $D \simeq (D^\vee)^\vee$ on F-vector spaces for D in $\mathrm{Gr}_{F,f}$ are morphisms in Gr_F. Observe also that a map in Gr_F is an isomorphism if and only if it is a linear isomorphism in each separate degree.

Definition 2.3.2. The covariant functor $\underline{\mathrm{D}} = \underline{\mathrm{D}}_K : \mathrm{Rep}_{\mathbf{C}_K}(G_K) \to \mathrm{Gr}_K$ is

$$\underline{\mathrm{D}}(W) = \oplus_q W\{q\} = \oplus_q (\mathbf{C}_K(q) \otimes_{\mathbf{C}_K} W)^{G_K}.$$

This functor is visibly left-exact.

In general, the Serre–Tate lemma says that $\underline{\mathrm{D}}$ takes values in $\mathrm{Gr}_{K,f}$ and more specifically that $\dim_K \underline{\mathrm{D}}(W) \le \dim_{\mathbf{C}_K} W$ with equality if and only if W is Hodge–Tate. As a simple example, the Tate–Sen theorem gives that $\underline{\mathrm{D}}(\mathbf{C}_K(r)) = K\langle -r\rangle$ for all $r \in \mathbf{Z}$. The functor $\underline{\mathrm{D}}$ satisfies a useful exactness property on Hodge–Tate objects, as follows.

Proposition 2.3.3. *If $0 \to W' \to W \to W'' \to 0$ is a short exact sequence in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and W is Hodge–Tate then so are W' and W'', in which case the sequence*

$$0 \to \underline{\mathrm{D}}(W') \to \underline{\mathrm{D}}(W) \to \underline{\mathrm{D}}(W'') \to 0$$

in $\mathrm{Gr}_{K,f}$ is short exact (so the multiplicities for each Hodge–Tate weight are additive in short exact sequences of Hodge–Tate representations).

Proof. We have a left-exact sequence

$$0 \to \underline{\mathrm{D}}(W') \to \underline{\mathrm{D}}(W) \to \underline{\mathrm{D}}(W'') \tag{2.3.1}$$

with $\dim_K \underline{\mathrm{D}}(W') \le \dim_{\mathbf{C}_K}(W')$ and similarly for W and W''. But equality holds for W by the Hodge–Tate property, so

$$\begin{aligned} \dim_{\mathbf{C}_K} W = \dim_K \underline{\mathrm{D}}(W) &\le \dim_K \underline{\mathrm{D}}(W') + \dim_K \underline{\mathrm{D}}(W'') \\ &\le \dim_{\mathbf{C}_K} W' + \dim_{\mathbf{C}_K} W'' \\ &= \dim_{\mathbf{C}_K} W, \end{aligned}$$

forcing equality throughout. In particular, W' and W'' are Hodge–Tate and so for K-dimension reasons the left-exact sequence (2.3.1) is right-exact too. ■

Example 2.3.4. Although Proposition 2.3.3 says that any subrepresentation or quotient representation of a Hodge–Tate representation is again Hodge–Tate, the converse is false in the sense that if W' and W'' are Hodge–Tate then W can fail to have this property. To give a counterexample, we recall that $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K) \neq 0$ by Theorem 2.1.5. This gives a non-split exact sequence

$$0 \to \mathbf{C}_K \to W \to \mathbf{C}_K \to 0 \tag{2.3.2}$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, and we claim that such a W cannot be Hodge–Tate. To see this, applying the left-exact functor $\underline{\mathrm{D}}$ to the exact sequence above gives a left exact sequence

$$0 \to K\langle 0\rangle \to \underline{\mathrm{D}}(W) \to K\langle 0\rangle$$

of graded K-vector spaces, so in particular $\underline{\mathrm{D}}(W) = W\{0\} = W^{G_K}$. If W were Hodge–Tate then by Proposition 2.3.3 this left exact sequence of graded K-vector spaces would be short exact, so there would exist some $w \in W^{G_K}$ with nonzero image in $K\langle 0\rangle$. We would then get a $\mathbf{C}_K$-linear G_K-equivariant section $\mathbf{C}_K \to W$ via $c \mapsto cw$. This splits (2.3.2) in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, contradicting the non-split property of (2.3.2). Hence, W cannot be Hodge–Tate.

The functor $\underline{\mathrm{D}} = \underline{\mathrm{D}}_K$ is useful when studying how the Hodge–Tate property interacts with basic operations such as a finite scalar extension on K, tensor products, duality, and replacing K with $\widehat{K^{\mathrm{un}}}$ (i.e., replacing G_K with I_K), as we now explain.

Theorem 2.3.5. *For any $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the natural map $K' \otimes_K \underline{\mathrm{D}}_K(W) \to \underline{\mathrm{D}}_{K'}(W)$ in $\mathrm{Gr}_{K',f}$ is an isomorphism for all finite extensions K'/K contained in $\overline{K} \subseteq \mathbf{C}_K$. Likewise, the natural map $\widehat{K^{\mathrm{un}}} \otimes_K \underline{\mathrm{D}}_K(W) \to \underline{\mathrm{D}}_{\widehat{K^{\mathrm{un}}}}(W)$ in $\mathrm{Gr}_{\widehat{K^{\mathrm{un}}},f}$ is an isomorphism.*

In particular, for any finite extension K'/K inside of $\overline{K}$, an object W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is Hodge–Tate if and only if it is Hodge–Tate when viewed in $\mathrm{Rep}_{\mathbf{C}_K}(G_{K'})$, and similarly W is Hodge–Tate in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ if and only if it is Hodge–Tate when viewed in $\mathrm{Rep}_{\mathbf{C}_K}(G_{\widehat{K^{\mathrm{un}}}}) = \mathrm{Rep}_{\mathbf{C}_K}(I_K)$.

This theorem says that the Hodge–Tate property is insensitive to replacing K with a finite extension or restricting to the inertia group (i.e., replacing K with $\widehat{K^{\mathrm{un}}}$). This is a prototype for a class of results that will arise in several later contexts (with properties that refine the Hodge–Tate property). The insensitivity to inertial restriction is a good feature of the Hodge–Tate property, but the insensitivity to finite (possibly ramified) extensions is a bad feature, indicating that the Hodge–Tate property is not sufficiently fine (e.g., to distinguish between good reduction and potentially good reduction for elliptic curves).

Proof. By a transitivity argument, the case of finite extensions is easily reduced to the case when K'/K is Galois. We first treat the finite Galois case, and then will need to do some work to adapt the method to handle the extension $\widehat{K^{\mathrm{un}}}/K$ that is generally not algebraic but should be thought of as being approximately algebraic (with Galois group $G_K/I_K = G_k$). Observe that $\mathrm{Gal}(K'/K)$ naturally acts semilinearly on the finite-dimensional K'-vector space $\underline{\mathrm{D}}_{K'}(W)$ with invariant subspace $\underline{\mathrm{D}}_K(W)$ over K, and likewise $G_K/I_K =$

G_k naturally acts semilinearly on the finite-dimensional $\widehat{K^{\mathrm{un}}}$-vector space $\underline{\mathrm{D}}_{\widehat{K^{\mathrm{un}}}}(W)$ with invariant subspace $\underline{\mathrm{D}}_K(W)$ over K.

Hence, for the case of finite (Galois) extensions our problem is a special case of *classical Galois descent for vector spaces*: if F'/F is a finite Galois extension of fields and D' is a finite-dimensional F'-vector space endowed with a semilinear action by $\mathrm{Gal}(F'/F)$ then the natural map

$$F' \otimes_F (D'^{\mathrm{Gal}(F'/F)}) \to D' \tag{2.3.3}$$

is an isomorphism. (See [8, Ch. II, Lemma 5.8.1] for a proof, resting on the non-vanishing of discriminants for finite Galois extensions.) This has an easy generalization to arbitrary Galois extensions F'/F with possibly infinite degree: we just need to impose the additional "discreteness" hypothesis that each element of D' has an open stabilizer in $\mathrm{Gal}(F'/F)$ (so upon choosing an F'-basis of D' there is an open normal subgroup $\mathrm{Gal}(F'/F_1)$ that fixes the basis vectors and hence reduces our problem to the finite case via the semilinear $\mathrm{Gal}(F_1/F)$-action on the F_1-span of the chosen F'-basis of D').

For the case of $\widehat{K^{\mathrm{un}}}$, we have to modify the preceding argument since $\widehat{K^{\mathrm{un}}}/K$ is generally not algebraic and the group of isometric automorphisms $\mathrm{Aut}(\widehat{K^{\mathrm{un}}}/K) = \mathrm{Gal}(K^{\mathrm{un}}/K) = G_K/I_K = G_k$ generally acts on the space of I_K-invariants in W with stabilizer groups that are closed but not open. Hence, we require a variant of the Galois descent isomorphism (2.3.3) subject to a (necessary) auxiliary continuity hypothesis.

First we check that the natural semilinear action on $D' := D_{\widehat{K^{\mathrm{un}}}}(W)$ by the profinite group $G_k = G_K/I_K$ is continuous relative to the natural topology on D' as a finite-dimensional $\widehat{K^{\mathrm{un}}}$-vector space. It suffices to check such continuity on the finitely many nonzero graded pieces D'_q separately, and $\mathbf{C}_K(-q) \otimes_{\widehat{K^{\mathrm{un}}}} D'_q$ with its G_K-action is naturally embedded in W (by the Serre–Tate injection ξ_W). Since G_K acts continuously on W by hypothesis and the natural topology on D'_q coincides with its subspace topology from naturally sitting in the $\mathbf{C}_K$-vector space $\mathbf{C}_K(-q) \otimes_{\widehat{K^{\mathrm{un}}}} D'_q$, we get the asserted continuity property for the action of $G_k = G_K/I_K$ on D'_q.

Although G_k acts $\widehat{K^{\mathrm{un}}}$-semilinearly rather than $\widehat{K^{\mathrm{un}}}$-linearly on D', since $\widehat{K^{\mathrm{un}}}$ is the fraction field of a complete discrete valuation ring $\mathscr{O} := \mathscr{O}_{\widehat{K^{\mathrm{un}}}}$ the proof of Lemma 1.2.6 easily adapts (using continuity of the semilinear G_k-action on D') to construct a G_k-stable $\mathscr{O}$-lattice $\Lambda \subseteq D'$. Consider the natural $\mathscr{O}$-linear G_k-equivariant map

$$\mathscr{O} \otimes_{\mathscr{O}_K} \Lambda^{G_k} \to \Lambda. \tag{2.3.4}$$

We shall prove that this is an isomorphism with Λ^{G_k} a finite free $\mathscr{O}_K$-module. Once this is proved, inverting p on both sides will give the desired isomorphism $\widehat{K^{\mathrm{un}}} \otimes_K \underline{\mathrm{D}}_K(W) \simeq D' = \underline{\mathrm{D}}_{\widehat{K^{\mathrm{un}}}}(W)$.

To verify the isomorphism property for (2.3.4), we shall argue via successive approximation by lifting from the residue field $\overline{k}$ of $\widehat{K^{\mathrm{un}}}$. Let $\pi \in \mathscr{O}_K$ be a uniformizer, so it is also a uniformizer of $\mathscr{O} = \mathscr{O}_{\widehat{K^{\mathrm{un}}}}$ and G_k acts trivially on π. The quotient $\Lambda/\pi\Lambda$ is a vector space over $\overline{k}$ with dimension equal to $d = \mathrm{rank}_{\mathscr{O}}\Lambda = \dim_{\widehat{K^{\mathrm{un}}}} D'$ and it is endowed with a natural semilinear action by $G_k = \mathrm{Gal}(\overline{k}/k)$ that has open stabilizers for all vectors (due to the continuity of the G_k-action on D' and the fact that Λ gets the π-adic topology as its subspace topology from D'). Hence, classical Galois descent in (2.3.3) (applied to $\overline{k}/k$) gives that $\Lambda/\pi\Lambda = \overline{k} \otimes_k \Delta$ in $\mathrm{Rep}_{\overline{k}}(G_k)$ for the d-dimensional k-vector space $\Delta = (\Lambda/\pi\Lambda)^{G_k}$. In particular, $\Lambda/\pi\Lambda \simeq \overline{k}^d$ compatibly with G_k-actions, so $\mathrm{H}^1(G_k, \Lambda/\pi\Lambda)$ vanishes since $\mathrm{H}^1(G_k, \overline{k}) = 0$. Since π is G_k-invariant, a successive approximation argument with continuous 1-cocycles (see [6, §1.2, Lemma 3], applied successively to increasing finite quotients of G_k) then gives that $\mathrm{H}^1_{\mathrm{cont}}(G_k, \Lambda) = 0$. Hence, passing to G_k-invariants on the exact sequence

$$0 \to \Lambda \xrightarrow{\pi} \Lambda \to \Lambda/\pi\Lambda \to 0$$

gives an exact sequence

$$0 \to \Lambda^{G_k} \xrightarrow{\pi} \Lambda^{G_k} \to (\Lambda/\pi\Lambda)^{G_k} \to 0.$$

That is, we have $\Lambda^{G_k}/\pi \cdot \Lambda^{G_k} \simeq (\Lambda/\pi\Lambda)^{G_k}$ as k-vector spaces.

Since Λ^{G_k} is a closed $\mathscr{O}_K$-submodule of the finite free $\mathscr{O}_{\widehat{K^{\mathrm{un}}}}$-module Λ of rank d and we have just proved that $\Lambda^{G_k}/\pi\Lambda^{G_k}$ is finite-dimensional of dimension d over $k = \mathscr{O}_K/(\pi)$, a simple approximation argument gives that any lift of a k-basis of $\Lambda^{G_k}/\pi\Lambda^{G_k}$ to a subset of Λ^{G_k} is an $\mathscr{O}_K$-spanning set of Λ^{G_k} of size d. Thus, Λ^{G_k} is a finitely generated torsion-free $\mathscr{O}_K$-module, so it is free of rank d since its reduction modulo π is d-dimensional over k. Our argument shows that the map (2.3.4) is a map between finite free $\mathscr{O}$-modules of the same rank and that this map becomes an isomorphism modulo π, so it is an isomorphism. ∎

Further properties of $\underline{\mathrm{D}}$ are best expressed by recasting the definition of $\underline{\mathrm{D}}$ in terms of a "period ring" formalism. This rests on the following innocuous-looking definition whose mathematical (as opposed to linguistic) importance will only be appreciated after some later developments.

Definition 2.3.6. The *Hodge–Tate ring* of K is the $\mathbf{C}_K$-algebra $B_{\mathrm{HT}} = \oplus_{q\in\mathbf{Z}}\mathbf{C}_K(q)$ in which multiplication is defined via the natural maps $\mathbf{C}_K(q) \otimes_{\mathbf{C}_K} \mathbf{C}_K(q') \simeq \mathbf{C}_K(q+q')$.

Observe that B_{HT} is a graded $\mathbf{C}_K$-algebra in the sense that its graded pieces are $\mathbf{C}_K$-subspaces with respect to which multiplication is additive in the degrees, and that the natural G_K-action respects the gradings and the ring structure (and is semilinear over $\mathbf{C}_K$). Concretely, if we choose a basis t of $\mathbf{Z}_p(1)$ then we can identify B_{HT} with the Laurent polynomial ring $\mathbf{C}_K[t, t^{-1}]$ with the evident grading (by monomials in t) and G_K-action (via $g(t^i) = \chi(g)^i t^i$ for $i \in \mathbf{Z}$ and $g \in G_K$).

By the Tate–Sen theorem, we have $B_{\mathrm{HT}}^{G_K} = K$. For any $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$, we have

$$\underline{\mathrm{D}}(W) = \oplus_q(\mathbf{C}_K(q) \otimes_{\mathbf{C}_K} W)^{G_K} = (B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W)^{G_K}$$

in Gr_K, where the grading is induced from the one on B_{HT}. Since B_{HT} compatibly admits all three structures of interest ($\mathbf{C}_K$-vector space structure, G_K-action, grading), we can go in the reverse direction (from graded K-vector spaces to $\mathbf{C}_K$-representations of G_K) as follows.

Let D be in $\mathrm{Gr}_{K,f}$, so $B_{\mathrm{HT}} \otimes_K D$ is a graded $\mathbf{C}_K$-vector space with typically infinite $\mathbf{C}_K$-dimension:

$$\mathrm{gr}^n(B_{\mathrm{HT}} \otimes_K D) = \oplus_q \,\mathrm{gr}^q(B_{\mathrm{HT}}) \otimes_K D_{n-q} = \oplus_q \mathbf{C}_K(q) \otimes_K D_{n-q}.$$

Moreover, the G_K-action on $B_{\mathrm{HT}} \otimes_K D$ arising from that on B_{HT} respects the grading since such compatibility holds in B_{HT}, so we get the object

$$\underline{V}(D) := \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_K D) = \oplus_q \mathbf{C}_K(-q) \otimes_K D_q \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$$

since D_q vanishes for all but finitely many q and is finite-dimensional over K for all q (as $D \in \mathrm{Gr}_{K,f}$). By inspection $\underline{V}(D)$ is a Hodge–Tate representation, and obviously $\underline{V} : \mathrm{Gr}_{K,f} \to \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is a covariant exact functor.

Example 2.3.7. For each $r \in \mathbf{Z}$, recall that $K\langle r\rangle$ denotes the 1-dimensional K-vector space K endowed with unique nontrivial graded piece in degree r. It is easy to check that $\underline{V}(K\langle r\rangle) = \mathbf{C}_K(-r)$. In particular, $\underline{V}(K\langle 0\rangle) = \mathbf{C}_K$.

For any W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the multiplicative structure on B_{HT} defines a natural B_{HT}-linear composite *comparison morphism*

$$\gamma_W : B_{\mathrm{HT}} \otimes_K \underline{\mathrm{D}}(W) \hookrightarrow B_{\mathrm{HT}} \otimes_K (B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W) \to B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W \tag{2.3.5}$$

that respects the G_K-actions (from B_{HT} on both sides and from W) and the gradings (from B_{HT} on both sides and from $\underline{\mathrm{D}}(W)$) since the second step in (2.3.5) rests on the multiplication in B_{HT} which is G_K-equivariant and respects the grading of B_{HT}. The Serre–Tate lemma admits the following powerful reformulation:

Lemma 2.3.8. *For W in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, the comparison morphism γ_W is injective. It is an isomorphism if and only if W is Hodge–Tate, in which case there is a natural isomorphism*

$$\underline{V}(\underline{\mathrm{D}}(W)) = \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_K \underline{\mathrm{D}}(W)) \overset{\gamma_W}{\simeq} \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W) = \mathrm{gr}^0(B_{\mathrm{HT}}) \otimes_{\mathbf{C}_K} W = W$$

in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

Proof. The map γ_W on gr^n's is the $\mathbf{Q}_p(n)$-twist of ξ_W. ∎

We have seen above that if D is an object in $\mathrm{Gr}_{K,f}$ then $\underline{V}(D)$ is a Hodge–Tate object in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, so by Lemma 2.3.8 we obtain a B_{HT}-linear comparison isomorphism

$$\gamma_{\underline{V}(D)} : B_{\mathrm{HT}} \otimes_K \underline{\mathrm{D}}(\underline{V}(D)) \simeq B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} \underline{V}(D)$$

respecting G_K-actions and gradings. Since $B_{\mathrm{HT}}^{G_K} = K$ and the G_K-action on the target of $\gamma_{\underline{V}(D)}$ respects the grading induced by $B_{\mathrm{HT}} = \oplus \mathbf{C}_K(r)$, by passing to G_K-invariants on the source and target of $\gamma_{\underline{V}(D)}$ we get an isomorphism

$$\underline{D}(\underline{V}(D)) \simeq \oplus_r (\underline{V}(D)(r))^{G_K}$$

in Gr_K with $\underline{V}(D)(r) \simeq \oplus_q \mathbf{C}_K(r-q) \otimes_K D_q$. Hence, $(\underline{V}(D)(r))^{G_K} = D_r$ by the Tate–Sen theorem, so we get an isomorphism

$$\underline{D}(\underline{V}(D)) \simeq \oplus_r D_r = D$$

in Gr_K. This proves the first part of:

Theorem 2.3.9. *The covariant functors $\underline{D}$ and $\underline{V}$ between the categories of Hodge–Tate representations in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and finite-dimensional objects in Gr_K are quasi-inverse equivalences.*

For any W, W' in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ the natural map

$$\underline{D}(W) \otimes \underline{D}(W') \to \underline{D}(W \otimes W')$$

in Gr_K induced by the G_K-equivariant map

$$(B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W) \otimes_{\mathbf{C}_K} (B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} W') \to B_{\mathrm{HT}} \otimes_{\mathbf{C}_K} (W \otimes_{\mathbf{C}_K} W')$$

defined by multiplication in B_{HT} is an isomorphism when W and W' are Hodge–Tate. Likewise, if W is Hodge–Tate then the natural map

$$\underline{D}(W) \otimes_K \underline{D}(W^\vee) \to \underline{D}(W \otimes W^\vee) \to \underline{D}(\mathbf{C}_K) = K\langle 0 \rangle$$

in Gr_K is a perfect duality (between $W\{q\}$ and $W^\vee\{-q\}$ for all q), so the induced map $\underline{D}(W^\vee) \to \underline{D}(W)^\vee$ is an isomorphism in $\mathrm{Gr}_{K,f}$. In other words, $\underline{D}$ is compatible with tensor products and duality on Hodge–Tate objects.

Similar compatibilities hold for $\underline{V}$ with respect to tensor products and duality.

Proof. For the tensor product and duality claims for $\underline{D}$, one first checks that both sides have compatible evident functorial behavior with respect to direct sums in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Hence, we immediately reduce to the special case $W = \mathbf{C}_K(q)$ and $W' = \mathbf{C}_K(q')$ for some $q, q' \in \mathbf{Z}$, and this case is an easy calculation. Likewise, to analyze the natural map $\underline{V}(D) \otimes_{\mathbf{C}_K} \underline{V}(D') \to \underline{V}(D \otimes D')$ we can reduce to the easy special case of the graded objects $D = K\langle r \rangle$ and $D' = K\langle r' \rangle$ for $r, r' \in \mathbf{Z}$; the case of duality goes similarly. ■

Definition 2.3.10. Let $\mathrm{Rep}_{\mathrm{HT}}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ be the full subcategory of objects V that are Hodge–Tate (i.e., $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is Hodge–Tate in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$), and define the functor $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Gr}_{K,f}$ by

$$D_{\mathrm{HT}}(V) = \underline{D}_K(\mathbf{C}_K \otimes_{\mathbf{Q}_p} V) = (B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V)^{G_K}$$

with grading induced by that on B_{HT}.

Our results in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ show that $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ is stable under tensor product, duality, subrepresentations, and quotients (but not extensions) in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, and that the formation of D_{HT} naturally commutes with finite extension on K as well as with scalar extension to $\widehat{K^{\mathrm{un}}}$. Also, our preceding results show that on $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ the functor D_{HT} is exact and is compatible with tensor products and duality. The comparison morphism

$$\gamma_V : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$$

for $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is an isomorphism precisely when V is Hodge–Tate (apply Lemma 2.3.8 to $W = \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$), and hence $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathrm{HT}}(G_K) \to \mathrm{Gr}_{K,f}$ is a faithful functor.

Example 2.3.11. Theorem 2.1.3 can now be written in the following more appealing form: if X is a smooth proper K-scheme then for $n \geq 0$ the representation $V := \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is in $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ with $D_{\mathrm{HT}}(V) \simeq \mathrm{H}^n_{\mathrm{Hodge}}(X/K) := \oplus_q \mathrm{H}^{n-q}(X, \Omega^q_{X/K})$. Thus, the comparison morphism γ_V takes the form of a B_{HT}-linear G_K-equivariant isomorphism

$$B_{\mathrm{HT}} \otimes_K \mathrm{H}^n_{\mathrm{Hodge}}(X/K) \simeq B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} \mathrm{H}^n(X_{\overline{K}}, \mathbf{Q}_p) \tag{2.3.6}$$

in Gr_K.

This is reminiscent of the deRham isomorphism

$$\mathrm{H}^n_{\mathrm{dR}}(M) \simeq \mathbf{R} \otimes_{\mathbf{Q}} \mathrm{H}_n(M, \mathbf{Q})^\vee$$

for smooth manifolds M, which in the case of finite-dimensional cohomology is described by the matrix $(\int_{\sigma_j} \omega_i)$ for an $\mathbf{R}$-basis $\{\omega_i\}$ of $\mathrm{H}^n_{\mathrm{dR}}(M)$ and a $\mathbf{Q}$-basis $\{\sigma_j\}$ of $\mathrm{H}_n(M, \mathbf{Q})$. The numbers $\int_\sigma \omega$ are classically called *periods* of M, and to define the deRham isomorphism relating deRham cohomology to topological cohomology we must use the coefficient ring $\mathbf{R}$ on the topological side. For this reason, the ring B_{HT} that serves as a coefficient ring for Faltings' comparison isomorphism (2.3.6) between Hodge and étale cohomologies is called a *period ring*. Likewise, the more sophisticated variants on B_{HT} introduced by Fontaine as a means of passing between other pairs of p-adic cohomology theories are all called period rings.

Whereas $\underline{\mathrm{D}}$ on the category of Hodge–Tate objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ is fully faithful into $\mathrm{Gr}_{K,f}$, D_{HT} on the category $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ of Hodge–Tate representations of G_K over $\mathbf{Q}_p$ is *not* fully faithful. For example, if $\eta : G_K \to \mathbf{Z}_p^\times$ has finite order then $D_{\mathrm{HT}}(\mathbf{Q}_p(\eta)) \simeq K\langle 0 \rangle = D_{\mathrm{HT}}(\mathbf{Q}_p)$ by the Tate–Sen theorem, but $\mathbf{Q}_p(\eta)$ and $\mathbf{Q}_p$ have no nonzero maps between them when $\eta \neq 1$. This lack of full faithfulness is one reason that the functor $\mathrm{Rep}_{\mathrm{HT}}(G_K) \to \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ given by $V \rightsquigarrow \mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is a drastic operation and needs to be replaced by something more sophisticated.

To improve on D_{HT} so as to get a *fully faithful* functor from a nice category of p-adic representations of G_K into a category of semilinear algebra objects, we need to do two things: we must refine B_{HT} to a ring with more structure (going beyond a mere grading with a compatible G_K-action) and we need to introduce a target semilinear algebra category that is richer than $\mathrm{Gr}_{K,f}$. As a warm-up, we will next turn to the category of étale φ-modules. This involves a digression away from studying p-adic representations of G_K (it really involves representations of the closed subgroup G_{K_∞} for certain infinitely ramified algebraic extensions K_∞/K inside of $\overline{K}$), but it will naturally motivate some of the objects of semilinear algebra that have to be considered in any reasonable attempt to refine the theory of Hodge–Tate representations.

3. Étale φ-modules

We now switch themes to describe p-adic representations of G_E for arbitrary fields E of characteristic $p > 0$; later this will be applied with $E = k(\!(u)\!)$ for a perfect field k of characteristic p, so in particular E must be allowed to be imperfect. The reason such Galois groups will be of interest to us was sketched in Example 1.3.4. In contrast with the case of Hodge–Tate representations in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, for which there was an equivalence with the relatively simple category $\mathrm{Gr}_{K,f}$ of finite-dimensional graded K-vector spaces, in our new setting we will construct an equivalence between various categories of representations of G_E and some interesting categories of modules equipped with an endomorphism that is semilinear over a "Frobenius" operator on the coefficient ring.

We shall work our way up to $\mathbf{Q}_p$-representation spaces for G_E by first studying $\mathbf{F}_p$-representation spaces for G_E, then general torsion $\mathbf{Z}_p$-representation spaces for G_E, and finally $\mathbf{Z}_p$-lattice representations of G_E (from which the $\mathbf{Q}_p$-case will be analyzed via Lemma 1.2.6).

Throughout this section we work with a fixed field E that is arbitrary with $\mathrm{char}(E) = p > 0$ and we fix a separable closure E_s. We let $G_E = \mathrm{Gal}(E_s/E)$. We emphasize that E is *not* assumed to be perfect, so the p-power endomorphisms of E and E_s are generally not surjective.

3.1. **p-torsion representations.** We are first interested in the category $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ of continuous representations of G_E on finite-dimensional $\mathbf{F}_p$-vector spaces V_0 (so continuity means that the G_E-action on V_0 factors through an action by $\mathrm{Gal}(E'/E)$ for some finite Galois extension E'/E contained in E_s that may depend on V_0). The role of the ring B_{HT} in §2.3 will now be played by E_s, and the relevant structures that this ring admits are twofold: a G_E-action and the p-power endomorphism $\varphi_{E_s} : E_s \to E_s$ (i.e., $x \mapsto x^p$). These two structures on E_s respectively play roles analogous to the G_K-action on B_{HT} and the grading on B_{HT}, and the properties $B_{\mathrm{HT}}^{G_K} = K$ and $\mathrm{gr}^0(B_{\mathrm{HT}}) = \mathbf{C}_K$ have as their respective analogues the identities $E_s^{G_E} = E$ and $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$. The compatibility of the G_K-action and grading on B_{HT} has as its analogue the evident fact that the G_E-action on E_s commutes with the endomorphism $\varphi_{E_s} : x \mapsto x^p$ (i.e., $g(x^p) = g(x)^p$ for all $x \in E_s$ and $g \in G_E$). We write $\varphi_E : E \to E$ to denote the p-power endomorphism of E, so $\varphi_{E_s}|_E = \varphi_E$.

Whereas in Theorem 2.3.9 we used B_{HT} to set up inverse equivalences $\underline{\mathrm{D}}$ and $\underline{V}$ between the category of Hodge–Tate objects in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and the category $\mathrm{Gr}_{K,f}$ of graded K-vector spaces, now we will use E_s to set up an equivalence between the category $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and a certain category of finite-dimensional E-vector spaces equipped with a suitable Frobenius semilinear endomorphism.

The following category of semilinear algebra objects to later be identified with $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ looks complicated at first, but we will soon see that it is not too bad. Below, we write $\varphi_E^*(M_0)$ for an E-vector space M_0 to denote the scalar extension $E \otimes_{\varphi_E, E} M_0$ with its natural E-vector space structure via the left tensor factor.

Definition 3.1.1. An *étale φ-module over E* is a pair (M_0, φ_{M_0}) where M_0 is a finite-dimensional E-vector space and $\varphi_{M_0} : M_0 \to M_0$ is a φ_E-semilinear endomorphism that is étale in the sense that its E-linearization $\varphi_E^*(M_0) \to M_0$ is an isomorphism (i.e., $\varphi_{M_0}(M_0)$ spans M_0 over E, or equivalently the "matrix" of φ_{M_0} relative to a choice of E-basis of M_0 is invertible). The notion of morphism between étale φ-modules over E is defined in the evident manner, and the category of étale φ-modules over E is denoted $\Phi M_E^{\text{ét}}$.

Remark 3.1.2. The reason for the word "étale" in the terminology is that an algebraic scheme X over a field k of characteristic $p > 0$ is étale if and only if the relative Frobenius map $F_{X/k} : X \to X^{(p)} = k \otimes_{\varphi_k, k} X$ over k is an isomorphism.

We often write M_0 rather than (M_0, φ_{M_0}) to denote an object in the category $\Phi M_E^{\text{ét}}$. The simplest interesting example of an object in $\Phi M_E^{\text{ét}}$ is $M_0 = E$ endowed with $\varphi_{M_0} = \varphi_E$; this object will simply be denoted as E. It may not be immediately evident how to make more interesting objects in $\Phi M_E^{\text{ét}}$, but shortly we will associate such an object to every object in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$.

We now give some basic constructions for making new objects out of old ones. There is an evident notion of tensor product in $\Phi M_E^{\text{ét}}$. The notion of duality is defined as follows. For $M_0 \in \Phi M_E^{\text{ét}}$, the dual $M_0^\vee$ has as its underlying E-vector space the usual E-linear dual of M_0, and $\varphi_{M_0^\vee} : M_0^\vee \to M_0^\vee$ carries an E-linear functional $\ell : M_0 \to E$ to the composite of the E-linear pullback functional $\varphi_E^*(\ell) : \varphi_E^*(M_0) \to E$ and the inverse $M_0 \simeq \varphi_E^*(M_0)$ of the E-linearized isomorphism $\varphi_E^*(M_0) \simeq M_0$ induced by φ_{M_0}. To show that this is an étale Frobenius structure, the problem is to check that $\varphi_{M_0^\vee}$ linearizes to an isomorphism. A slick method to establish this is via the alternative description of $\varphi_{M_0^\vee}$ that is provided in the following exercise.

Exercise 3.1.3. Prove that $\varphi_{M_0^\vee} : M_0^\vee \to M_0^\vee$ is the φ_E-semilinear map whose E-linearization is the isomorphism

$$\varphi_E^*(M_0^\vee) \simeq (\varphi_E^*(M_0))^\vee \simeq M_0^\vee$$

with the final isomorphism defined to be inverse to the linear dual of the E-linear isomorphism $\varphi_E^*(M_0) \simeq M_0$ induced by linearization of φ_{M_0}.

In concrete terms, if we choose an E-basis for M_0 and use the dual basis for $M_0^\vee$, then the resulting "matrices" that describe the φ_E-semilinear endomorphisms φ_{M_0} and $\varphi_{M_0^\vee}$ are transpose to each other. It is easy to check that the notions of tensor product and duality as defined in $\Phi M_E^{\text{ét}}$ satisfy the usual relations (e.g., the natural double duality isomorphism $M_0 \simeq M_0^{\vee\vee}$ is an isomorphism in $\Phi M_E^{\text{ét}}$, and the evaluation pairing $M_0 \otimes M_0^\vee \to E$ is a morphism in $\Phi M_E^{\text{ét}}$).

Lemma 3.1.4. *The category $\Phi M_E^{\text{ét}}$ is abelian. More specifically, if $h : M' \to M$ is a morphism in $\Phi M_E^{\text{ét}}$ then the induced Frobenius endomorphisms of* $\ker h$, $\mathrm{im}\, h$, *and* $\mathrm{coker}\, h$ *are étale (i.e., have E-linearization that is an isomorphism).*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \varphi_E^*(M') \arrow[r, "\varphi_E^*(h)"] \arrow[d, "\simeq"'] & \varphi_E^*(M) \arrow[d, "\simeq"] \\ M' \arrow[r, "h"'] & M \end{tikzcd}$$

This induces isomorphisms between kernels, cokernels and images formed in the horizontal directions, and the formation of kernels, cokernels, and images of linear maps commutes with arbitrary ground field extension (such as $\varphi_E : E \to E$). Hence, the desired étaleness properties are obtained. ■

We now use E_s equipped with its *compatible* G_E-action and φ_E-semilinear endomorphism φ_{E_s} to define covariant functors $\mathbf{D}_E$ and $\mathbf{V}_E$ between $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and $\Phi M_E^{\text{ét}}$.

Definition 3.1.5. For any $V_0 \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, define $\mathbf{D}_E(V_0)$ to be the E-vector space $(E_s \otimes_{\mathbf{F}_p} V_0)^{G_E}$ equipped with the φ_E-semilinear endomorphism $\varphi_{\mathbf{D}_E(V_0)}$ induced by $\varphi_{E_s} \otimes 1$. (This makes sense since φ_{E_s} commutes with the G_E-action on E_s.)

For any $M_0 \in \Phi M_E^{\text{ét}}$ we define $\mathbf{V}_E(M_0)$ to be the $\mathbf{F}_p$-vector space $(E_s \otimes_E M_0)^{\varphi=1}$ with its evident G_E-action induced by the G_E-action on E_s; here, $\varphi = \varphi_{E_s} \otimes \varphi_{M_0}$.

Some work is needed to check that $\mathbf{D}_E$ takes values in $\Phi M_E^{\text{ét}}$ and that $\mathbf{V}_E$ takes values in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$. Indeed, it is not at all obvious that $\mathbf{D}_E(V_0)$ is finite-dimensional over E in general, nor that that E-linearization of $\varphi_{\mathbf{D}_E(V_0)}$ is an isomorphism (so $\mathbf{D}_E(V_0) \in \Phi M_E^{\text{ét}}$). Likewise, it is not obvious that $\mathbf{V}_E(M_0)$ is finite-dimensional over $\mathbf{F}_p$, though it is clear from the definition that each element of $\mathbf{V}_E(M_0)$ has an open stabilizer in G_E (since such an element is a finite sum of elementary tensors in $E_s \otimes_E M_0$, and a finite intersection of open subgroups is open). Thus, once finite-dimensionality over $\mathbf{F}_p$ is established then $\mathbf{V}_E(M_0)$ will be an object in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$.

Example 3.1.6. There are two trivial examples that can be worked out by hand. We have $\mathbf{D}_E(\mathbf{F}_p) = E$ with Frobenius endomorphism φ_E and $\mathbf{V}_E(E) = \mathbf{F}_p$ with trivial G_E-action.

Remark 3.1.7. It is sometimes convenient to use contravariant versions $\mathbf{D}_E^*$ and $\mathbf{V}_E^*$ of the functors $\mathbf{D}_E$ and $\mathbf{V}_E$. These may be initially defined in an *ad hoc* way via

$$\mathbf{D}_E^*(V_0) = \mathbf{D}_E(V_0^\vee), \;\; \mathbf{V}_E^*(M_0) = \mathbf{V}_E(M_0^\vee)$$

but the real usefulness is due to an alternative formulation: since $E_s \otimes_{\mathbf{F}_p} V_0^* \simeq \mathrm{Hom}_{\mathbf{F}_p}(V_0, E_s)$ compatibly with the φ_{E_s}-actions and the G_E-actions (defined in the evident way on the Hom-space, namely $(g.\ell)(v) = g(\ell(g^{-1}v)))$, by passing to G_E-invariants we naturally get $\mathbf{D}_E^*(V_0) \simeq \mathrm{Hom}_{\mathbf{F}_p[G_E]}(V_0, E_s)$ as E-vector spaces equipped with a φ_E-semilinear endomorphism. Likewise, we naturally have an $\mathbf{F}_p[G_E]$-linear identification $\mathbf{V}_E^*(M_0) \simeq \mathrm{Hom}_{E,\varphi}(M_0, E_s)$ onto the space of E-linear Frobenius-compatible maps from M_0 into E_s.

Let us begin our study of $\mathbf{D}_E$ and $\mathbf{V}_E$ by checking that they take values in the expected target categories.

Lemma 3.1.8. *For any $V_0 \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, the E-vector space $\mathbf{D}_E(V_0)$ is finite-dimensional with dimension equal to $\dim_{\mathbf{F}_p} V_0$, and the E-linearization of $\varphi_{\mathbf{D}_E(V_0)}$ is an isomorphism. In particular, $\mathbf{D}_E(V_0)$ lies in $\Phi M_E^{\text{ét}}$ with E-rank equal to the $\mathbf{F}_p$-rank of V_0.*

For any $M_0 \in \Phi M_E^{\text{ét}}$, the $\mathbf{F}_p$-vector space $\mathbf{V}_E(M_0)$ is finite-dimensional with dimension at most $\dim_E M_0$. In particular, $\mathbf{V}_E(M_0)$ lies in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ with $\mathbf{F}_p$-rank at most $\dim_E M_0$.

The upper bound for $\dim_{\mathbf{F}_p} \mathbf{V}_E(M_0)$ in this lemma will be proved to be an equality in Theorem 3.1.9, but for now it is simpler (and sufficient) to just prove the upper bound.

Proof. Observe that $E_s \otimes_{\mathbf{F}_p} V_0$ equipped with its diagonal G_E-action is a finite-dimensional E_s-vector space equipped with a semilinear G_E-action that is continuous for the discrete topology in the sense that each element has an open stabilizer (as this is true for each element of E_s and V_0, and hence for finite sums of elementary tensors). Thus, the classical theorem on Galois descent for vector spaces in (2.3.3) (applied to E_s/E) implies that $E_s \otimes_{\mathbf{F}_p} V_0$ is naturally identified with the scalar extension to E_s of its E-vector subspace of G_E-invariant vectors. That is, the natural E_s-linear G_E-equivariant map

$$E_s \otimes_E \mathbf{D}_E(V_0) = E_s \otimes_E (E_s \otimes_{\mathbf{F}_p} V_0)^{G_E} \to E_s \otimes_{\mathbf{F}_p} V_0 \tag{3.1.1}$$

induced by multiplication in E_s is an isomorphism. In particular, $\mathbf{D}_E(V_0)$ has finite E-dimension equal to $\dim_{\mathbf{F}_p} V_0$. (This isomorphism is an analogue of the comparison morphism γ_W in (2.3.5) defined via multiplication in B_{HT} in our study of Hodge–Tate representations.)

A crucial observation is that (3.1.1) satisfies a further compatibility property beyond the E_s-linearity and G_E-actions, namely that it respects the natural Frobenius endomorphisms of both sides (as is clear from the definition). To exploit this, we first recall that for any vector space D over any field F of characteristic $p > 0$, if $\varphi_D : D \to D$ is a φ_F-semilinear endomorphism (with $\varphi_F : F \to F$ denoting $x \mapsto x^p$) then the F-linearization $\varphi_F^*(D) \to D$ of φ_D is compatible with arbitrary extension of the ground field $j : F \to F'$ (as the reader may easily check, ultimately because φ_F and $\varphi_{F'}$ are compatible via j). Applying this to the field extension $E \to E_s$, we see that the E-linearization of $\varphi_{\mathbf{D}_E(V_0)}$ is an isomorphism if and only if the E_s-linearization of $\varphi_{E_s} \otimes \varphi_{\mathbf{D}_E(V_0)}$ is an isomorphism. But Frobenius-compatibility of the E_s-linear isomorphism (3.1.1) renders this property equivalent to the assertion that for any finite-dimensional $\mathbf{F}_p$-vector space V_0 the E_s-linearization of the Frobenius endomorphism $\varphi_{E_s} \otimes 1$ of $E_s \otimes_{\mathbf{F}_p} V_0$ is an isomorphism. By unravelling definitions we see that this E_s-linearization is naturally identified with the identity map of $E_s \otimes_{\mathbf{F}_p} V_0$, so it is indeed an isomorphism. Hence, we have proved the claims concerning $\mathbf{D}_E(V_0)$.

Now we turn to the task of proving that $\mathbf{V}_E(M_0)$ has finite $\mathbf{F}_p$-dimension at most $\dim_E M_0$ (and in particular, it is finite). To do this, we will prove that the natural E_s-linear G_E-compatible and Frobenius-compatible map

$$E_s \otimes_{\mathbf{F}_p} \mathbf{V}_E(M_0) = E_s \otimes_{\mathbf{F}_p} (E_s \otimes_E M_0)^{\varphi=1} \to E_s \otimes_E M_0 \tag{3.1.2}$$

induced by multiplication in E_s is injective. (This map is another analogue of the comparison morphism for Hodge–Tate representations.) Such injectivity will give $\dim_{\mathbf{F}_p} \mathbf{V}_E(M_0) \leq \dim_E M_0$ as desired.

Since any element in the left side of (3.1.2) is a finite sum of elementary tensors, even though $\mathbf{V}_E(M_0)$ is not yet known to be finite-dimensional over $\mathbf{F}_p$ it suffices to prove that if $v_1, \dots, v_r \in \mathbf{V}_E(M_0) = (E_s \otimes_E M_0)^{\varphi=1}$ are $\mathbf{F}_p$-linearly independent then in $E_s \otimes_E M_0$ they are E_s-linearly independent. We assume to the contrary and choose a least $r \geq 1$ for which there is a counterexample, say $\sum a_i v_i = 0$ with $a_i \in E_s$ not all zero. By minimality we have $a_i \neq 0$ for all i, and we may therefore apply $E_s^\times$-scaling to arrange that $a_1 = 1$. Hence, $v_1 = -\sum_{i>1} a_i v_i$. But $v_1 = \varphi(v_1)$ since $v_1 \in \mathbf{V}_E(M_0)$, so

$$v_1 = -\sum_{i>1} \varphi_{E_s}(a_i)\varphi(v_i) = -\sum_{i>1} \varphi_{E_s}(a_i) v_i$$

since $v_i \in \mathbf{V}_E(M_0)$ for all $i > 1$. Hence,

$$\sum_{i>1}(a_i - \varphi_{E_s}(a_i))v_i = 0.$$

By minimality of r we must have $a_i = \varphi_{E_s}(a_i)$ for all $i > 1$, so $a_i \in E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$ for all $i > 1$. Thus, the identity $v_1 = -\sum_{i>1} a_i v_i$ has coefficients in $\mathbf{F}_p$, so we have contradicted the assumption that the v_j's are $\mathbf{F}_p$-linearly independent. ■

By Lemma 3.1.8, we have covariant functors $\mathbf{D}_E : \mathrm{Rep}_{\mathbf{F}_p}(G_E) \to \Phi M_E^{\text{ét}}$ and $\mathbf{V}_E : \Phi M_E^{\text{ét}} \to \mathrm{Rep}_{\mathbf{F}_p}(G_E)$, and $\mathbf{D}_E$ is rank-preserving. Also, since

$$(E_s \otimes_{\mathbf{F}_p} V_0)^{\varphi=1} = E_s^{\varphi=1} \otimes_{\mathbf{F}_p} V_0 = V_0, \ \ (E_s \otimes_E M_0)^{G_E} = E_s^{G_E} \otimes_E M_0 = M_0$$

(use an $\mathbf{F}_p$-basis of V_0 and an E-basis of M_0 respectively), passing to Frobenius-invariants on the isomorphism (3.1.1) defines an isomorphism $\mathbf{V}_E(\mathbf{D}_E(V_0)) \to V_0$ in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and passing to G_E-invariants on the injection (3.1.2) defines an injection $\mathbf{D}_E(\mathbf{V}_E(M_0)) \hookrightarrow M_0$ in $\Phi M_E^{\text{ét}}$.

Theorem 3.1.9. *Via the natural map* $\mathbf{V}_E \circ \mathbf{D}_E \simeq \mathrm{id}$ *and the functorial inclusion* $\mathbf{D}_E \circ \mathbf{V}_E \hookrightarrow \mathrm{id}$*, the covariant functors* $\mathbf{D}_E$ *and* $\mathbf{V}_E$ *are exact rank-preserving quasi-inverse equivalences of categories, and each functor is naturally compatible with tensor products and duality.*

Proof. The isomorphism (3.1.1) implies that $\mathbf{D}_E$ is an exact functor (as it becomes exact after scalar extension from E to E_s). For any two objects V_0 and V_0' in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$, the natural map

$$\mathbf{D}_E(V_0) \otimes_E \mathbf{D}_E(V_0') \to \mathbf{D}_E(V_0 \otimes_{\mathbf{F}_p} V_0')$$

induced by the Frobenius-compatible and G_E-equivariant map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_E (E_s \otimes_{\mathbf{F}_p} V_0') \to E_s \otimes_E (V_0 \otimes V_0')$$

arising from multiplication on E_s is a map in $\Phi M_E^{\text{ét}}$. This map is an isomorphism (and so $\mathbf{D}_E$ is naturally compatible with the formation of tensor products) because we may apply scalar extension from E to E_s and use the isomorphism (3.1.1) to convert this into the obvious claim that the natural map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_{E_s} (E_s \otimes_{\mathbf{F}_p} V_0') \to E_s \otimes_{\mathbf{F}_p} (V_0 \otimes_{\mathbf{F}_p} V_0')$$

is an isomorphism.

Similarly we get that $\mathbf{D}_E$ is compatible with the formation of duals: we claim that the natural map

$$\mathbf{D}_E(V_0) \otimes_E \mathbf{D}_E(V_0^\vee) \simeq \mathbf{D}_E(V_0 \otimes_{\mathbf{F}_p} V_0^\vee) \to \mathbf{D}_E(\mathbf{F}_p) = E \tag{3.1.3}$$

(with second step induced by functoriality of $\mathbf{D}_E$ relative to the evaluation morphism $V_0 \otimes V_0^\vee \to \mathbf{F}_p$ in $\mathrm{Rep}_{\mathbf{F}_p}(G_E)$) is a perfect E-bilinear duality between $\mathbf{D}_E(V_0)$ and $\mathbf{D}_E(V_0^\vee)$, or equivalently the induced morphism $\mathbf{D}_E(V_0^\vee) \to \mathbf{D}_E(V_0)^\vee$ that is easily checked to be a morphism in $\Phi M_E^{\text{ét}}$ is an isomorphism. To verify this perfect duality claim it suffices to check it after scalar extension from E to E_s, in which case via (3.1.1) the pairing map is identified with the natural map

$$(E_s \otimes_{\mathbf{F}_p} V_0) \otimes_{E_s} (E_s \otimes_{\mathbf{F}_p} V_0^\vee) \simeq E_s \otimes_{\mathbf{F}_p} (V_0 \otimes_{\mathbf{F}_p} V_0^\vee) \to E_s$$

that is obviously a perfect E_s-bilinear duality pairing.

To carry out our analysis of $\mathbf{V}_E$ and $\mathbf{D}_E \circ \mathbf{V}_E$, the key point is to check that $\mathbf{V}_E$ is rank-preserving. That is, we have to show that if $\dim_E M_0 = d$ then $\dim_{\mathbf{F}_p} \mathbf{V}(M_0) = d$. Once this is proved, the injective map (3.1.2) is an isomorphism for E_s-dimension reasons and so passing to G_E-invariants on this isomorphism gives that $\mathbf{D}_E \circ \mathbf{V}_E \to \mathrm{id}$ is an isomorphism. The compatibility of $\mathbf{V}_E$ with respect to tensor products and duality can then be verified exactly as we did for $\mathbf{D}_E$ by replacing (3.1.1) with (3.1.2) and using $\mathbf{V}_E(E) = \mathbf{F}_p$ to replace the above use of the identification $\mathbf{D}_E(\mathbf{F}_p) = E$.

Our problem is now really one of counting: we must prove that the inequality $\#\mathbf{V}_E(M_0) \le p^d$ for $d := \dim_E M_0$ is an equality. Arguing as in Remark 3.1.7 with $M_0^\vee$ in the role of M_0 and using double duality gives $\mathbf{V}_E(M_0) \simeq \mathrm{Hom}_{E,\varphi}(M_0^\vee, E_s)$. The key idea is to interpret this set of maps in terms of a system of étale polynomial equations in d variables. Choose a basis $\{m_1, \dots, m_d\}$ of M_0, so $M_0^\vee$ has a dual basis $\{m_i^\vee\}$ and $\varphi_{M_0^\vee}(m_j^\vee) = \sum_i c_{ij} m_i^\vee$ with $(c_{ij}) \in \mathrm{Mat}_{d\times d}(E)$ an invertible matrix. A general E-linear map $M_0^\vee \to E_s$ is given by $m_i^\vee \mapsto x_i \in E_s$ for each i, and Frobenius-compatibility for this map amounts to the system of equations $x_j^p = \sum_i c_{ij} x_i$ for all j. Hence, we have the identification $\mathbf{V}_E(M_0) = \mathrm{Hom}_{E\text{-alg}}(A, E_s)$, where

$$A = E[X_1, \dots, X_d]/(X_j^p - \sum_i c_{ij} X_i)_{1 \le j \le d}.$$

Clearly A is a finite E-algebra with rank p^d, and we wish to prove that its set of E_s-valued points has size equal to $p^d = \dim_E A$. In other words, we claim that A is an étale E-algebra in the sense of commutative algebra. This property amounts to the vanishing of $\Omega^1_{A/E}$, and by direct calculation

$$\Omega^1_{A/E} = (\oplus A \mathrm{d}X_i)/(\sum_j c_{ij} \mathrm{d}X_j)_{1 \le j \le d}.$$

Since $\det(c_{ij}) \in E^\times \subseteq A^\times$, the vanishing is clear. ■

3.2. **Torsion and lattice representations.** We wish to improve on the results in §3.1 by describing the entire category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ of continuous G_E-representations on finitely generated (not necessarily free) $\mathbf{Z}_p$-modules, and then passing to $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ by a suitable localization process. The basic strategy is to first handle torsion objects using $\mathbf{Z}_p$-length induction (and using the settled p-torsion case from §3.1 to get inductive arguments off the ground), and then pass to the inverse limit to handle general objects in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ (especially those that are finite free as $\mathbf{Z}_p$-modules). One difficulty at the outset is that since we are lifting our coefficients from $\mathbf{F}_p$ to $\mathbf{Z}_p$ on the G_E-representation side, we need to lift the E-coefficients in characteristic p on the semilinear algebra side to some ring of characteristic 0 admitting a natural endomorphism lifting φ_E (as well as an analogue for E_s so as to get a suitable lifted "period ring"). Since E is generally not perfect, we cannot work with the Witt ring $W(E)$ (which is generally quite bad if E is imperfect).

Thus, we impose the following hypothesis involving additional auxiliary data that will be fixed for the remainder of the present discussion: we assume that we are given a complete discrete valuation ring $\mathscr{O}_{\mathscr{E}}$ with characteristic 0, uniformizer p, and residue field E, and we assume moreover that there is specified an endomorphism $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ lifting φ_E on the residue field E. We write $\mathscr{E}$ to denote the fraction field $\mathscr{O}_{\mathscr{E}}[1/p]$ of $\mathscr{O}_{\mathscr{E}}$. Abstract commutative algebra (the theory of Cohen rings) ensures that if we drop the Frobenius-lifting hypothesis then there is such an $\mathscr{O}_{\mathscr{E}}$ and it is unique up to non-canonical isomorphism. It can be proved via a direct analysis with explicit Cohen rings that such a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ always exists. However, the present discussion is generally only applied with a special class of fields E for which we can write down an explicit such pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$, and such an explicit pair is useful for some purposes. Hence, we shall now construct such a pair in a special case, and then for the remainder of this section we return to the general case and assume that such an abstract pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ has been given to us.

Example 3.2.1. Assume that $E = k(\!(u)\!)$ with k perfect of characteristic $p > 0$. Let $W(k)$ denote the ring of Witt vectors of k. This is the unique absolutely unramified complete discrete valuation ring with mixed characteristic $(0, p)$ and residue field k; see §4.2. (If k is finite, $W(k)$ is the valuation ring of the corresponding finite unramified extension of $\mathbf{Q}_p$.) In this case an explicit pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ satisfying the above axioms can be constructed as follows.

Let $\mathfrak{S} = W(k)[\![u]\!]$; this is a 2-dimensional regular local ring in which (p) is a prime ideal at which the residue field is $k(\!(u)\!) = E$. Since the localization $\mathfrak{S}_{(p)}$ at the prime ideal (p) is a 1-dimensional regular local ring, it is a discrete valuation ring with uniformizer p. But u is a unit in this localized ring (since $u \notin (p)$ in $\mathfrak{S}$), so $\mathfrak{S}_{(p)}$ is identified with the localization of the Dedekind domain $\mathfrak{S}[1/u]$ at the prime ideal generated by p. Hence, the completion $\mathfrak{S}_{(p)}^{\wedge}$ of this discrete valuation ring is identified with the p-adic completion of the Laurent-series ring $\mathfrak{S}[1/u]$ over $W(k)$. In other words, this completion is a ring of Laurent series over $W(k)$ with a decay condition on coefficients in the negative direction:

$$\mathfrak{S}_{(p)}^{\wedge} \simeq \left\{ \sum_{n \in \mathbf{Z}} a_n u^n \mid a_n \in W(k) \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\}.$$

We define $\mathscr{O}_{\mathscr{E}} = \mathfrak{S}_{(p)}^{\wedge}$. The endomorphism $\sum a_n u^n \mapsto \sum \sigma(a_n) u^{np}$ of $\mathfrak{S}$ (with σ the unique Frobenius-lift on $W(k)$) uniquely extends to a local endomorphism of $\mathfrak{S}_{(p)}$ and hence to a local endomorphism φ of the completion $\mathscr{O}_{\mathscr{E}}$.

Fix a choice of a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ as required above. Since $\mathscr{O}_{\mathscr{E}}$ is a complete discrete valuation ring with residue field E and we have fixed a separable closure E_s of E, the maximal unramified extension (i.e., strict henselization) $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ of $\mathscr{O}_{\mathscr{E}}$ with residue field E_s makes sense and is unique up to unique isomorphism. It is a strictly henselian (generally not complete) discrete valuation ring with uniformizer p, so its fraction field $\mathscr{E}^{\mathrm{un}}$ is $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}[1/p]$. By the universal property of the maximal unramified extension (or rather, of the strict henselization), if $f : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ is a local map (such as φ or the identity) whose reduction $\overline{f} : E \to E$ is endowed with a specified lifting $\overline{f}' : E_s \to E_s$ then there is a unique local map $f' : \mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \to \mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ over f lifting $\overline{f}'$. By uniqueness, the formation of such an f' is compatible with composition.

By taking $f = \varphi$ and $\overline{f}' = \varphi_{E_s}$, we get a unique local endomorphism of $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ again denoted φ that extends the given abstract endomorphism φ of $\mathscr{O}_{\mathscr{E}}$ and lifts the p-power map on E_s. Additionally, by taking f to be the identity map and considering varying $\overline{f}' \in G_E = \mathrm{Gal}(E_s/E)$ we get an induced action of G_E on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ that is simply the classical identification of $\mathrm{Aut}_{\mathscr{O}_{\mathscr{E}}}(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}) = \mathrm{Gal}(\mathscr{E}^{\mathrm{un}}/\mathscr{E})$ with the Galois group G_E of the residue field. Moreover, this G_E-action on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ is clearly continuous and it commutes with φ on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ because the uniqueness of our lifting procedure reduces this to the obvious compatibility of the G_E-action and Frobenius endomorphism on both $\mathscr{O}_{\mathscr{E}}$ and E_s. In particular, the induced G_E-action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$ is continuous and commutes with the induced Frobenius endomorphism.

Definition 3.2.2. The category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ of *étale φ-modules over $\mathscr{O}_{\mathscr{E}}$* consists of pairs $(\mathscr{M}, \varphi_{\mathscr{M}})$ where $\mathscr{M}$ is a finitely generated $\mathscr{O}_{\mathscr{E}}$-module and $\varphi_{\mathscr{M}}$ is a φ-semilinear endomorphism of $\mathscr{M}$ whose $\mathscr{O}_{\mathscr{E}}$-linearization $\varphi^*(\mathscr{M}) \to \mathscr{M}$ is an isomorphism.

Obviously $\Phi M_E^{\text{ét}}$ is the full subcategory of p-torsion objects in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. Note that in the preceding definition we do not require $\mathscr{M}$ to be a finite free module over $\mathscr{O}_{\mathscr{E}}$ or over one of its artinian quotients $\mathscr{O}_{\mathscr{E}}/(p^n)$; this generality is essential for the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ to have nice stability properties. In particular, the étaleness condition in Definition 3.2.2 that $\varphi_{\mathscr{M}}$ linearize to an isomorphism cannot generally be described by a matrix condition. Since $\varphi^*(\mathscr{M})$ and $\mathscr{M}$ have the same $\mathscr{O}_{\mathscr{E}}$-rank and the same invariant factors (due to the uniformizer p being fixed by φ), the linearization of $\varphi_{\mathscr{M}}$ is a linear map between two abstractly isomorphic finitely generated $\mathscr{O}_{\mathscr{E}}$-modules, whence it is an isomorphism if and only if it is surjective. But surjectivity can be checked modulo p, so we conclude that the étaleness property on $\varphi_{\mathscr{M}}$ can be checked by working with the finite-dimensional vector space $\mathscr{M}/p\mathscr{M}$ over $\mathscr{O}_{\mathscr{E}}/(p) = E$.

The category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ has a good notion of tensor product, as well as duality functors $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Z}_p)$ on the respective full subcategories of objects that are of finite $\mathbf{Z}_p$-length and finite free over $\mathbf{Z}_p$.

There are similar tensor and duality functors in the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. Indeed, tensor products $\mathscr{M} \otimes \mathscr{M}'$ are defined in the evident manner using the $\mathscr{O}_{\mathscr{E}}$-module tensor product $\mathscr{M} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{M}'$ and the Frobenius endomorphism $\varphi_{\mathscr{M}} \otimes \varphi_{\mathscr{M}'}$, and it is easy to check that this really is an étale φ-module; i.e., the $\mathscr{O}_{\mathscr{E}}$-linearization of the tensor product Frobenius endomorphism is an isomorphism (since this $\mathscr{O}_{\mathscr{E}}$-linearization is identified with the tensor product of the $\mathscr{O}_{\mathscr{E}}$-linearizations of $\varphi_{\mathscr{M}}$ and $\varphi_{\mathscr{M}'}$). For duality, we use the functor $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{O}_{\mathscr{E}})$ on objects that are finite free over $\mathscr{O}_{\mathscr{E}}$ and the Frobenius endomorphism of this linear dual is defined similarly to the p-torsion case over E. That is, for $\ell \in \mathscr{M}^\vee = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\mathscr{M}, \mathscr{O}_{\mathscr{E}})$ the element $\varphi_{\mathscr{M}^\vee}(\ell) \in \mathscr{M}^\vee$ is the composite of the $\mathscr{O}_{\mathscr{E}}$-linear pullback functional $\varphi^*(\ell) : \varphi^*(\mathscr{M}) \to \mathscr{O}_{\mathscr{E}}$ and the inverse $\mathscr{M} \simeq \varphi^*(\mathscr{M})$ of the $\mathscr{O}_{\mathscr{E}}$-linearization of $\varphi_{\mathscr{M}}$. To verify that this Frobenius structure is étale (i.e., it linearizes to an isomorphism $\varphi^*(\mathscr{M}) \simeq \mathscr{M}$) one can establish an alternative description of $\varphi_{\mathscr{M}^\vee}$ exactly as in Exercise 3.1.3.

Likewise, on the full subcategory $\Phi M^{\text{ét,tor}}_{\mathscr{O}_{\mathscr{E}}}$ of objects of finite $\mathscr{O}_{\mathscr{E}}$-length we use the duality functor $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$ on which we define a φ-semilinear endomorphism akin to the finite free case, now using the natural Frobenius structure on $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$ to identify $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$ with its own scalar extension by $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$. To see that this is really an étale Frobenius structure one again works out an alternative description akin to Exercise 3.1.3, but now it is necessary to give some thought (left to the reader) to justifying that scalar extension by $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ commutes with the formation of the $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$-valued dual (hint: the scalar extension φ is flat since it is a local map between discrete valuation rings with a common uniformizer).

It is an important point to check at the outset that $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ is an abelian category. The content of this verification is to check the étaleness property for the linearized Frobenius maps between kernels, cokernels, and images. Since the formation of cokernels is right exact (and so commutes with reduction modulo p), the case of cokernels follows from Lemma 3.1.4 and the observed sufficiency of checking the étaleness property modulo p. Thus, if $f : \mathscr{M}' \to \mathscr{M}$ is a map in $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ then $\operatorname{coker} f$ has an étale Frobenius endomorphism. Since $\varphi : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}}$ is flat, the formation of $\operatorname{im} f$ and $\ker f$ commutes with scalar extension by φ. That is, $\operatorname{im} \varphi^*(f) \simeq \varphi^*(\operatorname{im} f)$ and similarly for kernels. Since the image of a linear map in a "module category" is naturally identified with the kernel of projection to the cokernel, the known isomorphism property for the linearizations of the Frobenius endomorphisms of $\mathscr{M}$ and $\operatorname{coker} f$ thereby implies the same for $\operatorname{im} f$. Repeating the same trick gives the result for $\ker f$ due to the étaleness property for $\mathscr{M}'$ and $\operatorname{im} f$.

Fontaine discovered that by using the completion $\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}}$ as a "period ring", one can define inverse equivalences of categories between $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ and $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ recovering the inverse equivalences $\mathbf{D}_E$ and $\mathbf{V}_E$ between p-torsion subcategories in Theorem 3.1.9. To make sense of this, we first require a replacement for the basic identities $E_s^{G_E} = E$ and $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p$ that lay at the bottom of our work in the p-torsion case in §3.1.

Lemma 3.2.3. *The natural inclusions* $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}}^{G_E}$ *and* $\mathscr{E} \to (\widehat{\mathscr{E}^{\text{un}}})^{G_E}$ *are equalities, and likewise* $\mathbf{Z}_p = (\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}})^{\varphi=1}$ *and* $\mathbf{Q}_p = (\widehat{\mathscr{E}^{\text{un}}})^{\varphi=1}$.

The successive approximation method used to prove this lemma will arise again later, but for now we hold off on axiomatizing it to a wider context.

Proof. Since G_E and φ fix p, and $\widehat{\mathscr{E}^{\text{un}}} = \widehat{\mathscr{O}_{\mathscr{E}}}[1/p]$, the integral claims imply the field claims. Hence, we focus on the integral claims. The evident inclusions $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}}^{G_E}$ and $\mathbf{Z}_p \to (\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}})^{\varphi=1}$ are local maps between p-adically separated and complete rings, so it clearly suffices to prove surjectivity modulo p^n for all $n \geq 1$. We shall verify this by induction on n, so we first check the base case $n = 1$.

By left-exactness of the formation of G_E-invariants, the exact sequence

$$0 \to \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \xrightarrow{p} \widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}} \to E_s \to 0$$

of $\mathscr{O}_{\mathscr{E}}$-modules gives a linear injection $(\widehat{\mathscr{O}^{\text{un}}_{\mathscr{E}}})^{G_E}/(p) \hookrightarrow E_s^{G_E} = E$ of nonzero modules over $\mathscr{O}_{\mathscr{E}}/(p) = E$, so this injection is bijective for E-dimension reasons. In particular, the natural map $\mathscr{O}_{\mathscr{E}} \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}/(p)$ is surjective. Since $E_s^{\varphi_{E_s}=1} = \mathbf{F}_p = \mathbf{Z}_p/(p)$, a similar argument gives that $\mathbf{Z}_p \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1}/(p)$ is surjective. This settles the case $n = 1$.

Now consider $n > 1$ and assume that $\mathscr{O}_{\mathscr{E}} \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}/(p^{n-1})$ is surjective. Choose any $\xi \in (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}$; we seek $x \in \mathscr{O}_{\mathscr{E}}$ such that $\xi \equiv x \bmod p^n$. We can choose $c \in \mathscr{O}_{\mathscr{E}}$ such that $\xi \equiv c \bmod p^{n-1}$, so $\xi - c = p^{n-1}\xi'$ with $\xi' \in (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E}$. By the settled case $n = 1$ there exists $c' \in \mathscr{O}_{\mathscr{E}}$ such that $\xi' \equiv c' \bmod p$, so $\xi \equiv c + p^{n-1}c' \bmod p^n$ with $c + p^{n-1}c' \in \mathscr{O}_{\mathscr{E}}$. The case of φ-invariants goes similarly. ■

Theorem 3.2.4 (Fontaine)**.** *There are covariant naturally quasi-inverse equivalences of abelian categories*

$$\mathbf{D}_{\mathscr{E}} : \mathrm{Rep}_{\mathbf{Z}_p}(G_E) \to \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}, \ \ \mathbf{V}_{\mathscr{E}} : \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}} \to \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$$

defined by

$$\mathbf{D}_{\mathscr{E}}(V) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{G_E}, \ \ \mathbf{V}_{\mathscr{E}}(M) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M)^{\varphi=1}.$$

(The operator $\varphi_{\mathbf{D}_{\mathscr{E}}(V)}$ is induced by φ on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$.) These functors preserve rank and invariant factors over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$ (in particular, they are length-preserving over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$ for torsion objects and preserve the property of being finite free modules over $\mathscr{O}_{\mathscr{E}}$ and $\mathbf{Z}_p$), and are compatible with tensor products.

The functors $\mathbf{D}_{\mathscr{E}}$ and $\mathbf{V}_{\mathscr{E}}$ are each naturally compatible with the formation of the duality functors $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ on torsion objects, as well as with the formation of the duality functors $\mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(\cdot, \mathscr{O}_{\mathscr{E}})$ and $\mathrm{Hom}_{\mathbf{Z}_p}(\cdot, \mathbf{Z}_p)$ on finite free module objects.

We emphasize that it is not evident from the definitions that $\mathbf{D}_{\mathscr{E}}(V)$ is finitely generated over $\mathscr{O}_{\mathscr{E}}$ for every V in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$, let alone that its Frobenius endomorphism (induced by the Frobenius of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$) is étale. Likewise, it is not evident that $\mathbf{V}_{\mathscr{E}}(M)$ is finitely generated over $\mathbf{Z}_p$ for every M in $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$, nor that the G_E-action on this (arising from the G_E-action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$) is continuous for the p-adic topology. These properties will be established in the course of proving Theorem 3.2.4.

Before we prove Theorem 3.2.4 we dispose of the problem of $\mathscr{O}_{\mathscr{E}}$-module finiteness of $\mathbf{D}_{\mathscr{E}}(V)$ for $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ via the following lemma that is a generalization of the completed unramified descent for finite free modules that was established in the course of proving Theorem 2.3.4.

Lemma 3.2.5. *Let R be a complete discrete valuation ring with residue field k. Choose a separable closure k_s of k and let $R' = \widehat{R^{\mathrm{un}}}$ be the completion of the associated maximal unramified extension R^{un} of R (with residue field k_s). Let $G_k = \mathrm{Gal}(k_s/k)$ act on R' over R in the canonical manner.*

Let M be a finitely generated R'-module equipped with a semilinear G_k-action that is continuous with respect to the natural topology on M. The R-module M^{G_k} is finitely generated, and the natural map

$$\alpha_M : R' \otimes_R (M^{G_k}) \to M$$

is an isomorphism, so M^{G_k} has the same rank and invariant factors over R as M does over R'. In particular, $M \rightsquigarrow M^{G_k}$ is an exact functor and M^{G_k} is a free R-module if and only if M is a free R'-module.

This lemma goes beyond the completed unramified descent result that was established (for the special case $R = \mathscr{O}_K$ but using general methods) in the proof of Theorem 2.3.4 because we now allow M to have nonzero torsion. This requires some additional steps in the argument.

Proof. Once the isomorphism result is established, the exactness of M^{G_k} in M follows from the faithful flatness of $R \to R'$.

Let π be a uniformizer of R, so it is also a uniformizer of R' and is fixed by the G_k-action. We first treat the case when M has finite R'-length, which is to say that it is killed by π^r for some $r \geq 1$. We shall induct on r in this case. If $r = 1$ then M is a finite-dimensional k_s-vector space equipped with a semilinear action of G_k having open stabilizers, so classical Galois descent for vector spaces as in (2.3.3) implies that the natural map $k_s \otimes_k M^{G_k} \to M$ is an isomorphism. (In particular, M^{G_k} is a finite-dimensional k-vector space.) This is the desired result in the π-torsion case.

Now suppose $r > 1$ and that the result is known in the π^{r-1}-torsion case. Let $M' = \pi^{r-1}M$ and $M'' = M/M''$. Clearly M' is π-torsion and M'' is π^{r-1}-torsion. In particular, the settled π-torsion case gives that M' is G_k-equivariantly isomorphic to a product of finitely many copies of k_s, so $\mathrm{H}^1(G_k, M') = 0$. Hence, the left-exact sequence of R-modules

$$0 \to M'^{G_k} \to M^{G_k} \to M''^{G_k} \to 0$$

is exact. The flat extension of scalars $R \to R'$ gives exactness of the top row in the following commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & R' \otimes_R M'^{G_k} \arrow[r] \arrow[d, "\alpha_{M'}", "\simeq"'] & R' \otimes_R M^{G_k} \arrow[r] \arrow[d, "\alpha_M"] & R' \otimes_R M''^{G_k} \arrow[r] \arrow[d, "\alpha_{M''}"', "\simeq"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

in which the outer vertical maps $\alpha_{M'}$ and $\alpha_{M''}$ are isomorphisms by the inductive hypothesis. Thus, the middle map α_M is an isomorphism. This settles the case when M is a torsion R'-module. In particular, the functor $M \rightsquigarrow M^{G_k}$ is exact in the torsion case.

In the general case we shall pass to inverse limits from the torsion case. Fix $n \geq 1$. For all $m \geq n$ we have an R'-linear G_k-equivariant right exact sequence

$$M/(\pi^m) \xrightarrow{\pi^n} M/(\pi^m) \to M/(\pi^n) \to 0 \tag{3.2.1}$$

of torsion objects, so applying the exact functor of G_k-invariants gives a right-exact sequence of finite-length R-modules. But $M^{G_k} \simeq \varprojlim (M/(\pi^m))^{G_k}$ since $M = \varprojlim (M/(\pi^m))$, and passage to inverse limits is exact on sequences of finite-length R-modules, so passing to the inverse limit (over m) on the right-exact sequence of G_k-invariants of (3.2.1) gives the right-exact sequence

$$M^{G_k} \stackrel{\pi^n}{\to} M^{G_k} \to (M/(\pi^n))^{G_k} \to 0$$

for all $n \geq 1$. In other words, the natural R-module map $M^{G_k}/(\pi^n) \to (M/(\pi^n))^{G_k}$ is an isomorphism for all $n \geq 1$.

In the special case $n = 1$, we have just shown that $M^{G_k}/(\pi) \simeq (M/(\pi))^{G_k}$, and our results in the π-torsion case ensure that $(M/(\pi))^{G_k}$ is finite-dimensional over k. Hence, $M^{G_k}/(\pi)$ is finite-dimensional over $k = R/(\pi)$ in general. Since M^{G_k} is a closed R-submodule of the finitely generated R'-module M, the R-module M^{G_k} is π-adically separated and complete. Thus, if we choose elements of M^{G_k} lifting a finite k-basis of $M^{G_k}/(\pi)$ then a π-adic successive approximation argument shows that such lifts span M^{G_k} over R. In particular, M^{G_k} is a finitely generated R-module in general.

Now consider the natural map $\alpha_M : R' \otimes_R M^{G_k} \to M$. This is a map between finitely generated R'-modules, so to show that it is an isomorphism it suffices to prove that the reduction modulo π^n is an isomorphism for all $n \geq 1$. But $\alpha_M \bmod \pi^n$ is identified with $\alpha_{M/(\pi^n)}$ due to the established isomorphism $M^{G_k}/(\pi^n) \simeq (M/(\pi^n))^{G_k}$. Hence, the settled isomorphism result in the general torsion case completes the argument. ■

Now we are ready to take up the proof of Theorem 3.2.4.

Proof. For $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$, consider the natural $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$-linear "comparison morphism"

$$\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathbf{D}_{\mathscr{E}}(V) = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{G_E} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V. \tag{3.2.2}$$

This is clearly compatible with the natural G_E-action and Frobenius endomorphism on both sides. Setting $M = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V$, the semilinear action of G_E on M is clearly continuous (due to the hypothesis that G_E acts continuously on V and the evident continuity of its action on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$). Thus, we can apply Lemma 3.2.5 with $R = \mathscr{O}_{\mathscr{E}}$ to deduce that $\mathbf{D}_{\mathscr{E}}(V) = M^{G_E}$ is a finitely generated $\mathscr{O}_{\mathscr{E}}$-module and that (3.2.2) is an isomorphism.

We immediately obtain some nice consequences. First of all, the Frobenius structure on $\mathbf{D}_{\mathscr{E}}(V)$ is étale (i.e., its $\mathscr{O}_{\mathscr{E}}$-linearization is an isomorphism) because it suffices to check this after the faithfully flat Frobenius-compatible scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, whereupon the isomorphism (3.2.2) reduces this étaleness claim to the obvious fact that the Frobenius endomorphism $\varphi \otimes 1$ on the target of (3.2.2) linearizes to an isomorphism. Hence, have shown that $\mathbf{D}_{\mathscr{E}}$ does indeed take values in the category $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. As such, we claim that $\mathbf{D}_{\mathscr{E}}$ is an exact functor that preserves rank and invariant factors (of $\mathbf{Z}_p$-modules and $\mathscr{O}_{\mathscr{E}}$-modules) and is naturally compatible with tensor products (in a manner analogous to the tensor compatibility that we have already established in the p-torsion case in Theorem 3.1.9). It suffices to check these properties after faithfully flat scalar extension to $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, and after applying such a scalar extension we may use (3.2.2) to transfer the claims to their analogues for the functor $V \rightsquigarrow \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V$, all of which are obvious.

Now we can establish half of the claim concerning inverse functors: for any V in $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ we claim that $\mathbf{V}_{\mathscr{E}}(\mathbf{D}_{\mathscr{E}}(V))$ is naturally $\mathbf{Z}_p[G_E]$-linearly isomorphic to V (but we have not yet proved that $\mathbf{V}_{\mathscr{E}}$ carries general étale φ-modules over $\mathscr{O}_{\mathscr{E}}$ into $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$!). By passing to φ-invariants on the isomorphism (3.2.2) we get a natural $\mathbf{Z}_p[G_E]$-linear isomorphism

$$\mathbf{V}_{\mathscr{E}}(\mathbf{D}_{\mathscr{E}}(V)) \simeq (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{\varphi=1},$$

so we just have to show that the natural $\mathbf{Z}_p[G_E]$-linear map

$$V \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} V)^{\varphi=1}$$

defined by $v \mapsto 1 \otimes v$ is an isomorphism. To justify this, it suffices to show that the diagram

$$0 \to \mathbf{Z}_p \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \stackrel{\varphi-1}{\to} \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \to 0$$

is an exact sequence, since the rightmost term is $\mathbf{Z}_p$-flat (so applying $V \otimes_{\mathbf{Z}_p} (\cdot)$ then yields an exact sequence, giving the desired identification of V with a space of φ-invariants).

The identification of $\mathbf{Z}_p$ with $\ker(\varphi - 1)$ in $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$ follows from Lemma 3.2.3, so we just have to show that $\varphi - 1$ is surjective as a $\mathbf{Z}_p$-linear endomorphism of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. By p-adic completeness and separatedness of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$, along with the fact that $\varphi - 1$ commutes with multiplication by p, we can use successive approximation to reduce to checking the surjectivity on $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}/(p) = E_s$. But on E_s the self-map $\varphi - 1$ becomes $x \mapsto x^p - x$, which is surjective since E_s is separably closed.

We now turn our attention to properties of $\mathbf{V}_{\mathscr{E}}$, the first order of business being to show that it takes values in the category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$. Our analysis of $\mathbf{V}_{\mathscr{E}}$ rests on an analogue of Lemma 3.2.5:

Lemma 3.2.6. *For any D in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$, the $\mathbf{Z}_p$-module $\mathbf{V}_{\mathscr{E}}(D)$ is finitely generated and the natural $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$-linear G_E-equivariant Frobenius-compatible map*

$$\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} \mathbf{V}_{\mathscr{E}}(D) = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D)^{\varphi=1} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D$$

is an isomorphism. In particular, $\mathbf{V}_{\mathscr{E}}(D)$ is exact in D, it has the same rank and invariant factors as D, and its formation is naturally compatible with tensor products.

Proof. We will handle the case when D is a torsion object, and then the general case is deduced from this by passage to inverse limits as in the proof of Lemma 3.2.5. Hence, we assume that D is killed by p^r for some $r \geq 1$, and we shall induct on r. The case $r = 1$ is the known case of étale φ-modules over E that we worked out in the proof of Theorem 3.1.9. To carry out the induction, consider $r > 1$ such that the desired isomorphism result is known in the general p^{r-1}-torsion case. Letting $D' = p^{r-1}D$ and $D'' = D/D'$, we have an exact sequence

$$0 \to D' \to D \to D'' \to 0$$

in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ with D' killed by p and D'' killed by p^{r-1}. Applying the flat scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$ gives an exact sequence, and we just need to check that the resulting left-exact sequence

$$0 \to \mathbf{V}_{\mathscr{E}}(D') \to \mathbf{V}_{\mathscr{E}}(D) \to \mathbf{V}_{\mathscr{E}}(D'')$$

of φ-invariants is actually surjective on the right, for then we can do a diagram chase to infer the desired isomorphism property for D from the settled cases of D' and D'' much like in the proof of Lemma 3.2.5.

Consider the commutative diagram of exact sequences of $\mathbf{Z}_p$-modules

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D' \arrow[r] \arrow[d, "\varphi-1"'] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D \arrow[r] \arrow[d, "\varphi-1"] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D'' \arrow[r] \arrow[d, "\varphi-1"] & 0 \\ 0 \arrow[r] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D' \arrow[r] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D \arrow[r] & \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D'' \arrow[r] & 0 \end{tikzcd}$$

The kernels of the maps $\varphi - 1$ are the submodules of φ-invariants, so the induced diagram of kernels is the left-exact sequence that we wish to prove is short exact. Hence, by the snake lemma it suffices to show that the cokernel along the left side vanishes. Since D' is p-torsion, the left vertical map is the self-map $\varphi - 1$ of $E_s \otimes_E D'$, and we just need to show that this self-map is surjective. But D' is an étale φ-module over E, so our work in the p-torsion case (see (3.1.1)) gives the Frobenius-compatible $\mathbf{F}_p[G_E]$-linear comparison isomorphism

$$E_s \otimes_E D' \simeq E_s \otimes_{\mathbf{F}_p} V'$$

with $V' = \mathbf{V}_E(D') \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$. Hence, the surjectivity problem is reduced to the surjectivity of $\varphi_{E_s} - 1 : x \mapsto x^p - x$ on E_s, which is clear since E_s is separably closed. ■

Returning to the proof of Theorem 3.2.4, as an immediate application of Lemma 3.2.6 we can prove that the G_E-action on the finitely generated $\mathbf{Z}_p$-module $\mathbf{V}_{\mathscr{E}}(D)$ is continuous (for the p-adic topology). It just has to be shown that the action is discrete (i.e., has open stabilizers) modulo p^n for all $n \geq 1$, but the exactness in Lemma 3.2.6 gives $\mathbf{V}_{\mathscr{E}}(D)/(p^n) \simeq \mathbf{V}_{\mathscr{E}}(D/(p^n))$, so it suffices to treat the case when D is p^n-torsion for some $n \geq 1$. In this case $\mathbf{V}_{\mathscr{E}}(D)$ is the space of φ-invariants in $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D = \mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/(p^n) \otimes_{\mathscr{O}_{\mathscr{E}}/(p^n)} D$, so it suffices to prove that the G_E-action on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/(p^n)$ has open stabilizers. Even the action on $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ has open stabilizers, since $\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ is the rising union of finite étale extensions $\mathscr{O}_{\mathscr{E}} \to \mathscr{O}'_{\mathscr{E}}$ corresponding to finite separable extensions E'/E inside of E_s (with $\mathscr{O}'_{\mathscr{E}}/(p) = E'$) and such a finite étale extension is invariant by the action of the open subgroup $G_{E'} \subseteq G_E$ (as can be checked by inspecting actions on the residue field). Thus, we have shown that $\mathbf{V}_{\mathscr{E}}$ takes values in the expected category $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$.

If we pass to G_E-invariants on the isomorphism in Lemma 3.2.6 then we get an $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible isomorphism

$$\mathbf{D}_{\mathscr{E}}(\mathbf{V}_{\mathscr{E}}(D)) \simeq (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D)^{G_E}$$

for any $D \in \Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$. Let us now check that the target of this isomorphism is naturally isomorphic to D via the $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible map $h : D \to (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D)^{G_E}$ defined by $d \mapsto 1 \otimes d$. It suffices to check the isomorphism property after the faithfully flat scalar extension $\mathscr{O}_{\mathscr{E}} \to \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. By Lemma 3.2.5 with $M = \widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D$ and $R = \mathscr{O}_{\mathscr{E}}$, the $\mathscr{O}_{\mathscr{E}}$-module M^{G_E} is finitely generated and the natural map $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} M^{G_E} \to M$ is an isomorphism. But this isomorphism carries the scalar extension $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} h$ of h over to the identity map $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D = M$. Hence, the scalar extension of h is an isomorphism, so h is as well. This completes the verification that $\mathbf{V}_{\mathscr{E}}$ and $\mathbf{D}_{\mathscr{E}}$ are naturally quasi-inverse functors.

It remains to check the behavior of $\mathbf{D}_{\mathscr{E}}$ and $\mathbf{V}_{\mathscr{E}}$ with respect to duality functors. First consider the full subcategories $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)^{\mathrm{tor}}$ and $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét,tor}}$ of torsion objects, on which we use the respective duality functors $V^{\vee} = \mathrm{Hom}_{\mathbf{Z}_p}(V, \mathbf{Q}_p/\mathbf{Z}_p)$ and $D^{\vee} = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(D, \mathscr{E}/\mathscr{O}_{\mathscr{E}})$. In this torsion case the already established tensor compatibility of $\mathbf{D}_{\mathscr{E}}$ gives a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible map

$$\mathbf{D}_{\mathscr{E}}(V) \otimes \mathbf{D}_{\mathscr{E}}(V^{\vee}) \simeq \mathbf{D}_{\mathscr{E}}(V \otimes V^{\vee}) \to \mathbf{D}_{\mathscr{E}}(\mathbf{Q}_p/\mathbf{Z}_p),$$

where (i) we use the evaluation mapping $V \otimes V^{\vee} \to \mathbf{Q}_p/\mathbf{Z}_p$ in the category of $\mathbf{Z}_p[G_E]$-modules and (ii) for any $\mathbf{Z}_p[G_E]$-module W (such as $\mathbf{Q}_p/\mathbf{Z}_p$) we define $\mathbf{D}_{\mathscr{E}}(W) = (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbf{Z}_p} W)^{G_E}$ as an $\mathscr{O}_{\mathscr{E}}$-module endowed with a φ-semilinear Frobenius endomorphism via the G_E-equivariant Frobenius endomorphism of $\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}}$. Clearly $\mathbf{D}_{\mathscr{E}}(\mathbf{Q}_p/\mathbf{Z}_p) = (\widehat{\mathscr{E}^{\mathrm{un}}}/\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{G_E} = (\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})^{G_E}$, and the following lemma identifies this space of G_E-invariants.

Lemma 3.2.7. *The natural Frobenius-compatible map $\mathscr{E}/\mathscr{O}_{\mathscr{E}} \to (\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}})^{G_E}$ is an isomorphism.*

Proof. If we express $\mathscr{E}^{\mathrm{un}}/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ as the direct limit of its p^n-torsion levels $(\mathscr{O}_{\mathscr{E}}^{\mathrm{un}} \cdot p^{-n})/\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}$ for $n \to \infty$, it suffices to prove the analogous claim for the p^n-torsion level for each $n \geq 1$, and using multiplication by p^n converts this into the claim that $\mathscr{O}_{\mathscr{E}}/(p^n) \to (\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}/(p^n))^{G_E}$ is an isomorphism for all $n \geq 1$. The injectivity is clear, and the surjectivity was shown in the proof of Lemma 3.2.3. ∎

By Lemma 3.2.7, we get a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius compatible map

$$\mathbf{D}_{\mathscr{E}}(V) \otimes \mathbf{D}_{\mathscr{E}}(V^{\vee}) \to \mathscr{E}/\mathscr{O}_{\mathscr{E}} \tag{3.2.3}$$

for $V \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)^{\mathrm{tor}}$, so this in turn defines a natural $\mathscr{O}_{\mathscr{E}}$-linear Frobenius-compatible *duality comparison morphism*

$$\mathbf{D}_{\mathscr{E}}(V^{\vee}) \to \mathbf{D}_{\mathscr{E}}(V)^{\vee}.$$

We claim that this latter map in $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ is an isomorphism (or equivalently the $\mathscr{O}_{\mathscr{E}}$-bilinear $\mathscr{E}/\mathscr{O}_{\mathscr{E}}$-valued duality pairing (3.2.3) is a perfect pairing), thereby expressing the natural compatibility of $\mathbf{D}_{\mathscr{E}}$ with respect to duality functors on torsion objects. To establish this isomorphism property for torsion V, we observe that both sides of the duality comparison morphism are exact functors in V, whence we can reduce the isomorphism problem to the p-torsion case. But in this case it is easy to check that our duality pairing is precisely the one constructed for $\mathbf{D}_E$ in our study of étale φ-modules over E in the proof of Theorem 3.1.9 (using the natural Frobenius-compatible E-linear identification of $(\mathscr{E}/\mathscr{O}_{\mathscr{E}})[p]$ with E via the basis $1/p$), and in that earlier work we already established the perfectness of the duality pairing.

In a similar manner we can establish the compatibility of $\mathbf{V}_{\mathscr{E}}$ with duality functors on torsion objects, by considering the functor

$$\mathbf{V}_{\mathscr{E}} : D \rightsquigarrow (\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathscr{O}_{\mathscr{E}}} D)^{\varphi=1}$$

from the category of $\mathscr{O}_{\mathscr{E}}$-modules endowed with a φ-semilinear endomorphism to the category of $\mathbf{Z}_p[G_E]$-modules and verifying that

$$\mathbf{V}_{\mathscr{E}}(\mathscr{E}/\mathscr{O}_{\mathscr{E}}) = (\widehat{\mathscr{E}^{\mathrm{un}}}/\widehat{\mathscr{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1} \simeq \mathbf{Q}_p/\mathbf{Z}_p$$

via an analogue of Lemma 3.2.7. The details are left to the reader.

Finally, we consider the behavior with respect to duality on objects with finite free module structures over $\mathbf{Z}_p$ and $\mathscr{O}_{\mathscr{E}}$. In this case we use the duality functors $V^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(V, \mathbf{Z}_p)$ and $D^\vee = \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}(D, \mathscr{O}_{\mathscr{E}})$ (endowed with the evident G_E and Frobenius structures), and the tensor compatibility enables us to define duality pairings similarly to the torsion case, now resting on the identifications $\mathbf{D}_{\mathscr{E}}(\mathbf{Z}_p) = \mathscr{O}_{\mathscr{E}}$ and $\mathbf{V}_{\mathscr{E}}(\mathscr{O}_{\mathscr{E}}) = \mathbf{Z}_p$ from Lemma 3.2.3. We then get morphisms

$$\mathbf{D}_{\mathscr{E}}(V^\vee) \to \mathbf{D}_{\mathscr{E}}(V)^\vee, \quad \mathbf{V}_{\mathscr{E}}(D^\vee) \to \mathbf{V}_{\mathscr{E}}(D)^\vee$$

in $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ and $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ respectively which we want to prove are isomorphisms. In view of the finite freeness of the underlying module structures it suffices to check that these are isomorphisms modulo p, and the exactness of $\mathbf{V}_{\mathscr{E}}$ and $\mathbf{D}_{\mathscr{E}}$ identifies these mod-p reductions with the corresponding duality comparison morphisms from the p-torsion theory for $V/pV \in \mathrm{Rep}_{\mathbf{F}_p}(G_E)$ and $D/pD \in \Phi M^{\text{ét}}_E$. But we proved in our study of p-torsion objects that such p-torsion duality comparison morphisms are isomorphisms. ■

3.3. **$\mathbf{Q}_p$-representations of G_E.** We conclude our study of p-adic representations of G_E by using our results for $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ to describe the category $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ in a similar Frobenius-semilinear manner. Inspired by Lemma 1.2.6, the idea is that we should use finite-dimensional $\mathscr{E}$-vector spaces (equipped with suitable Frobenius semilinear automorphisms) rather than finite free $\mathscr{O}_{\mathscr{E}}$-modules. However, we will see that there is a subtlety, namely that we need to impose some integrality requirements on the Frobenius structure (whereas in the Galois case the analogous integrality condition, the existence of a Galois-stable $\mathbf{Z}_p$-lattice, is always automatically satisfied: Lemma 1.2.6). For clarity, we now write $\varphi_{\mathscr{O}_{\mathscr{E}}}$ to denote the Frobenius endomorphism of $\mathscr{O}_{\mathscr{E}}$ and $\varphi_{\mathscr{E}}$ to denote the induced endomorphism of its fraction field $\mathscr{E} = \mathscr{O}_{\mathscr{E}}[1/p]$.

To motivate the correct definition of an étale φ-module over $\mathscr{E}$, consider $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ and define the $\mathscr{E}$-vector space

$$\mathbf{D}_{\mathscr{E}}(V) = (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathbf{Q}_p} V)^{G_E}$$

equipped with the $\varphi_{\mathscr{E}}$-semilinear endomorphism $\varphi_{\mathbf{D}_{\mathscr{E}}(V)}$ induced by the G_E-equivariant Frobenius endomorphism of $\widehat{\mathscr{E}^{\mathrm{un}}}$. It may not be immediately evident if $\mathbf{D}_{\mathscr{E}}(V)$ is finite-dimensional over $\mathscr{E}$ or if its Frobenius structure $\mathscr{E}$-linearizes to an isomorphism, but by Lemma 1.2.6 both of these properties and more can be readily deduced from our work in the integral case:

Proposition 3.3.1. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ $D = \mathbf{D}_{\mathscr{E}}(V)$ has finite $\mathscr{E}$-dimension $\dim_{\mathscr{E}} D = \dim_{\mathbf{Q}_p} V$, and the $\mathscr{E}$-linearization $\varphi^*_{\mathscr{E}}(D) \to D$ of φ_D is an isomorphism. Moreover, there is a φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice $L \subseteq D$ such that the $\mathscr{O}_{\mathscr{E}}$-linearization $\varphi^*_{\mathscr{O}_{\mathscr{E}}}(L) \to L$ is an isomorphism.*

Proof. By Lemma 1.2.6, we have $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$ for $\Lambda \in \mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ that is finite free as a $\mathbf{Z}_p$-module. Thus, from the definition it is clear that

$$\mathbf{D}_{\mathscr{E}}(V) = \mathbf{D}_{\mathscr{E}}(\Lambda)[1/p] \simeq \mathscr{E} \otimes_{\mathscr{O}_{\mathscr{E}}} \mathbf{D}_{\mathscr{E}}(\Lambda)$$

as $\mathscr{E}$-vector spaces endowed with a $\varphi_{\mathscr{E}}$-semilinear endomorphism. Since $\mathbf{D}_{\mathscr{E}}(\Lambda) \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ and this is finite free as an $\mathscr{O}_{\mathscr{E}}$-module with rank equal to $\mathrm{rank}_{\mathbf{Z}_p}(\Lambda) = \dim_{\mathbf{Q}_p}(V)$, we are done (take $L = \mathbf{D}_{\mathscr{E}}(\Lambda)$). ■

Proposition 3.3.1 motivates the following definition.

Definition 3.3.2. An *étale φ-module over $\mathscr{E}$* is a finite-dimensional $\mathscr{E}$-vector space D equipped with a $\varphi_{\mathscr{E}}$-semilinear endomorphism $\varphi_D : D \to D$ whose linearization $\varphi^*_{\mathscr{E}}(D) \to D$ is an isomorphism and which admits a φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice $L \subseteq D$ such that $(L, \varphi_D|_L) \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ (i.e., the linearization $\varphi^*_{\mathscr{O}_{\mathscr{E}}}(L) \to L$ induced by φ_D is an isomorphism). The category of such pairs (D, φ_D) is denoted $\Phi M^{\text{ét}}_{\mathscr{E}}$.

The lattice L in Definition 3.3.2 is auxiliary data and is not at all canonical. In Definition 3.3.2 the existence of the φ_D-stable $L \in \Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$ forces φ_D to $\mathscr{E}$-linearize to an isomorphism, but it seems more elegant to impose this latter étaleness property on φ_D before we mention the hypothesis concerning the existence of the non-canonical L. Such $\mathscr{O}_{\mathscr{E}}$-lattices L are analogous to Galois-stable $\mathbf{Z}_p$-lattices in an object of $\mathrm{Rep}_{\mathbf{Q}_p}(\Gamma)$ for a profinite group Γ: their existence is a useful device in proofs, but they are not part of the intrinsic structure of immediate interest.

Example 3.3.3. The naive definition one may have initially imagined for an étale φ-module over $\mathscr{E}$ is a finite-dimensional $\mathscr{E}$-vector space D equipped with a $\varphi_{\mathscr{E}}$-semilinear endomorphism φ_D whose $\mathscr{E}$-linearization is an isomorphism. However, this is insufficient for getting an equivalence with $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ because such objects (D, φ_D) can fail to admit a Frobenius-stable (let alone étale) $\mathscr{O}_{\mathscr{E}}$-lattice L as in Proposition 3.3.1. The problem is that the Frobenius endomorphism φ_D can lack good integrality properties; there is no analogue of Lemma 1.2.6 on the Frobenius-semilinear module side.

To give a concrete example, let $D = \mathscr{E}$ and define $\varphi_D = p^{-1} \cdot \varphi_{\mathscr{E}}$. In this case for any nonzero $x \in D$ we have

$$\varphi_D(x) = p^{-1} \cdot \varphi_{\mathscr{E}}(x) = p^{-1} \cdot \frac{\varphi_{\mathscr{E}}(x)}{x} \cdot x.$$

Since the multiplier $\varphi_{\mathscr{E}}(x)/x$ lies in $\mathscr{O}^\times_{\mathscr{E}}$, the additional factor of $1/p$ prevents $\varphi_D(x)$ from being an $\mathscr{O}_{\mathscr{E}}$-multiple of x. The $\mathscr{O}_{\mathscr{E}}$-lattices in $\mathscr{E}$ are precisely the $\mathscr{O}_{\mathscr{E}}$-modules $\mathscr{O}_{\mathscr{E}} \cdot x$ for $x \in \mathscr{E}^\times$, so we conclude that there is no φ_D-stable $\mathscr{O}_{\mathscr{E}}$-lattice L in D (let alone one whose Frobenius endomorphism linearizes to a lattice isomorphism).

There is an evident functor $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}} \to \Phi M^{\text{ét}}_{\mathscr{E}}$ given by $L \rightsquigarrow L[1/p] = \mathscr{E} \otimes_{\mathscr{O}_{\mathscr{E}}} L$, and clearly

$$\mathrm{Hom}_{\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}}(L, L')[1/p] = \mathrm{Hom}_{\Phi M^{\text{ét}}_{\mathscr{E}}}(L[1/p], L'[1/p]),$$

so $\Phi M^{\text{ét}}_{\mathscr{E}}$ is identified with the "isogeny category" of $\Phi M^{\text{ét}}_{\mathscr{O}_{\mathscr{E}}}$. In particular, $\Phi M^{\text{ét}}_{\mathscr{E}}$ is abelian.

Theorem 3.3.4. *The functors* $\mathbf{D}_{\mathscr{E}}(V) := (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathbf{Q}_p} V)^{\varphi=1}$ *and* $\mathbf{V}_{\mathscr{E}}(D) := (\widehat{\mathscr{E}^{\mathrm{un}}} \otimes_{\mathscr{E}} D)^{G_E}$ *are rank-preserving exact quasi-inverse equivalences between* $\mathrm{Rep}_{\mathbf{Q}_p}(G_E)$ *and* $\Phi M_{\mathscr{E}}^{\text{ét}}$ *that naturally commute with the formation of tensor products and duals.*

Proof. If Λ is a G_E-stable $\mathbf{Z}_p$-lattice in V then we have seen that $\mathbf{D}_{\mathscr{E}}(V) = \mathbf{D}_{\mathscr{E}}(\Lambda)[1/p]$, and likewise if we choose (as we may by definition) an étale φ-module L that is a Frobenius-stable $\mathscr{O}_{\mathscr{E}}$-lattice in a chosen $D \in \Phi M_{\mathscr{E}}^{\text{ét}}$ then $\mathbf{V}_{\mathscr{E}}(D) = \mathbf{V}_{\mathscr{E}}(L)[1/p]$. Thus, everything is immediately obtained by p-localization on our results comparing $\mathrm{Rep}_{\mathbf{Z}_p}(G_E)$ and $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}}$ (using the full subcategories of objects with finite free module structures over $\mathbf{Z}_p$ and $\mathscr{O}_{\mathscr{E}}$). ■

4. First steps toward better period rings

4.1. **From gradings to filtrations.** The ring B_{HT} provides a convenient mechanism for working with Hodge–Tate representations, but the Hodge–Tate condition on a p-adic representation of the Galois group G_K of a p-adic field K is too weak to be really useful. What we seek is a class of p-adic representations that is broad enough to include the representations arising from algebraic geometry but also small enough to permit the existence of an equivalence of categories with (or at least a fully faithful exact tensor functor to) a category of semilinear algebra objects. Based on our experience with Hodge–Tate representations and étale φ-modules, we can expect that on the semilinear algebra side we will need to work with modules admitting some kind of structures like Frobenius endomorphisms and gradings (or filtrations). We also want the functor relating our "good" p-adic representations of G_K to semilinear algebra to be defined by a period ring that is "better" than B_{HT} and allows us to recover B_{HT} (i.e., whatever class of good representations we study should at least be of Hodge–Tate type).

The ring $B_{\mathrm{HT}} = \oplus_q \mathbf{C}_K(q)$ is a graded $\mathbf{C}_K$-algebra endowed with a compatible semilinear G_K-action. In view of the isomorphism (2.3.6) in Gr_K, the grading on B_{HT} is closely related to the grading on the Hodge cohomology $\mathrm{H}^n_{\mathrm{Hodge}}(X) = \oplus_{p+q=n}\mathrm{H}^p(X, \Omega^q_{X/K})$ for smooth proper K-schemes X. To motivate how we should refine B_{HT}, we can get a clue from the refinement of $\mathrm{H}^n_{\mathrm{Hodge}}(X)$ given by the algebraic deRham cohomology $\mathrm{H}^n_{\mathrm{dR}}(X/K)$. This is not the place to enter into the definition of algebraic deRham cohomology, but it is instructive to record some of its properties.

For any proper scheme X over any field k whatsoever, the algebraic deRham cohomologies $\mathrm{H}^n(X) = \mathrm{H}^n_{\mathrm{dR}}(X/k)$ are finite-dimensional k-vector spaces endowed with a natural decreasing (Hodge) filtration

$$\mathrm{H}^n(X) = \mathrm{Fil}^0(\mathrm{H}^n(X)) \supseteq \mathrm{Fil}^1(\mathrm{H}^n(X)) \supseteq \cdots \supseteq \mathrm{Fil}^{n+1}(\mathrm{H}^n(X)) = 0$$

by k-subspaces and $\mathrm{Fil}^q(\mathrm{H}^n(X))/\mathrm{Fil}^{q+1}(\mathrm{H}^n(X))$ is naturally a subquotient of $\mathrm{H}^{n-q}(X, \Omega^q_{X/k})$, with a natural equality

$$\mathrm{Fil}^q(\mathrm{H}^n(X))/\mathrm{Fil}^{q+1}(\mathrm{H}^n(X)) = \mathrm{H}^{n-q}(X, \Omega^q_{X/k})$$

if $\mathrm{char}(k) = 0$.

Definition 4.1.1. A *filtered module* over a commutative ring R is an R-module M endowed with a collection $\{\mathrm{Fil}^i M\}_{i\in\mathbf{Z}}$ of submodules that is decreasing in the sense that $\mathrm{Fil}^{i+1}(M) \subseteq \mathrm{Fil}^i(M)$ for all $i \in \mathbf{Z}$. If $\cup \mathrm{Fil}^i(M) = M$ then the filtration is *exhaustive* and if $\cap \mathrm{Fil}^i(M) = 0$ then the filtration is *separated*. For any filtered R-module M, the *associated graded module* is $\mathrm{gr}^\bullet(M) = \oplus_i(\mathrm{Fil}^i(M)/\mathrm{Fil}^{i+1}(M))$.

A *filtered ring* is a ring R equipped with an exhaustive and separated filtration $\{R^i\}$ by additive subgroups such that $1 \in R^0$ and $R^i \cdot R^j \subseteq R^{i+j}$ for all $i, j \in \mathbf{Z}$. The *associated graded ring* is $\mathrm{gr}^\bullet(R) = \oplus_i R^i/R^{i+1}$. If k is a ring then a *filtered k-algebra* is a k-algebra A equipped with a structure of filtered ring such that the filtered pieces A^i are k-submodules of A, and the *associated graded k-algebra* is $\mathrm{gr}^\bullet(A) = \oplus_i A^i/A^{i+1}$.

Example 4.1.2. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $A = \mathrm{Frac}(R)$. There is a natural structure of filtered ring on A via $A^i = \mathfrak{m}^i$ for $i \in \mathbf{Z}$. In this case the associated graded ring $\mathrm{gr}^\bullet(A)$ is a k-algebra that is non-canonically isomorphic to a Laurent polynomial ring $k[t, 1/t]$ upon choosing a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Note that canonically $\mathrm{gr}^\bullet(A) = \mathrm{gr}^\bullet(\widehat{A})$, where $\widehat{A}$ denotes the fraction field of the completion $\widehat{R}$ of R.

For a smooth proper $\mathbf{C}$-scheme X, Grothendieck constructed a natural $\mathbf{C}$-linear isomorphism $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C}) \simeq \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{H}^n_{\mathrm{top}}(X(\mathbf{C}), \mathbf{Q})$. Complex conjugation on the left tensor factor of the target defines a conjugate-linear automorphism $v \mapsto \overline{v}$ of $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C})$, and by Hodge theory this determines a canonical splitting of the Hodge filtration on $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C})$ via the $\mathbf{C}$-subspaces $H^{n-q,q} := \overline{F^{n-q}} \cap F^q$ where $F^j = \mathrm{Fil}^j(\mathrm{H}^n_{\mathrm{dR}}(X/\mathbf{C}))$; i.e., $H^{n-q,q} \simeq F^q/F^{q+1}$ for all q, so $F^j = \oplus_{q\geq j} H^{n-q,q}$. Moreover, in Hodge theory one constructs a natural isomorphism $H^{n-q,q} \simeq \mathrm{H}^{n-q}(X, \Omega^q_{X/\mathbf{C}})$. In particular, complex conjugation gives rise to a canonical splitting of the Hodge filtration when the ground field is $\mathbf{C}$.

In the general algebraic case over an arbitrary field k of characteristic 0, the best one has canonically is that for any smooth proper k-scheme X, the k-vector space $\mathrm{H}^n_{\mathrm{dR}}(X/k)$ is naturally endowed with an exhaustive and separated filtration whose associated graded vector space

$$\mathrm{gr}^\bullet(\mathrm{H}^n_{\mathrm{dR}}(X/k)) := \bigoplus_q \mathrm{Fil}^q(\mathrm{H}^n_{\mathrm{dR}}(X/k))/\mathrm{Fil}^{q+1}(\mathrm{H}^n_{\mathrm{dR}}(X/k))$$

is the Hodge cohomology $\oplus_q \mathrm{H}^{n-q}(X, \Omega^q_{X/k})$ of X.

A natural idea for improving Faltings' comparison isomorphism (2.3.6) between p-adic étale and graded Hodge cohomology via B_{HT} is to replace the graded $\mathbf{C}_K$-algebra B_{HT} with a filtered K-algebra B_{dR} endowed with a G_K-action respecting the filtration such that $\mathrm{Fil}^0(B_{\mathrm{dR}})/\mathrm{Fil}^1(B_{\mathrm{dR}}) \simeq \mathbf{C}_K$ as rings and such that there is a canonical G_K-equivariant isomorphism $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras. In this way we can hope that the functor $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ on $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ with values in (exhaustive and separated) filtered K-vector spaces is a finer invariant than $D_{\mathrm{HT}}(V)$ in the sense that on a reasonable class of V (within the Hodge–Tate class) the evident natural map

$$\mathrm{gr}^\bullet(D_{\mathrm{dR}}(V)) \to (\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)^{G_K} = (B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V)^{G_K} = D_{\mathrm{HT}}(V)$$

of graded K-vector spaces is an isomorphism.

Inspired by Example 4.1.2, we are led to seek a complete discrete valuation ring B_{dR}^{+} over K (with maximal ideal denoted $\mathfrak{m}$) endowed with a G_K-action such that the residue field is naturally G_K-equivariantly isomorphic to $\mathbf{C}_K$ and the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is naturally isomorphic to $\mathbf{C}_K(1)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Since there is a canonical isomorphism $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq (\mathfrak{m}/\mathfrak{m}^2)^{\otimes i}$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$ for all $i \in \mathbf{Z}$, for the fraction field B_{dR} of such a ring B_{dR}^{+} we would then canonically have $\mathrm{gr}^{\bullet}(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras with G_K-action.

A naive guess is to take $B_{\mathrm{dR}}^{+} = \mathbf{C}_K[\![t]\!]$ with a continuous G_K-action given by $g(\sum a_n t^n) = \sum g(a_n)\chi(g)^n t^n$. However, this does not lead to new concepts refining the theory of Hodge–Tate representations since the $\mathbf{C}_K$-algebra structure on $\mathbf{C}_K[\![t]\!]$ and the powers of the generator t of the maximal ideal $\mathfrak{m}$ allow us to canonically define a G_K-equivariant splitting of the filtration on $\mathfrak{m}^i/\mathfrak{m}^j$ for any $i, j \in \mathbf{Z}$ with $j > i$. In other words, for such a naive choice of complete discrete valuation ring the filtration is too closely related to a grading to give anything interesting (beyond what we already get from the Hodge–Tate theory).

A more promising idea is to imitate the procedure in commutative algebra whereby for perfect fields k of characteristic $p > 0$ there is a functorially associated complete discrete valuation ring $W(k)$ (of Witt vectors) that has uniformizer p and residue field k. (See §4.2.) A big difference is that now we want to functorially build a complete discrete valuation ring with residue field $\mathbf{C}_K$ of characteristic 0 (and we will not expect to have a canonical uniformizer). Thus, we cannot use a naive Witt construction (as in §4.2). Nonetheless, we shall see that an artful application of Witt-style ideas will give rise to the right equicharacteristic-0 complete discrete valuation ring B_{dR}^{+} for our purposes (and though any complete discrete valuation ring with residue field F of characteristic 0 is abstractly isomorphic to $F[\![t]\!]$ by commutative algebra, such a structure will not exist for B_{dR}^{+} in a G_K-*equivariant* manner).

We should emphasize at the outset that B_{dR}^{+} will differ from $\mathbf{C}_K[\![t]\!]$ (as complete discrete valuation rings with G_K-action and residue field $\mathbf{C}_K$) in at least two key respects. First, as we just noted, there will be no G_K-equivariant ring-theoretic section to the reduction map from B_{dR}^{+} onto its residue field $\mathbf{C}_K$. Second, even the quotient $B_{\mathrm{dR}}^{+}/\mathfrak{m}^2$ as an extension of $\mathbf{C}_K$ by $\mathbf{C}_K(1)$ will have no G_K-equivariant additive splitting.

Roughly speaking, the idea behind the construction of B_{dR}^{+} is as follows. Rather than try to directly make a canonical complete discrete valuation ring with residue field $\mathbf{C}_K$, we observe that $\mathbf{C}_K = \mathscr{O}_{\mathbf{C}_K}[1/p]$ with $\mathscr{O}_{\mathbf{C}_K} = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p^n) = \varprojlim \mathscr{O}_{\overline{K}}/(p^n)$ closely related to p-power torsion rings. Hence, it is more promising to try to adapt Witt-style constructions for $\mathscr{O}_{\mathbf{C}_K}$ than for $\mathbf{C}_K$. We will make a certain height-1 valuation ring R_K of equicharacteristic p whose fraction field $\mathrm{Frac}(R_K)$ is algebraically closed (hence perfect) such that there is a natural G_K-action on R_K and a natural surjective G_K-equivariant map

$$\theta : W(R_K) \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}.$$

(Note that $W(R_K) \subseteq W(\mathrm{Frac}(R_K))$, so $W(R_K)$ is a domain of characteristic 0.) We would then get a surjective G_K-equivariant map $\theta_K : W(R_K)[1/p] \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K$. Since R_K is like a 1-dimensional ring, $W(R_K)$ is like a 2-dimensional ring and so $W(R_K)[1/p]$ is like a 1-dimensional ring. The ring structure of $W(A)$ is generally pretty bad if A is not a perfect field of characteristic p, but as long as the maximal ideal $\ker\theta_K$ is principal and nonzero we can replace $W(R_K)[1/p]$ with its $\ker\theta_K$-adic completion to obtain a canonical complete discrete valuation ring B^+_{dR} having residue field $\mathbf{C}_K$ (and it will satisfy all of the other properties that we shall require).

4.2. **Witt vectors and universal Witt constructions.** Let k be a finite field of characteristic p and let A be the valuation ring of the finite unramified extension of $\mathbf{Z}_p$ with residue field k. Let $[\cdot] : k \to A$ be the multiplicative Teichmüller lifting (carrying 0 to 0 and sending $k^\times$ isomorphically onto $\mu_{q-1}(A)$ with $q = \#k$), so every element $a \in A$ admits a unique expansion $a = \sum_{n\geq 0}[c_n]p^n$ with $c_n \in k$. For any such $a \in A$ and $a' = \sum_{n\geq 0}[c'_n]p^n \in A$, it is natural to ask if we can compute the Teichmüller expansions of $a + a'$ and aa' by "universal formulas" (independent of k beyond the specification of the characteristic as p) involving only algebraic operations over $\mathbf{F}_p$ on the sequences $\{c_n\}$ and $\{c'_n\}$ in k. Since A is functorially determined by k it is not unreasonable to seek this kind of reconstruction of A in such a direct manner in terms of k.

One can work out such formulas in some low-degree Teichmüller coefficients, and then it becomes apparent that what really matters about k is not its finiteness but rather its perfectness. Rather than give a complete development of Witt vectors from scratch, we refer the reader to [7, Ch. II, §4–§6] for such a development and some aspects of this theory will be reviewed below as necessary. We assume that the reader has some previous experience with the ring of Witt vectors $W(A)$ for an arbitrary commutative ring A (not just for $\mathbf{F}_p$-algebras A).

Let A be a perfect $\mathbf{F}_p$-algebra (i.e., an $\mathbf{F}_p$-algebra for which $a \mapsto a^p$ is an automorphism of A). Observe that the additive multiplication map $p : W(A) \to W(A)$ is given by $(a_i) \mapsto (0, a_0^p, a_1^p, \dots)$, so it is injective and the subset $p^nW(A) \subseteq W(A)$ consists of Witt vectors (a_i) such that $a_0 = \cdots = a_{n-1} = 0$ since A is perfect, so we naturally have $W(A)/(p^n) \simeq W_n(A)$ by projection to the first n Witt components. Hence, the natural map $W(A) \to \varprojlim W(A)/(p^n)$ is an isomorphism. Thus, $W(A)$ for perfect $\mathbf{F}_p$-algebras A is a strict p-ring in the sense of the following definition.

Definition 4.2.1. A *p-ring* is a ring B that is separated and complete for the topology defined by a specified decreasing collection of ideals $\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \dots$ such that $\mathfrak{b}_n\mathfrak{b}_m \subseteq \mathfrak{b}_{n+m}$ for all $n, m \geq 1$ and $B/\mathfrak{b}_1$ is a perfect $\mathbf{F}_p$-algebra (so $p \in \mathfrak{b}_1$).

We say that B is a *strict p-ring* if moreover $\mathfrak{b}_i = p^iB$ for all $i \geq 1$ (i.e., B is p-adically separated and complete with B/pB a perfect $\mathbf{F}_p$-algebra) and $p : B \to B$ is injective.

In addition to $W(A)$ being a strict p-ring for perfect $\mathbf{F}_p$-algebras A, a wide class of (generally non-strict) p-rings is given by complete local noetherian rings with a perfect residue field of characteristic $p > 0$ (taking $\mathfrak{b}_i$ to be the ith power of the maximal ideal).

Lemma 4.2.2. *Let B be a p-ring. There is a unique set-theoretic section $r_B : B/\mathfrak{b}_1 \to B$ to the reduction map such that $r_B(x^p) = r_B(x)^p$ for all $x \in B/\mathfrak{b}_1$. Moreover, r_B is multiplicative and $r_B(1) = 1$.*

Proof. This proceeds by the same method as used in the development of the theory of Witt vectors, as follows. By perfectness of the $\mathbf{F}_p$-algebra $B/\mathfrak{b}_1$, we can make sense of $x^{p^{-n}}$ for all $x \in B/\mathfrak{b}_1$ and all $n \geq 1$. For any choice of lift $\widehat{x^{p^{-n}}} \in B$ of $x^{p^{-n}}$, the sequence of powers $\widehat{x^{p^{-n}}}^{p^n}$ is Cauchy for the $\mathfrak{b}_1$-adic topology. Indeed, for $n' \geq n$ we have $\widehat{x^{p^{-n'}}}^{p^{n'-n}} \equiv \widehat{x^{p^{-n}}} \bmod \mathfrak{b}_1$, so raising to the p^n-power gives $\widehat{x^{p^{-n'}}}^{p^{n'}} \equiv \widehat{x^{p^{-n}}}^{p^n} \bmod (p\mathfrak{b}_1^n, \mathfrak{b}_1^{p^n})$ since in general if $y \equiv y' \bmod J$ for an ideal J in a ring R with $p \in J$ (such as $J = \mathfrak{b}_1$ in $R = B$) then $y^{p^n} \equiv y'^{p^n} \bmod (pJ^n, J^{p^n})$ for all $n \geq 1$. Since $\mathfrak{b}_1^i \subseteq \mathfrak{b}_i$ for all $i \geq 1$ and B is assumed to be separated and complete for the topology defined by the $\mathfrak{b}_i$'s, there is a well-defined limit

$$r_B(x) = \lim_{n\to\infty} \widehat{x^{p^{-n}}}^{p^n} \in B$$

relative to this topology. Obviously $r_B(x^p) = r_B(x)^p$. If we make another choice of lifting $\widetilde{x^{p^{-n}}}$ then the congruence $\widehat{x^{p^{-n}}} \equiv \widetilde{x^{p^{-n}}} \bmod \mathfrak{b}_1$ implies $\widehat{x^{p^{-n}}}^{p^n} \equiv \widetilde{x^{p^{-n}}}^{p^n} \bmod (p\mathfrak{b}_1^n, \mathfrak{b}_1^{p^n})$ for all $n \geq 1$, whence the limit $\widetilde{r}_B(x)$ constructed using these other liftings satisfies $\widetilde{r}_B(x) \equiv r_B(x) \bmod \mathfrak{b}_n$ for all $n \geq 1$, so $\widetilde{r}_B(x) = r_B(x)$. In other words, $r_B(x)$ is independent of the choice of liftings $\widehat{x^{p^{-n}}}$.

In particular, if ρ_B is a p-power compatible section as in the statement of the lemma then we could choose $\widehat{x^{p^{-n}}} = \rho_B(x^{p^{-n}})$ for all $n \geq 1$ in the construction of $r_B(x)$, so

$$\widehat{x^{p^{-n}}}^{p^n} = \rho_B((x^{p^{-n}})^{p^n}) = \rho_B(x).$$

Passing to the limit gives $r_B(x) = \rho_B(x)$. This proves the uniqueness in the lemma, so it remains to check that r_B is multiplicative and $r_B(1) = 1$. The latter condition is clear from the construction, and for the multiplicativity we observe that $\widehat{(xy)^{p^{-n}}}$ can be chosen to be $r_B(x^{p^{-n}})r_B(y^{p^{-n}})$ in the construction of $r_B(xy)$, so passing to p^n-powers and then to the limit gives $r_B(x)r_B(y) = r_B(xy)$. ■

An immediate consequence of this lemma is that in a strict p-ring B endowed with the p-adic topology (relative to which it is separated and complete), each element $b \in B$ has the unique form $b = \sum_{n\geq 0} r_B(b_n)p^n$ with $b_n \in B/\mathfrak{b}_1 = B/pB$. This leads to the following useful universal property of certain Witt rings.

Proposition 4.2.3. *If A is a perfect $\mathbf{F}_p$-algebra and B is a p-ring, then the natural "reduction" map* $\mathrm{Hom}(W(A), B) \to \mathrm{Hom}(A, B/\mathfrak{b}_1)$ *(which makes sense since $A = W(A)/(p)$ and $p \in \mathfrak{b}_1$) is bijective. More generally, for any strict p-ring $\mathscr{B}$, the natural map*

$$\mathrm{Hom}(\mathscr{B}, B) \to \mathrm{Hom}(\mathscr{B}/(p), B/\mathfrak{b}_1)$$

is bijective for every p-ring B.

In particular, since $\mathscr{B}$ and $W(\mathscr{B}/(p))$ satisfy the same universal property in the category of p-rings for any strict p-ring $\mathscr{B}$, strict p-rings are precisely the rings of the form $W(A)$ for perfect $\mathbf{F}_p$-algebras A.

Proof. Elements $\beta \in \mathscr{B}$ have the unique form $\beta = \sum_n r_{\mathscr{B}}(\beta_n)p^n$ for $\beta_n \in \mathscr{B}/(p)$. By construction, the multiplicative sections r_B and $r_{\mathscr{B}}$ are functorial with respect to any ring map $h : \mathscr{B} \to B$ and the associated reduction $\overline{h} : \mathscr{B}/(p) \to B/\mathfrak{b}_1$, so

$$h(\beta) = \sum h(r_{\mathscr{B}}(\beta_n))p^n = \sum r_B(\overline{h}(\beta_n))p^n,$$

whence h is uniquely determined by $\overline{h}$. To go in reverse and lift ring maps, we have to show that if $\overline{h} : \mathscr{B}/(p) \to B/\mathfrak{b}_1$ is a given ring map then the map of sets $\mathscr{B} \to B$ defined by

$$\beta = \sum r_{\mathscr{B}}(\beta_n)p^n \mapsto \sum r_B(\overline{h}(\beta_n))p^n$$

is a ring map. This map obviously respects multiplicative identity elements, so we have to check additivity and multiplicativity. For this it suffices to prove quite generally that in an arbitrary p-ring C, the ring structure on a pair of elements $c = \sum r_C(c_n^{p^{-n}})p^n$ and $c' = \sum r_C({c'_n}^{p^{-n}})p^n$ with sequences $\{c_n\}$ and $\{c'_n\}$ in $C/\mathfrak{c}_1$ is given by formulas

$$c + c' = \sum r_C(S_n(c_0, \dots, c_n; c'_0, \dots, c'_n)^{p^{-n}})p^n, \;\; cc' = \sum r_C(P_n(c_0, \dots, c_n; c'_0, \dots, c'_n)^{p^{-n}})p^n$$

for universal polynomials $S_n, P_n \in \mathbf{Z}[X_0, \dots, X_n; Y_0, \dots, Y_n]$. In fact, we can take S_n and P_n to be the universal nth Witt addition and multiplication polynomials in the theory of Witt vectors. The validity of such universal formulas is proved by the same arguments as in the proof of uniqueness of such Witt polynomials. ■

Let us give two important applications of Proposition 4.2.3. First of all, for a p-adic field K with (perfect) residue field k we recover the theory of its maximal unramified subextension. Indeed, since $\mathscr{O}_K$ endowed with the filtration by powers $\{\mathfrak{m}^i\}_{i\geq 1}$ of its maximal ideal $\mathfrak{m}$ is a p-ring, there is a unique map of rings $W(k) \to \mathscr{O}_K$ lifting the identification $W(k)/(p) = k = \mathscr{O}_K/\mathfrak{m}$. Since p has nonzero image in the maximal ideal $\mathfrak{m}$ of the domain $\mathscr{O}_K$, this map $W(k) \to \mathscr{O}_K$ is local and injective. Moreover, $\mathscr{O}_K/(p)$ is thereby a vector space over $W(k)/(p) = k$ with basis $\{1, \pi, \dots, \pi^{e-1}\}$ for a uniformizer π and $e = \mathrm{ord}_K(p)$, so by successive approximation and p-adic completeness and separatedness of $\mathscr{O}_K$ it follows that $\{\pi^i\}_{0\leq i<e}$ is a $W(k)$-basis of $\mathscr{O}_K$. In particular, $\mathscr{O}_K$ is a finite free module over $W(k)$ of rank e, so likewise $K = \mathscr{O}_K[1/p]$ is a finite extension of $K_0 = W(k)[1/p]$ of degree e, and it must be totally ramified as such since the residue fields coincide. We call K_0 the *maximal unramified subfield* of K, and for finite k this coincides with the classical notion that goes by the same name.

Remark 4.2.4. Let $\overline{k}$ denote the algebraic closure of k given by the residue field of $\mathscr{O}_{\overline{K}}$. Although $\mathscr{O}_{\overline{K}}$ is not p-adically complete – so we cannot generally embed $W(\overline{k})$ into $\mathscr{O}_{\overline{K}}$ – the (non-noetherian) valuation ring $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete and there is a canonical local embedding $W(\overline{k}) \to \mathscr{O}_{\mathbf{C}_K}$. However, this is not directly constructed by the general formalism of p-rings since no quotient of $\mathscr{O}_{\mathbf{C}_K}$ modulo a proper ideal containing p is a perfect $\mathbf{F}_p$-algebra. Rather, since $K_0 \subseteq K$ with $[K : K_0] < \infty$, we have $\mathbf{C}_K = \mathbf{C}_{K_0}$ and $W(\overline{k})$ is the valuation ring of the completion $\widehat{K_0^{\mathrm{un}}}$ of the maximal unramified extension of K_0 (with residue field $\overline{k}$). In particular, $\mathscr{O}_{\overline{K}}/(p) = \mathscr{O}_{\mathbf{C}_K}/(p)$ is not only an algebra over $W(k)/(p) = k$ in a canonical manner, but also over $W(\overline{k})/(p) = \overline{k}$ (as can also be proved by other methods, such as Hensel's lemma).

For a second application of Proposition 4.2.3, we require some preparations. If A is any $\mathbf{F}_p$-algebra whatsoever (e.g., $A = \mathscr{O}_{\overline{K}}/(p)$) then we can construct a canonically associated perfect $\mathbf{F}_p$-algebra $R(A)$ as follows:

$$R(A) = \varprojlim_{x \mapsto x^p} A = \{(x_0, x_1, \dots) \in \prod_{n \geq 0} A \mid x_{i+1}^p = x_i \text{ for all } i\}$$

with the product ring structure. This is perfect because the additive pth power map on $R(A)$ is clearly surjective and it is injective since if $(x_i) \in R(A)$ satisfies $(x_i)^p = (0)$ then $x_{i-1} = x_i^p = 0$ for all $i \geq 1$, so $(x_i) = 0$. In terms of universal properties, observe that the map $R(A) \to A$ defined by $(x_i) \mapsto x_0$ is a map to A from a perfect $\mathbf{F}_p$-algebra, and it is easy to check that this is final among all maps to A from perfect $\mathbf{F}_p$-algebras. For example, if A is a perfect $\mathbf{F}_p$-algebra then the canonical map $R(A) \to A$ is an isomorphism (as is also clear by inspection in such cases). The functoriality of $R(A)$ in A is exhibited in the evident manner in terms of compatible p-power sequences.

We will be particularly interested in the perfect $\mathbf{F}_p$-algebra

$$R = R_K = R(\mathscr{O}_{\overline{K}}/(p)) = R(\mathscr{O}_{\mathbf{C}_K}/(p))$$

endowed with its natural G_K-action via functoriality. Since $\mathscr{O}_{\overline{K}}/(p)$ is canonically an algebra over the perfect field $\overline{k}$, likewise by functoriality we have a ring map

$$\overline{k} = R(\overline{k}) \to R(\mathscr{O}_{\overline{K}}/(p)) = R \tag{4.2.1}$$

described concretely by $c \mapsto (j(c), j(c^{1/p}), j(c^{1/p^2}), \dots)$ where $j : \overline{k} \to \mathscr{O}_{\overline{K}}/(p)$ is the canonical (even unique) k-algebra section to the reduction map $\mathscr{O}_{\overline{K}}/(p) \twoheadrightarrow \overline{k}$. Although $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete, $\mathscr{O}_{\mathbf{C}_K}/(p)$ is not perfect. If we ignore this for a moment, then the canonical G_K-equivariant map $R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $(x_n) \mapsto x_0$ would uniquely lift to a ring map

$$\theta : W(R) \to \mathscr{O}_{\mathbf{C}_K}$$

due to the universal property of $W(R)$ in Proposition 4.2.3. It will later be shown how to actually construct a canonical such G_K-equivariant surjection θ despite the fact that we actually cannot apply Proposition 4.2.3 in this way (due to $\mathscr{O}_{\mathbf{C}_K}/(p)$ not being perfect). The induced G_K-equivariant surjection $W(R)[1/p] \to \mathbf{C}_K$ via θ then solves our original motivating problem of expressing $\mathbf{C}_K$ as a G_K-equivariant quotient of a "one-dimensional" ring, and further work will enable us to replace $W(R)[1/p]$ with a canonical complete discrete valuation ring.

To proceed further (e.g., to prove that R is a valuation ring with algebraically closed fraction field and to actually construct θ as above), it is necessary to investigate the properties of the ring R. This is taken up in the next section.

4.3. **Properties of R.** Although $R = R(\mathscr{O}_{\mathbf{C}_K}/(p))$ for a p-adic field K is defined ring-theoretically in characteristic p as a ring of p-power compatible sequences, it is important that such sequences can be uniquely lifted to p-power compatible sequences in $\mathscr{O}_{\mathbf{C}_K}$ (but possibly not in $\mathscr{O}_{\overline{K}}$). This lifting process behaves well with respect to multiplication in R, but it expresses the additive structure of R in a slightly complicated manner. To explain how this lifting works, it is convenient to work more generally with any p-adically separated and complete ring (e.g., $\mathscr{O}_{\mathbf{C}_K}$ but not $\mathscr{O}_{\overline{K}}$).

Proposition 4.3.1. *Let $\mathscr{O}$ be a p-adically separated and complete ring. The multiplicative map of sets*

$$\varprojlim_{x \mapsto x^p} \mathscr{O} \to R(\mathscr{O}/p\mathscr{O}) \tag{4.3.1}$$

defined by $(x^{(n)})_{n \geq 0} \mapsto (x^{(n)} \bmod p)$ is bijective. Also, for any $x = (x_n) \in R(\mathscr{O}/p\mathscr{O})$ and arbitrary lifts $\widehat{x_r} \in \mathscr{O}$ of $x_r \in \mathscr{O}/p\mathscr{O}$ for all $r \geq 0$, the limit $\ell_n(x) = \lim_{m \to \infty} \widehat{x_{n+m}}^{p^m}$ exists in $\mathscr{O}$ for all $n \geq 0$ and is independent of the choice of lifts $\widehat{x_r}$. Moreover, the inverse to (4.3.1) *is given by $x \mapsto (\ell_n(x))$.*

In particular, $R(\mathscr{O}/p\mathscr{O})$ is a domain if $\mathscr{O}$ is a domain.

Proof. The given map of sets $\varprojlim \mathscr{O} \to R(\mathscr{O}/p\mathscr{O})$ makes sense and is multiplicative, and to make sense of the proposed inverse map we observe that for each $n \geq 0$ and $m' \geq m \geq 0$ we have

$$\widehat{x_{n+m'}}^{p^{m'-m}} \equiv \widehat{x_{n+m}} \bmod p\mathscr{O},$$

so $\widehat{x_{n+m'}}^{p^{m'}} \equiv \widehat{x_{n+m}}^{p^m} \bmod p^{m+1}\mathscr{O}$. Hence, the limit $\ell_n(x)$ makes sense for each $n \geq 0$, and the same argument as in the proof of Lemma 4.2.2 shows that $\ell_n(x)$ is independent of the choice of liftings $\widehat{x_r}$. The proposed inverse map $x \mapsto (\ell_n(x))$ is therefore well-defined, and in view of it being independent of the liftings we see that it is indeed an inverse map. ∎

In what follows, for any $x \in R(\mathscr{O}/p\mathscr{O})$ as in Proposition 4.3.1 we write $x^{(n)} \in \mathscr{O}$ to denote the limit $\ell_n(x) = \lim_{m \to \infty} \widehat{x_{n+m}}^{p^m}$ for all $n \geq 0$.

Remark 4.3.2. The bijection in Proposition 4.3.1 allows us to transfer the natural $\mathbf{F}_p$-algebra structure on $R(\mathscr{O}/p\mathscr{O})$ over to such a structure on the inverse limit set $\varprojlim \mathscr{O}$ of p-power compatible sequences $x = (x^{(n)})_{n \geq 0}$ in $\mathscr{O}$. The multiplicative structure is easy to translate through this bijection: $(xy)^{(n)} = x^{(n)}y^{(n)}$. For addition, the proof of the proposition gives

$$(x+y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

Also, if p is odd then $(-1)^p = -1$ in $\mathscr{O}$, so $(-x^{(n)})$ is a p-power compatible sequence for any x. Hence, from the description of the additive structure we see that $(-x)^{(n)} = -x^{(n)}$ for all $n \geq 0$ and all x when $p \neq 2$. This argument fails to work if $p = 2$, but then $(-x)^{(n)} = x^{(n)}$ for all $n \geq 0$ since $-x = x$ in such cases (as $R(\mathscr{O}/2\mathscr{O})$ is an $\mathbf{F}_2$-algebra if $p = 2$).

We now fix a p-adic field K and let R denote the domain $R(\mathscr{O}_{\overline{K}}/(p)) = R(\mathscr{O}_{\mathbf{C}_K}/(p))$ of characteristic p. An element $x \in R$ will be denoted $(x_n)_{n \geq 0}$ when we wish to view its p-power compatible components as elements of $\mathscr{O}_{\mathbf{C}_K}/(p)$ and we use the notation $(x^{(n)})_{n \geq 0}$ to denote its unique representation using a p-power compatible sequence of elements $x^{(n)} \in \mathscr{O}_{\mathbf{C}_K}$. An element $x \in R$ is a unit if and only if the component $x_0 \in \mathscr{O}_{\overline{K}}/(p)$ is a unit, so R is a local ring. Also, since every element of $\mathscr{O}_{\overline{K}}$ is a square, it is easy to check (e.g., via Proposition 4.3.1) that the nonzero maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m} = \mathfrak{m}^2$. In particular, R is not noetherian. The ring R has several non-obvious properties which are used throughout the development of p-adic Hodge theory, and the remainder of this section is devoted to stating and proving these properties (not all of which will be used in these notes).

Lemma 4.3.3. *Let $|\cdot|_p : \mathbf{C}_K \twoheadrightarrow p^{\mathbf{Q}} \cup \{0\}$ be the normalized absolute value satisfying $|p|_p = 1/p$. The map $|\cdot|_R : R \to p^{\mathbf{Q}} \cup \{0\}$ defined by $x = (x^{(n)}) \mapsto |x^{(0)}|_p$ is an absolute value on R that makes R the valuation ring for the unique valuation v_R on* $\mathrm{Frac}(R)$ *extending $-\log_p |\cdot|_R$ on R (and having value group $\mathbf{Q}$).*

Also, R is v_R-adically separated and complete, and the subfield $\overline{k}$ of R maps isomorphically onto the residue field of R.

Proof. Obviously $x^{(0)} = 0$ if and only if $x = 0$, and $|xy|_R = |x|_R |y|_R$ since $(xy)^{(0)} = x^{(0)} y^{(0)}$.

To show that $|x + y|_R \leq \max(|x|_R, |y|_R)$ for all $x, y \in R$, we may assume $x, y \neq 0$, so $x^{(0)}, y^{(0)} \neq 0$. By symmetry we may assume $|x^{(0)}|_p \leq |y^{(0)}|_p$, so for all $n \geq 0$ we have

$$|x^{(n)}|_p = |x^{(0)}|_p^{p^{-n}} \leq |y^{(0)}|_p^{p^{-n}} = |y^{(n)}|_p.$$

The ratios $x^{(n)}/y^{(n)}$ therefore lie in $\mathscr{O}_{\mathbf{C}_K}$ for $n \geq 0$ and form a p-power compatible sequence. This sequence is therefore an element $z \in R$ and clearly $yz = x$ in R, so $y|x$ in R. Hence,

$$|x + y|_R = |y(z + 1)|_R = |y|_R |z + 1|_R \leq |y|_R \leq \max(|x|_R, |y|_R).$$

The same argument shows that R is the valuation ring of v_R on $\mathrm{Frac}(R)$.

To prove $|\cdot|_R$-completeness of R, first note that if we let $v = -\log_p |\cdot|_p$ on $\mathbf{C}_K$ then $v_R(x) = v(x^{(0)}) = p^n v(x^{(n)})$ for $n \geq 0$. Thus, $v_R(x) \geq p^n$ if and only if $v(x^{(n)}) \geq 1$ if and only if $x^{(n)} \bmod p = 0$. Hence, if we let

$$\theta_n : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$$

denote the ring homomorphism $x = (x_m)_{m \geq 0} \mapsto x_n$ then $\{x \in R \mid v_R(x) \geq p^n\} = \ker \theta_n$. In view of how the inverse limit R sits within the product space $\prod_{m \geq 0} (\mathscr{O}_{\mathbf{C}_K}/(p))$, or more specifically since $x_n = 0$ implies $x_m = 0$ for all $m \leq n$, we conclude that the v_R-adic topology on R coincides with its subspace topology within $\prod_{m \geq 0} (\mathscr{O}_{\mathbf{C}_K}/(p))$ where the factors are given the discrete topology, so the v_R-adic completeness is clear (as R is closed in this product space due to the definition of $R = R(\mathscr{O}_{\mathbf{C}_K}/(p))$).

Finally, the definition of the k-embedding of $\overline{k}$ into R in (4.2.1) implies that $\theta_0 : R \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}/(p)$ is a $\overline{k}$-algebra map, but θ_0 is local and so induces an injection on residue fields. Since $\overline{k} \to \mathscr{O}_{\mathbf{C}_K}/(p)$ induces an isomorphism on residue fields, we are done. ∎

For $x = (x^{(n)})$ and $y = (y^{(n)})$ in R, we have $x^{(n)} \equiv y^{(n)} \bmod p$ if and only if $x^{(i)} \equiv y^{(i)} \bmod p^{n-i+1}$ for all $0 \leq i \leq n$, so the v_R-adic topology on R also coincides with its closed subspace topology from sitting as a multiplicative inverse limit within $\prod_{n \geq 0} \mathscr{O}_{\mathbf{C}_K}$ where each factor is given the p-adic topology. This gives an alternative way of seeing the v_R-adic completeness of R.

Example 4.3.4. An important example of an element of R is

$$\varepsilon = (\varepsilon^{(n)})_{n\geq 0} = (1, \zeta_p, \zeta_{p^2}, \dots)$$

with $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$ (so $\varepsilon^{(1)} = \zeta_p$ is a primitive pth root of unity and hence $\varepsilon^{(n)}$ is a primitive p^nth root of unity for all $n \geq 0$). Any two such elements are $\mathbf{Z}_p^\times$-powers of each other. For any such choice of element we claim that

$$v_R(\varepsilon - 1) = \frac{p}{p-1}.$$

To see this, by definition we have $v_R(\varepsilon - 1) = v((\varepsilon-1)^{(0)})$ where $v = \mathrm{ord}_p = -\log_p|\cdot|_p$, so we need to describe $(\varepsilon - 1)^{(0)} \in \mathscr{O}_{\mathbf{C}_K}$. By Remark 4.3.2, in $\mathscr{O}_{\mathbf{C}_K}$ we have

$$(\varepsilon - 1)^{(0)} = \lim_{n\to\infty} (\varepsilon^{(n)} + (-1)^{(n)})^{p^n},$$

with $\varepsilon^{(n)} = \zeta_{p^n}$ a primitive p^nth root of unity in $\overline{K}$ and $(-1)^{(n)} = -1$ if $p \neq 2$ whereas $(-1)^{(n)} = 1$ if $p = 2$. We shall separately treat the cases of odd p and $p = 2$.

If p is odd then

$$v_R(\varepsilon - 1) = \lim_{n\to\infty} p^n \,\mathrm{ord}_p(\zeta_{p^n} - 1) = \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1}.$$

If $p = 2$ then

$$v_R(\varepsilon - 1) = \lim_{n\to\infty} 2^n \,\mathrm{ord}_2(\zeta_{2^n} + 1) = \lim_{n\to\infty} 2^n \,\mathrm{ord}_2((\zeta_{2^n} - 1) + 2).$$

Since $\mathrm{ord}_2(\zeta_{2^n} - 1) = 1/2^{n-1} < \mathrm{ord}_2(2)$ for $n > 1$, we have $\mathrm{ord}_2((\zeta_{2^n} - 1) + 2) = \mathrm{ord}_2(\zeta_{2^n} - 1)$ for $n > 1$, so we may conclude as for odd p.

Theorem 4.3.5. *The field* $\mathrm{Frac}(R)$ *of characteristic p is algebraically closed.*

Proof. Since R is a perfect valuation ring of characteristic p, its fraction field is a perfect field of characteristic p. Hence, our problem is to prove that it is separably closed. Via the valuation we see that it suffices to prove that any monic polynomial $P \in R[X]$ that is separable over $\mathrm{Frac}(R)$ has a root in R when $\deg P > 0$. Since P and its derivative P' are relatively prime over $\mathrm{Frac}(R)$ by separability, clearing denominators gives $U, V \in R[X]$ such that

$$PU + P'V = r \in R - \{0\}.$$

The value group of v_R is $\mathbf{Q}$, so further scaling of U and V by a common nonzero element of R allows us to arrange that $v_R(r) \in \mathbf{Z}^+$. Let $m = v_R(r) \geq 1$. We will construct a Cauchy sequence $\{\rho_n\}$ in R such that $P(\rho_n) \to 0$, so the limit $\rho = \lim \rho_n \in R$ (which exists by the completeness in Lemma 4.3.3) is a root of P. The construction of $\{\rho_n\}$ rests on the following lemma:

Lemma 4.3.6. *With $m = v_R(r) \geq 1$ as defined above, if $n \geq 2m+1$ and $\xi \in R$ satisfies $v_R(P(\xi)) \geq n$ then there exists $y \in R$ such that $v_R(y) \geq n - m$ and $v_R(P(\xi + y)) \geq n + 1$.*

Granting this lemma for a moment, and assuming furthermore that there exists some $\rho_1 \in R$ satisfying $v_R(P(\rho_1)) \geq 2m+1$, we may take $\xi = \rho_1$ in the lemma (with $n = 2m+1$) to find $y_1 \in R$ such that $v_R(y_1) \geq m+1$ and $v_R(P(\rho_1 + y_1)) \geq 2m+2$. We then apply the lemma to $\xi = \rho_2 := \rho_1 + y_1$ and $n = 2m+2$, and so on, to construct sequences $\{\rho_i\}$ and $\{y_i\}$ in R such that $\rho_{i+1} = \rho_i + y_i$ for all $i \geq 1$, $v_R(y_i) \geq m+i$, and $v_R(P(\rho_i)) \geq 2m+i$. Since $v_R(\rho_{i+1} - \rho_i) = v_R(y_i) \geq m+i$, the sequence $\{\rho_i\}$ is Cauchy. Hence, the limit $\rho = \lim \rho_i$ exists in R and $v_R(P(\rho)) = +\infty$, which is to say $P(\rho) = 0$ as desired. It therefore suffices to find such a ρ_1 and to then prove Lemma 4.3.6.

To find ρ_1 as just used, for each $j \geq 1$ consider the ring map $\theta_j : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $x = (x_i) \mapsto x_j$. Since

$$\ker \theta_j = \{x \in R \,|\, v_R(x) \geq p^j\},$$

we seek $\rho_1 \in R$ such that $\theta_j(P(\rho_1)) = 0$ for a fixed j large enough so that $p^j \geq 2m+1$. More generally, for any $n \geq 1$ we shall construct $t \in R$ such that $\theta_n(P(t)) = 0$. Consider the induced map $\theta_n : R[X] \to (\mathscr{O}_{\mathbf{C}_K}/(p))[X]$ given by θ_n on coefficients. This carries P to a monic polynomial Q with positive degree, so upon lifting Q to a monic polynomial $\widetilde{Q}$ in $\mathscr{O}_{\mathbf{C}_K}[X]$ we may choose a root $z \in \mathscr{O}_{\mathbf{C}_K}$ of $\widetilde{Q}$ (as $\mathbf{C}_K$ is algebraically closed and $\widetilde{Q}$ is a monic polynomial with positive degree over its valuation ring). The map $\theta_n : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ is surjective (again using that $\mathbf{C}_K$ is algebraically closed), so we can pick $t \in \theta_n^{-1}(z \bmod p)$. Clearly $\theta_n(P(t)) = (\theta_n(P))(\theta_n(t)) = Q(z \bmod p) = \widetilde{Q}(z) \bmod p = 0$, as desired.

Now we turn to the task of proving Lemma 4.3.6. Recall that $v_R(r) = m \geq 1$. To find the required $y \in R$, consider the algebraic expansion

$$P(X+Y) = P(X) + YP'(X) + \sum_{j \geq 2} Y^j P_j(X)$$

for suitable $P_j \in R[X]$. For any $y \in R$ with $v_R(y) \geq n - m$ we have

$$v_R(P(\xi + y)) \geq \min(v_R(P(\xi) + P'(\xi)y_n), v_R(y^j P_j(\xi))_{j \geq 2}),$$

and for any $j \geq 2$ clearly $v_R(y^j P_j(\xi)) \geq j v_R(y) \geq 2 v_R(y) \geq 2(n-m) \geq n+1$. Hence, we can ignore the contribution from y-degrees beyond the first in our search for y and we just need to find $y \in R$ such that

$$v_R(P(\xi) + yP'(\xi)) \geq n+1.$$

The idea is to take $y = -P(\xi)/P'(\xi)$, except that we do not know this lies in R (let alone that $v_R(y) \geq n-m$) nor that the denominator $P'(\xi)$ is nonzero. It is enough to check that $v_R(P'(\xi)) \leq m$ (so $P'(\xi) \neq 0$), for then $y = P(\xi)/P'(\xi)$ makes sense in $\mathrm{Frac}(R)$ and satisfies

$$v_R(y) = v_R(P(\xi)) - v_R(P'(\xi)) \geq n - m$$

as required.

To prove the upper bound $v_R(P'(\xi)) \leq m$, we evaluate the identity $UP + VP' = r$ in $R[X]$ (with $v_R(r) = m$) at $X = \xi$ to get

$$U(\xi)P(\xi) + V(\xi)P'(\xi) = r$$

in R. But $v_R(U(\xi)P(\xi)) \geq v_R(P(\xi)) \geq n > m = v_R(r)$, so $v_R(V(\xi)P'(\xi)) = v_R(r) = m$. Hence, $V(\xi)$ and $P'(\xi)$ are nonzero, and moreover $v_R(P'(\xi)) \leq m$ as desired. ■

Consider an element $\varepsilon \in R$ as in Example 4.3.4 (so $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$). Thus, $\theta_0(\varepsilon) = 1 \in \mathscr{O}_{\mathbf{C}_K}/(p)$, so the image of ε in the residue field $\overline{k}$ of R is 1. Hence, $\varepsilon - 1$ lies in the maximal ideal $\mathfrak{m}_R$ of R, which we knew anyway from Example 4.3.4 since there we proved $v_R(\varepsilon - 1) = p/(p-1) > 0$. By the completeness of R, we get a unique local k-algebra map $k[\![u]\!] \to R$ satisfying $u \mapsto \varepsilon - 1 \neq 0$. This map depends on the choice of ε, but its image does not:

Lemma 4.3.7. *The image of $k[\![u]\!]$ in R is independent of ε.*

Proof. Consider a second choice ε', so $\varepsilon' = \varepsilon^a$ for some $a \in \mathbf{Z}_p^\times$. (Note that ε lies in the multiplicative group $1 + \mathfrak{m}_R$ that is p-adically separated and complete, so $\mathbf{Z}_p$-exponentiation on here makes sense.) Letting $x = \varepsilon - 1$ and $x' = \varepsilon' - 1$ in $\mathfrak{m}_R$, we can compute formally

$$x' = \varepsilon^a - 1 = (1+x)^a - 1 = ax + \dots$$

in R. Rigorously, the unique local k-algebra self-map of $k[\![u]\!]$ satisfying $u \mapsto (1+u)^a - 1$ carries the map $k[\![u]\!] \to R$ resting on ε to the one resting on ε'. But this self-map is an automorphism since $(1+u)^a - 1 = au + \dots$ with $a \in \mathbf{Z}_p^\times$. ■

In view of the lemma, we may define the canonical subfield $E \subseteq \mathrm{Frac}(R)$ to be the fraction field of the canonical image of $k[\![u]\!]$ in R for any choice of ε as in Lemma 4.3.7. By Theorem 4.3.5, the separable closure E_s of E within $\mathrm{Frac}(R)$ is a separable closure of E. The action of the Galois group G_K on R extends uniquely to an action on $\mathrm{Frac}(R)$, and this does not fix the image $\varepsilon - 1$ of u. However, for the extension $K_\infty = K(\mu_{p^\infty})$ generated by the components $\varepsilon^{(n)}$ of ε (for all choices of ε) we see that the subgroup $G_{K_\infty} \subseteq G_K$ is the isotropy group of $\varepsilon - 1 \in R$ and so is the isotropy group of the intrinsic subfield $E \subseteq \mathrm{Frac}(R)$. Hence, G_{K_∞} preserves the separable closure $E_s \subseteq \mathrm{Frac}(R)$, so we get a group homomorphism

$$G_{K_\infty} \to \mathrm{Aut}(E_s/E) = G_E.$$

Lemma 4.3.8. *The map of Galois groups $G_{K_\infty} \to G_E$ is continuous.*

Proof. Fix a finite Galois extension E' of E inside of $E_s \subseteq \mathrm{Frac}(R)$. We may choose a primitive element $x \in E'^\times$ for E' over E. By replacing x with $1/x$ if necessary, we can arrange that $x \in R$. The algebraicity of x over E implies that the G_{K_∞}-orbit of x is finite, say $\{x = x_1, \dots, x_n\}$, with all $x_i \in R$. To find an open subgroup of G_{K_∞} that has trivial image in $\mathrm{Gal}(E'/E)$, or equivalently lands in $G_{E'} \subseteq G_E$, we just need to show that if $g \in G_{K_\infty}$ is sufficiently close to 1 then $g(x)$ is distinct from the finitely many elements $x_2, \dots, x_n$ that are distinct from x (forcing $g(x) = x$). The existence of such a neighborhood of the identity is immediate from the continuity of the action of G_K on the Hausdorff space R. ■

A much deeper fact (not used in these notes) that is best understood as part of the theory of norm fields of Fontaine and Wintenberger is that the continuous map in Lemma 4.3.8 is in fact bijective and so is a topological isomorphism. (The theory of norm fields even gives a functorial equivalence between the categories of finite separable extensions of K_∞ and of E.) This is the concrete realization of a special case of the abstract isomorphism in (1.3.1).

4.4. **The field of p-adic periods B_{dR}.** We have now assembled enough work to carry out the first important refinement on the graded ring B_{HT}, namely the construction of the field of p-adic periods B_{dR} as promised in the discussion following Example 4.1.2. Inspired by the universal property of Witt vectors in Proposition 4.2.3 and the perfectness of the $\mathbf{F}_p$-algebra R, we seek to lift the G_K-equivariant surjective ring map $\theta_0 : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ defined by $(x_i) \mapsto x_0$ to a G_K-equivariant surjective ring map $\theta : W(R) \to \mathscr{O}_{\mathbf{C}_K}$. As we have already observed, although $\mathscr{O}_{\mathbf{C}_K}$ is p-adically separated and complete, we cannot use Proposition 4.2.3 because $\mathscr{O}_{\mathbf{C}_K}/(p)$ is not perfect. Nonetheless, we will construct such a θ in a canonical (in particular, G_K-equivariant) manner.

In the end, the formula for θ will be very simple and explicit:

$$\theta(\sum [c_n]p^n) = \sum c_n^{(0)} p^n.$$

(Recall that $W(R)$ is a strict p-ring with $W(R)/(p) = R$, so each of its elements has the unique form $\sum [c_n]p^n$ with $c_n \in R$.) This is very much in the spirit of the proof of Proposition 4.2.3 since $c^{(0)} = \lim_{m\to\infty} \widehat{c_m}^{p^m}$ for any $c \in R$ using any choice of lift $\widehat{c_m} \in \mathscr{O}_{\mathbf{C}_K}$ of $c_m \in \mathscr{O}_{\mathbf{C}_K}/(p)$ (with $\{c_m\}$ a compatible sequence of p-power roots of $c_0 \in \mathscr{O}_{\mathbf{C}_K}/(p)$). In terms of the Witt coordinatization $(r_0, r_1, \dots) = \sum p^n [r_n^{p^{-n}}]$ we expect to have $\theta : (r_0, r_1, \dots) \mapsto \sum (r_n^{p^{-n}})^{(0)} p^n$, but for any $r \in R$ we have $(r^{p^{-n}})^{(0)} = ((r^{p^{-n}})^{(n)})^{p^n} = r^{(n)}$ in $\mathscr{O}_{\mathbf{C}_K}$ since $r \mapsto r^{(n)}$ is multiplicative. Hence, we expect to have the formula $\theta : (r_0, r_1, \dots) \mapsto \sum r_n^{(n)} p^n$. It is a pain to prove by hand that this explicit formula defines a ring map, so we will proceed in a more indirect manner.

Since $x_{n+1}^p = x_n$ in $\mathscr{O}_{\mathbf{C}_K}/(p)$ for $x = (x_i) \in R$, the projection maps $\theta_n : R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ given by $x \mapsto x_n$ satisfy $\mathrm{Frob} \circ \theta_{n+1} = \theta_n$ with $\mathrm{Frob} : \mathscr{O}_{\mathbf{C}_K}/(p) \to \mathscr{O}_{\mathbf{C}_K}/(p)$ denoting the p-power map. Applying the Witt functor on arbitrary commutative rings to this compatibility gives a commutative diagram

$$\begin{tikzcd} W(R) \arrow[d, "W(\theta_{n+1})"'] \arrow[dr, "W(\theta_n)"] & \\ W(\mathscr{O}_{\mathbf{C}_K}/(p)) \arrow[r, "\varphi"'] & W(\mathscr{O}_{\mathbf{C}_K}/(p)) \end{tikzcd}$$

where the bottom side is the usual Frobenius endomorphism $\varphi = W(\mathrm{Frob})$ of the Witt vectors of any $\mathbf{F}_p$-algebra. Let $\Theta_n : W(R) \to W_n(\mathscr{O}_{\mathbf{C}_K}/(p))$ denote the composition of $W(\theta_n)$ with the projection $W \to W_n$ to length-n truncated Witt vectors (on $\mathscr{O}_{\mathbf{C}_K}/(p)$-valued points), so $\Theta_n = f_n \circ \Theta_{n+1}$ where $f_n : W_{n+1}(\mathscr{O}_{\mathbf{C}_K}/(p)) \to W_n(\mathscr{O}_{\mathbf{C}_K}/(p))$ is the map $(a_0, \dots, a_n) \mapsto (a_0^p, \dots, a_{n-1}^p)$. Thus, we get a canonical map of rings

$$\alpha : W(R) \to \varprojlim_{f_n} W_n(\mathscr{O}_{\mathbf{C}_K}/(p)).$$

We claim that α is bijective. For $x = (x^{(m)})_{m\geq 0} \in R$ with $\{x^{(m)}\}$ a p-power compatible sequence in $\mathscr{O}_{\mathbf{C}_K}$, the map $\Theta_n : W(R) \to W_n(\mathscr{O}_{\mathbf{C}_K}/(p))$ carries the Teichmüller digit

$$[x] = (x, 0, 0, \dots)$$

to $[x^{(n)} \bmod p]$. Likewise, Θ_n carries a general Witt vector $w = (w_0, w_1, \dots) \in W(R)$ (with $w_n = (w_n^{(m)})_{m\geq 0} \in R$) to

$$(w_0^{(n)} \bmod p, \dots, w_{n-1}^{(n)} \bmod p).$$

Hence, from the definition of α it is a bijective map. Since the valuation v_R on R satisfies $v_R(x) = \mathrm{ord}_p(x^{(0)}) = p^n \,\mathrm{ord}_p(x^{(n)})$, it is easy to check that α is a topological isomorphism when we give each $\mathscr{O}_{\mathbf{C}_K}/(p)$ its discrete topology, R its v_R-adic topology, and $W(A) = \prod_{n\geq 0} A$ the product topology for any topological ring A.

Now comes the key point. We want to construct a (continuous G_K-equivariant) ring homomorphism $\theta : W(R) \to \mathscr{O}_{\mathbf{C}_K}$ as if $\mathscr{O}_{\mathbf{C}_K}$ were a strict p-ring (which it isn't), so in particular we hope to have the formula $\theta : (r_n) \mapsto \sum p^n r_n^{(n)}$. But we have a topological and G_K-equivariant identification $\mathscr{O}_{\mathbf{C}_K} = \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p^n)$ and we just constructed a topological ring isomorphism $\alpha : W(R) \simeq \varprojlim_{f_n} W_n(\mathscr{O}_{\mathbf{C}_K}/(p))$ that is also visibly G_K-equivariant. Hence, it suffices to construct compatible G_K-equivariant maps

$$\psi_n : W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \to \mathscr{O}_{\mathbf{C}_K}/(p^n)$$

inducing the desired θ in the inverse limit as $n \to \infty$. Such maps ψ_n can be constructed as follows.

From the theory of Witt vectors, the set-theoretic map $\mathbf{w}_n : W_{n+1}(A) \to A$ given by

$$(a_0, \dots, a_n) \mapsto a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n$$

is a ring map for *any* ring A. Taking $A = \mathscr{O}_{\mathbf{C}_K}$, we can use this to define a canonical map $W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \to \mathscr{O}_{\mathbf{C}_K}/(p^n)$:

Lemma 4.4.1. *The map $\psi_n : W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \to \mathscr{O}_{\mathbf{C}_K}/(p^n)$ defined by*

$$(c_0, \dots, c_{n-1}) \mapsto \mathbf{w}_n(\widehat{c}_0, \dots, \widehat{c}_{n-1}, 0) = \sum_{j=0}^{n-1} p^j \widehat{c}_j^{\,p^{n-j}} \bmod p^n$$

for arbitrary lifts $\widehat{c}_j \in \mathscr{O}_{\mathbf{C}_K}$ of $c_j \in \mathscr{O}_{\mathbf{C}_K}/(p)$ is well-defined and a map of rings; it uniquely fits into the commutative diagram

$$\begin{array}{ccc} W_{n+1}(\mathscr{O}_{\mathbf{C}_K}) & \xrightarrow{\mathbf{w}_n} & \mathscr{O}_{\mathbf{C}_K} \\ \downarrow & & \downarrow \\ W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) & \xrightarrow[\psi_n]{} & \mathscr{O}_{\mathbf{C}_K}/(p^n) \end{array}$$

with surjective vertical maps, using componentwise reduction in the left side.

Proof. It is clear by hand that ψ_n is a well-defined map of sets and that it fits into the given commutative diagram (in view of how $\mathbf{w}_n$ is defined), whence ψ_n must be a ring map since the left side is surjective. Alternatively, one can prove directly that the kernel of the surjective left side is killed by the composite along the top and right sides since the kernel consists of Witt vectors $(pb_0, \dots, pb_{n-1}, b_n)$. ■

By inspection, the visibly G_K-equivariant diagram

$$\begin{tikzcd} W_{n+1}(\mathscr{O}_{\mathbf{C}_K}/(p)) \arrow[r, "\psi_{n+1}"] \arrow[d, "f_n"'] & \mathscr{O}_{\mathbf{C}_K}/(p^{n+1}) \arrow[d] \\ W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \arrow[r, "\psi_n"'] & \mathscr{O}_{\mathbf{C}_K}/(p^n) \end{tikzcd}$$

(using reduction along the right side) commutes, whence passing to the inverse limit in n defines a continuous G_K-equivariant ring homomorphism

$$\theta : W(R) \simeq \varprojlim_{f_n} W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \to \varprojlim \mathscr{O}_{\mathbf{C}_K}/(p^n) = \mathscr{O}_{\mathbf{C}_K}$$

(where continuity is relative to the p-adic topology on $\mathscr{O}_{\mathbf{C}_K}$ and the product topology on $W(R)$ relative to the v_R-adic topology on R). To unravel this, we compute on Teichmüller lifts: for $r = (r^{(n)})_{n\geq 0} \in R$ and $[r] = (r, 0, 0, \dots) \in W(R)$,

$$\begin{aligned} \theta([r]) = \varprojlim \psi_n(\Theta_n([r])) = \varprojlim \psi_n([r^{(n)} \bmod p]) &= \varprojlim((r^{(n)})^{p^n} \bmod p^n) \\ &= \varprojlim r^{(0)} \bmod p^n \\ &= r^{(0)}. \end{aligned}$$

Hence, on a general Witt vector $(r_0, r_1, \dots) = \sum p^n [r_n^{p^{-n}}]$,

$$\theta((r_n)) = \sum p^n \theta([r_n^{p^{-n}}]) = \sum p^n (r_n^{p^{-n}})^{(0)} = \sum p^n r_n^{(n)},$$

as desired. This explicit formula makes it evident that θ is surjective (since $R \to \mathscr{O}_{\mathbf{C}_K}/(p)$ via $r \mapsto r^{(n)}$ is surjective for each $n \geq 0$). In concrete terms, the formula shows that θ fits into the following family of commutative diagrams:

$$\begin{tikzcd} W(R) \arrow[rr, "\theta"] \arrow[d] & & \mathscr{O}_{\mathbf{C}_K} \arrow[dr] & \\ W_n(R) \arrow[r, "\Theta_n"'] & W_n(\mathscr{O}_{\mathbf{C}_K}/(p)) \arrow[rr, "\psi_n"'] & & \mathscr{O}_{\mathbf{C}_K}/(p^n) \end{tikzcd}$$

Proposition 4.4.2. *The continuous surjective G_K-equivariant map $\theta : W(R) \to \mathscr{O}_{\mathbf{C}_K}$ is open. Also, using the canonical k-algebra map $j : \overline{k} \to R$ to make $W(R)$ into a $W(\overline{k})$-algebra via $W(j)$, θ is a $W(\overline{k})$-algebra map via the natural $W(\overline{k})$-algebra structure on $\mathscr{O}_{\mathbf{C}_K}$.*

Proof. To prove openness, using the product of the valuation topology from R on $W(R)$ and the p-adic topology on $\mathscr{O}_{\mathbf{C}_K}$, we just have to show that if J is an open ideal in R then the image under θ of the additive subgroup of vectors (r_i) with $r_0, \dots, r_n \in J$ (for fixed n) is open in $\mathscr{O}_{\mathbf{C}_K}$. This image is $J^{(0)} + pJ^{(1)} + \cdots + p^{n-1}J^{(n-1)}$, where $J^{(m)}$ is the image of J under the map of sets $R \to \mathscr{O}_{\mathbf{C}_K}$ defined by $r \mapsto r^{(m)}$. Since $\mathscr{O}_{\mathbf{C}_K}$ has the p-adic topology, it suffices to show that $J^{(m)}$ is open in $\mathscr{O}_{\mathbf{C}_K}$ for each $m \geq 0$. But $J^{(m)} = (J^{p^m})^{(0)}$, so to prove that θ is open we just have to show that if J is an open ideal in R then $J^{(0)}$ is open in $\mathscr{O}_{\mathbf{C}_K}$. It is enough to work with J's running through a base of open ideals, so we take $J = \{r \in R \,|\, v_R(r) \geq c\}$ with $c \in \mathbf{Q}$. Since $v_R(r) = v(r^{(0)})$ and the map $r \mapsto r^{(0)}$ is a surjection from R onto $\mathscr{O}_{\mathbf{C}_K}$, clearly for such J we have that $J^{(0)} = \{t \in \mathscr{O}_{\mathbf{C}_K} \,|\, v(t) \geq c\}$, which is certainly open in $\mathscr{O}_{\mathbf{C}_K}$. This concludes the proof that θ is an open map.

Next, consider the claim that θ is a map of $W(\overline{k})$-algebras. Recall that $\mathscr{O}_{\mathbf{C}_K}$ is made into a $W(\overline{k})$-algebra via the unique continuous $W(k)$-algebra map $h : W(\overline{k}) \to \mathscr{O}_{\mathbf{C}_K}$ lifting the identity map on $\overline{k}$ at the level of residue fields. (By such continuity and the p-adic separatedness and completeness of $\mathscr{O}_{\mathbf{C}_K}$, the existence and uniqueness of such an h is reduced to the case when $\overline{k}$ is replaced with a finite extension k'/k, and the unique $W(k)$-algebra map $W(k') \to \mathscr{O}_{\mathbf{C}_K}$ lifting the inclusion $k' \to \overline{k}$ is built as follows: by $W(k)$-finiteness it must land in the valuation ring of a *finite* extension of K if it exists, so we can pass to the case when the target is a complete discrete valuation ring, whence the universal property of $W(k')$ can be used. Concretely, $W(k')$ is just a finite unramified extension of $W(k)$ within $\overline{K}$, the point being that the map on residue fields uniquely determines the map in characteristic 0.) Using p-adic continuity, it is enough to chase Teichmüller digits.

Our problem is now to show that for each $c \in \overline{k}$ the image $h([c])$ is equal to $\theta([j(c)])$, where $j : \overline{k} \to R$ is the canonical k-algebra map defined by $c \mapsto (c^{1/p^m})_{m \geq 0} \in R(\mathscr{O}_{\overline{K}}/(p)) = R$ and we view $\mathscr{O}_{\overline{K}}/(p)$ as a $\overline{k}$-algebra over its k-algebra structure via Hensel's Lemma. The key point is that c viewed in $\mathscr{O}_{\overline{K}}/(p) = \mathscr{O}_{\mathbf{C}_K}/(p)$ is just $h([c]) \bmod p$ (check!), so $j(c) = (h([c^{1/p^m}]) \bmod p) \in R$. Since the sequence of elements $h([c^{1/p^m}])$ in $\mathscr{O}_{\mathbf{C}_K}$ is p-power compatible, $j(c)^{(0)} = h([c])$. Thus, $\theta([j(c)]) = j(c)^{(0)} = h([c])$. ■

We now have a G_K-equivariant surjective ring homomorphism

$$\theta_K : W(R)[1/p] \twoheadrightarrow \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K,$$

but the source ring is not a complete discrete valuation ring. We shall replace $W(R)[1/p]$ with its $\ker\theta_K$-adic completion, and the reason this works is that $\ker\theta_K = (\ker\theta)[1/p]$ turns out to be a principal ideal. We now record some facts about $\ker\theta$.

Proposition 4.4.3. *Choose $\pi \in R$ such that $\pi^{(0)} = p$ (i.e., $\pi = (p, p^{1/p}, p^{1/p^2}, \dots) \in \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbf{C}_K} = R$, so $v_R(\pi) = 1$) and let $\xi = \xi_\pi = [\pi] - p = (\pi, -1, \dots) \in W(R)$.*

(1) *The ideal $\ker\theta \subseteq W(R)$ is the principal ideal generated by ξ.*
(2) *An element $w = (r_0, r_1, \dots) \in \ker\theta$ is a generator of $\ker\theta$ if and only if $r_1 \in R^\times$.*

A defect of ξ, despite its explicitness, is that G_K does not act on ξ in a nice way (but it does preserve $\xi \cdot W(R) = \ker\theta$). This will be remedied after replacing $W(R)[1/p]$ with its $\ker\theta_K$-adic completion.

Proof. Clearly $\theta(\xi) = \theta([\pi]) - p = \pi^{(0)} - p = 0$ and $\ker\theta \cap p^n W(R) = p^n \cdot \ker\theta$ since $W(R)/(\ker\theta) = \mathscr{O}_{\mathbf{C}_K}$ has no nonzero p-torsion. Since $W(R)$ is p-adically separated and complete (as R is a perfect domain, so the p-adic topology on $W(R)$ is just the product topology on $W(R)$ using the discrete topology of R), to prove that ξ is a principal generator of $\ker\theta$ it therefore suffices to show $\ker\theta \subseteq (\xi, p) = ([\pi], p)$. But if $w = (r_0, r_1, \dots) \in \ker\theta$ then $r_0^{(0)} \equiv 0 \bmod p$, so $v_R(r_0) = \mathrm{ord}_p(r_0^{(0)}) \geq 1 = v_R(\pi)$ and hence $r_0 \in \pi R$. We conclude that $w \in ([r_0], p) \subseteq ([\pi], p)$, as desired.

A general element $w = (r_0, r_1, \dots) \in \ker\theta$ has the form

$$w = \xi \cdot (r_0', r_1', \dots) = (\pi, -1, \dots)(r_0', r_1', \dots) = (\pi r_0', \pi^p r_1' - r_0'^{\,p}, \dots),$$

so $r_1 = \pi^p r_1' - r_0'^{\,p}$. Hence, $r_1 \in R^\times$ if and only if $r_0' \in R^\times$, and this final unit condition is equivalent to the multiplier $(r_0', r_1', \dots)$ being a unit in $W(R)$, which amounts to w being a principal generator of $\ker\theta$ (since $W(R)$ is a domain). ■

Corollary 4.4.4. *For all $j \geq 1$,*

$$W(R) \cap (\ker\theta_K)^j = (\ker\theta)^j.$$

Also, $\cap(\ker\theta)^j = \cap(\ker\theta_K)^j = 0$.

Proof. By a simple induction on j and chasing multiples of ξ, to prove the displayed equality it suffices to check the case $j = 1$. This case is clear since $W(R)/(\ker\theta) = \mathscr{O}_{\mathbf{C}_K}$ has no nonzero p-torsion.

Since any element of $W(R)[1/p]$ admits a p-power multiple in $W(R)$, we conclude that

$$\cap(\ker\theta_K)^j = (\cap(\ker\theta)^j)[1/p].$$

To prove this vanishes, it suffices to consider an arbitrary $w = (r_0, r_1, \dots) \in W(R)$ lying in $\cap(\ker\theta)^j$. Thus, w is divisible by arbitrarily high powers of $\xi = [\pi] - p = (\pi, -1, \dots)$, so r_0 is divisible by arbitrarily high powers of π in R. But $v_R(\pi) = 1 > 0$, so by v_R-adic separatedness of R we see that $r_0 = 0$. This says that $w = pw'$ for some $w' \in W(R)$ since R is a perfect $\mathbf{F}_p$-algebra. Hence, $w' \in (\cap(\ker\theta)^j)[1/p] = \cap(\ker\theta_K)^j$. Thus, $w' \in W(R) \cap (\ker\theta_K)^j = (\ker\theta)^j$ for all j. This shows that each element of $\cap(\ker\theta)^j$ in $W(R)$ lies in $\cap p^n W(R)$, and this vanishes since $W(R)$ is a strict p-ring. ■

We conclude that $W(R)[1/p]$ injects into the inverse limit

$$B_{\mathrm{dR}}^+ := \varprojlim_j W(R)[1/p]/(\ker\theta_K)^j \tag{4.4.1}$$

whose transition maps are G_K-equivariant, so B_{dR}^+ has a natural G_K-action that is compatible with the action on its subring $W(R)[1/p]$. (Beware that in (4.4.1) we cannot move the p-localization outside of the inverse limit: algebraic localization and inverse limit do not generally commute with each other, as is most easily seen when comparing the t-adic completion $\mathbf{Q}_p[\![t]\!]$ of $\mathbf{Q}_p[t] = \mathbf{Z}_p[t][1/p]$ with its subring $\mathbf{Z}_p[\![t]\!][1/p]$ of power series with "bounded denominators".) The inverse limit B_{dR}^+ maps G_K-equivariantly onto each quotient $W(R)[1/p]/(\ker\theta_K)^j$ via the evident natural map, and in particular for $j = 1$ the map θ_K induces a natural G_K-equivariant surjective map $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbf{C}_K$. It is clear that $\ker\theta_{\mathrm{dR}}^+ \cap W(R) = \ker\theta$ and $\ker\theta_{\mathrm{dR}}^+ \cap W(R)[1/p] = \ker\theta_K$ since θ_{dR}^+ restricts to θ_K on the subring $W(R)[1/p]$.

Proposition 4.4.5. *The ring B_{dR}^+ is a complete discrete valuation ring with residue field $\mathbf{C}_K$, and any generator of $\ker\theta_K$ in $W(R)[1/p]$ is a uniformizer of B_{dR}^+. The natural map $B_{\mathrm{dR}}^+ \to W(R)[1/p]/(\ker\theta_K)^j$ is identified with the projection to the quotient modulo the jth power of the maximal ideal for all $j \geq 1$.*

Proof. Since $\ker\theta_K$ is a nonzero principal maximal ideal (with residue field $\mathbf{C}_K$) in the domain $W(R)[1/p]$, for $j \geq 1$ it is clear that $W(R)[1/p]/(\ker\theta_K)^j$ is an artin local ring whose only ideals are $(\ker\theta_K)^i/(\ker\theta_K)^j$ for $0 \leq i \leq j$. In particular, an element of B_{dR}^+ is a unit if and only if it has nonzero image under θ_{dR}^+. In other words, the maximal ideal $\ker\theta_{\mathrm{dR}}^+$ consists of precisely the non-units, so B_{dR}^+ is a local ring.

Consider a non-unit $b \in B_{\mathrm{dR}}^+$, so its image in each $W(R)[1/p]/(\ker\theta_K)^j$ has the form $b_j\xi$ with b_j uniquely determined modulo $(\ker\theta_K)^{j-1}$ (with ξ as above). In particular, the residue classes $b_j \bmod (\ker\theta_K)^{j-1}$ are a compatible sequence and so define an element $b' \in B_{\mathrm{dR}}^+$ with $b = \xi b'$. The construction of b' shows that it is unique. Hence, the maximal ideal of B_{dR}^+ has the principal generator ξ, and ξ is not a zero divisor in B_{dR}^+.

It is now clear that for each $j \geq 1$ the multiples of ξ^j in B_{dR}^+ are the elements killed by the surjective projection to $W(R)[1/p]/(\ker\theta_K)^j$. In particular, B_{dR}^+ is ξ-adically separated, so it is a discrete valuation ring with uniformizer ξ. We have identified the construction of B_{dR}^+ as the inverse limit of its artinian quotients, so it is a complete discrete valuation ring. ∎

The Frobenius automorphism φ of $W(R)[1/p]$ does not naturally extend to B_{dR}^+ since it does not preserve $\ker\theta_K$; for example, $\varphi(\xi) = [\pi^p] - p \notin \ker\theta_K$. There is no natural Frobenius structure on B_{dR}^+. Nonetheless, we do have a filtration via powers of the maximal ideal, and this is a G_K-stable filtration. We get the same on the fraction field:

Definition 4.4.6. The *field of p-adic periods* (or the *de Rham period ring*) is $B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^+)$ equipped with its natural G_K-action and G_K-stable filtration via the $\mathbf{Z}$-powers of the maximal ideal of B_{dR}^+.

To show that the filtered field B_{dR} is an appropriate refinement of B_{HT}, we wish to prove that the associated graded algebra $\mathrm{gr}^\bullet(B_{\mathrm{dR}})$ over the residue field $\mathbf{C}_K$ of B_{dR}^+ (see Example 4.1.2) is G_K-equivariantly identified with the graded $\mathbf{C}_K$-algebra B_{HT}. This amounts to proving that the Zariski cotangent space of B_{dR}^+, which is 1-dimensional over the residue field $\mathbf{C}_K$, admits a canonical copy of $\mathbf{Z}_p(1)$; this would be a canonical $\mathbf{Z}_p$-line on which G_K acts by the p-adic cyclotomic character, and identifies the Zariski cotangent space with $\mathbf{C}_K(1)$ as required.

We will do better: we shall prove that B_{dR}^+ admits a uniformizer t, canonical up to $\mathbf{Z}_p^\times$-multiple, on which G_K acts by the cyclotomic character, and that the set of such t's is naturally $\mathbf{Z}_p^\times$-equivariantly bijective with the set of $\mathbf{Z}_p$-bases of $\mathbf{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$. (Such elements t do not live in $W(R)[1/p]$, so it is essential to have passed to the completion B_{dR}^+ to find such a uniformizer on which there is such a nice G_K-action.) The construction of t rests on elements $\varepsilon \in R$ from Example 4.3.4 as follows.

Choose $\varepsilon \in R$ with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$, so $\theta([\varepsilon] - 1) = \varepsilon^{(0)} - 1 = 0$. Hence, $[\varepsilon] - 1 \in \ker \theta \subseteq \ker \theta^+_{\mathrm{dR}}$, so $[\varepsilon] = 1 + ([\varepsilon] - 1)$ is a 1-unit in the complete discrete valuation ring B^+_{dR} over K. We can therefore make sense of the logarithm

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon] - 1)) = \sum_{n \geq 1} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \in B^+_{\mathrm{dR}}.$$

This clearly lies in the maximal ideal of B^+_{dR}. Note that if we make another choice ε' then $\varepsilon' = \varepsilon^a$ for a unique $a \in \mathbf{Z}_p^\times$ using the natural $\mathbf{Z}_p$-module structure on 1-units in R. Hence, by continuity of the Teichmüller map $R \to W(R)$ relative to the v_R-adic topology of R we have $[\varepsilon'] = [\varepsilon]^a$ in $W(R)$. Thus, $t' = \log([\varepsilon']) = \log([\varepsilon]^a)$.

We wish to claim that $\log([\varepsilon]^a) = a \cdot \log([\varepsilon])$, but this requires an argument because the logarithm is defined as a convergent sum relative to a topology on B^+_{dR} that "ignores" the v_R-adic topology of R whereas the exponentiation procedure $[\varepsilon]^a$ involves the v_R-adic topology of R in an essential manner. A good way to deal with this is to introduce a topological ring structure on B^+_{dR} that is finer than its discrete valuation topology and relative to which the natural map $W(R) \to B^+_{\mathrm{dR}}$ is continuous. We leave this to the reader in the form of the following important multi-part exercise.

Exercise 4.4.7. This exercise introduces a topological ring structure on $W(R)[1/p]$ that induces the natural v_R-adic product topology on the subring $W(R)$ and extends it to a natural topological ring structure on B^+_{dR} whose induced quotient topology on the residue field $\mathbf{C}_K$ is the natural valuation topology. Roughly speaking, for $W(R)[1/p]$ the idea is to impose a topology using controlled decay of coefficients of Laurent series in p. The situation is fundamentally different from topologizing $\mathbf{Q}_p = \mathbf{Z}_p[1/p]$ from the topology on $\mathbf{Z}_p$ because $pW(R)$ is *not* open in $W(R)$ (in contrast with $p\mathbf{Z}_p \subseteq \mathbf{Z}_p$) when R is given its v_R-adic (rather than its discrete) topology.

(1) For any open ideal $\mathfrak{a} \subseteq R$ and $N \geq 0$, let

$$U_{N,\mathfrak{a}} = \bigcup_{j > -N} (p^{-j} W(\mathfrak{a}^{p^j}) + p^N W(R)) \subseteq W(R)[1/p],$$

where $W(J)$ for an ideal J of R means the ideal of Witt vectors in $W(R)$ whose components all lie in J. Prove that $U_{N,\mathfrak{a}}$ is a G_K-stable $W(R)$-submodule of $W(R)[1/p]$.

(2) Prove $U_{N+M,\mathfrak{a}+\mathfrak{b}} \subseteq U_{N,\mathfrak{a}} \cap U_{M,\mathfrak{b}}$ and that $U_{N,\mathfrak{a}} \cdot U_{N,\mathfrak{a}} \subseteq U_{N,\mathfrak{a}}$. Deduce that $W(R)[1/p]$ has a unique structure of topological ring with the $U_{N,\mathfrak{a}}$'s a base of open neighborhoods of 0, and that the G_K-action on $W(R)[1/p]$ is continuous.

(3) Prove that $U_{N,\mathfrak{a}} \cap W(R) = W(\mathfrak{a}) + p^N W(R)$, and deduce that $W(R)$ endowed with its product topology using the v_R-adic topology on R is a closed topological subring of $W(R)[1/p]$. Conclude that $K_0 = W(k)[1/p] \subseteq W(R)[1/p]$ is a closed subfield with its usual p-adic topology (hint: k is a discrete subring of R).

(4) For each $N \geq 0$, prove that $p^N \mathscr{O}_{\mathbf{C}_K} \subseteq \theta_K(U_{N,\mathfrak{a}})$ and show that this containment gets arbitrarily close to an equality for the p-adic topology (i.e., $\theta_K(U_{N,\mathfrak{a}})$ is contained in $p^{N+a} \mathscr{O}_{\mathbf{C}_K}$ for arbitrarily small $a > 0$) by taking $\mathfrak{a}$ to be sufficiently small. In particular, deduce that $\theta_K : W(R)[1/p] \to \mathbf{C}_K$ is a continuous open map.

(5) Prove that the multiplication map $\xi : W(R)[1/p] \to W(R)[1/p]$ is a closed embedding, so all ideals $(\ker \theta_K)^j = \xi^j W(R)[1/p]$ are closed. Conclude that with the quotient topology on each $W(R)[1/p]/(\ker \theta_K)^j$, the inverse limit topology on B_{dR}^+ makes it a Hausdorff topological ring relative to which the powers of the maximal ideal are closed, $W(R)$ is a closed subring, the multiplication map by ξ on B_{dR}^+ is a closed embedding, and the residue field $\mathbf{C}_K$ inherits its *valuation topology* as the quotient topology.

We now use the final part of the preceding exercise. Let $U_R \subseteq 1 + \mathfrak{m}_R$ be the subgroup of elements x such that $x^{(0)} = 1$ (such as any choice of ε). We claim that the logarithm $\log([x]) \in B_{\mathrm{dR}}^+$ formed as a convergent sum for the discrete valuation topology is continuous in x relative to the v_R-adic topology of the topological group $U_R \subseteq 1+\mathfrak{m}_R$ and the topological ring structure just constructed on B_{dR}^+. Since $x \mapsto \log([x])$ is an abstract homomorphism $U_R \to B_{\mathrm{dR}}^+$ between topological groups, it suffices to check continuity at the identity. If $\mathfrak{a} \subseteq R$ is an ideal and $x \in (1 + \mathfrak{a}) \cap U_R$ then working in $W(R/\mathfrak{a})$ shows that $[x] - 1 \in W(\mathfrak{a})$, so $([x] - 1)^n/n \in p^{-j} W(\mathfrak{a}^{p^j})$ with $j = \mathrm{ord}_p(n)$ for all $n \geq 1$. This gives the required continuity, in view of how the topology on B_{dR}^+ is defined.

For any $a \in \mathbf{Z}_p$ and $x \in U_R$ we have $x^a \in U_R$ by continuous extension from the case $a \in \mathbf{Z}^+$ via the tautological continuity of the map $x \mapsto x^{(0)}$ from R to $\mathscr{O}_{\mathbf{C}_K}$. Likewise, by continuity of $\log : U_R \to B_{\mathrm{dR}}^+$, for any $a \in \mathbf{Z}_p$ and $x \in U_R$ we have $\log([x^a]) = a \log([x])$ by continuous extension from the case $a \in \mathbf{Z}^+$. Hence, for $\varepsilon' = \varepsilon^a$ with $a \in \mathbf{Z}_p^\times$ we have $t' := \log([\varepsilon']) = a \log([\varepsilon]) = at$.

In other words, the line $\mathbf{Z}_p t$ in the maximal ideal of B_{dR}^+ is intrinsic (i.e., independent of the choice of ε) and making a choice of $\mathbf{Z}_p$-basis of this line is the same as making a choice of ε. Also, choosing ε is literally a choice of $\mathbf{Z}_p$-basis of $\mathbf{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$. For any $g \in G_K$ we have $g(\varepsilon) = \varepsilon^{\chi(g)}$ in R since $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for the primitive p^nth roots of unity $\varepsilon^{(n)} \in \mathscr{O}_{\overline{K}}$ for all $n \geq 0$. Thus, by the G_K-equivariance of the logarithm on 1-units of B_{dR}^+,

$$g(t) = \log(g([\varepsilon])) = \log([g(\varepsilon)]) = \log([\varepsilon^{\chi(g)}]) = \log([\varepsilon]^{\chi(g)}) = \chi(g)t.$$

We conclude that $\mathbf{Z}_p t$ is a canonical copy of $\mathbf{Z}_p(1)$ as a G_K-stable line in B_{dR}^+. Intuitively, this line is viewed as an analogue of the $\mathbf{Z}$-line $\mathbf{Z}(1) := \ker(\exp) \subseteq \mathbf{C}$, and in particular the choice of a $\mathbf{Z}_p$-basis element t is analogous to a choice of $2\pi i$ in complex analysis.

The key fact concerning such elements t is that they are uniformizers of B_{dR}^+, and hence we get a canonical isomorphism $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \simeq B_{\mathrm{HT}}$. We now prove this uniformizer property.

Proposition 4.4.8. *The element $t = \log([\varepsilon])$ in B_{dR}^+ is a uniformizer.*

Proof. By construction of t, $\theta_{\mathrm{dR}}^+(t) = 0$. Hence, t is a non-unit. We have to prove that t is not in the square of the maximal ideal. In view of its definition as an infinite series in powers $([\varepsilon] - 1)^n/n$ with $[\varepsilon] - 1$ in the maximal ideal, all such terms with $n \geq 2$ can be ignored. Thus, we just have to check that $[\varepsilon] - 1$ is not in the square of the maximal ideal. But the projection from B_{dR}^+ onto the quotient modulo the square of its maximal ideal is the same as the natural map onto $W(R)[1/p]/(\ker \theta_K)^2$, so we have to prove that $[\varepsilon] - 1$ is not contained in $(\ker\theta_K)^2$, or equivalently is not contained in $W(R)\cap(\ker\theta_K)^2 = (\ker\theta)^2 = \xi^2 W(R)$ with $\xi = [\pi] - p$ for $\pi \in R$ defined by a compatible sequence of p-power roots of p.

To show that $[\varepsilon] - 1$ is not a $W(R)$-multiple of ξ^2, it suffices to project into the 0th component of $W(R)$ and show that $\varepsilon - 1$ is not an R-multiple of π^2. That is, it suffices to prove $v_R(\varepsilon - 1) < v_R(\pi^2) = 2$. But $v_R(\varepsilon - 1) = p/(p-1)$ by Example 4.3.4, so for $p > 2$ we have a contradiction. Now suppose $p = 2$. In this case we will work in $W_2(R)$. Since $\xi^2 = [\pi^2] - 2[\pi] + 4 = (\pi^2, 0, \dots)$ in $W(R)$, for any $w = (r_0, r_1, \dots) \in W(R)$ we compute $\xi^2 w = (r_0\pi^2, r_1\pi^4, \dots)$. However, for $p = 2$ we have $-1 = (1, 1, \dots)$ in $\mathbf{Z}_2 = W(\mathbf{F}_2)$ since $-1 = 1 + 2\cdot 1 \bmod 4$, so $[\varepsilon] - 1 = (\varepsilon - 1, \varepsilon - 1, \dots)$ in $W(R)$. Thus, if $[\varepsilon] - 1$ were a $W(R)$-multiple of ξ^2 for $p = 2$ then $\varepsilon - 1 = r_1\pi^4$ for some $r_1 \in R$. This says $v_R(\varepsilon - 1) \geq v_R(\pi^4) = 4$, a contradiction since $v_R(\varepsilon - 1) = p/(p-1) = 2$. ∎

Remark 4.4.9. Note that the construction of B_{dR}^+ only involves the field K through its completed algebraic closure $\mathbf{C}_K$. More specifically, if $K' \subseteq \mathbf{C}_K$ is a complete discretely-valued subfield (so it is a p-adic field, as its residue field k' is perfect due to sitting between k and $\overline{k}$) then we get the same ring B_{dR}^+ whether we use K or K'. The actions of G_K and $G_{K'}$ on this common ring are related in the evident manner, namely via the inclusion $G_{K'} \hookrightarrow G_K$ as subgroups of the isometric automorphism group of $\mathbf{C}_K$. For example, replacing K with $\widehat{K^{\mathrm{un}}}$ does not change B_{dR}^+ but replaces the G_K-action with the underlying I_K-action. Likewise, the ring B_{dR}^+ is unaffected by replacing K with a finite extension within $\overline{K}$.

We end our preliminary discussion of B_{dR}^+ by recording some important properties that are not easily seen from its explicit construction. First of all, whereas $W(R)[1/p]$ does not contain any nontrivial finite totally ramified extension of $K_0 = W(k)[1/p]$ (as it lies inside of the absolutely unramified p-adic field $W(\mathrm{Frac}(R))[1/p]$), the completion B_{dR}^+ contains a unique copy of $\overline{K}$ as a subfield over K_0 compatibly with the action of G_K (and even G_{K_0}). This is due to Hensel's Lemma: since B_{dR}^+ is a complete discrete valuation ring over K_0, and moreover $\overline{K}$ is a subfield of the residue field $\mathbf{C}_K$ that is separable algebraic over K_0, it follows that $\overline{K}$ uniquely lifts to a subfield over K_0 in B_{dR}^+. The uniqueness of the lifting ensures that this is a G_K-equivariant lifting. This canonical $\overline{K}$-structure on B_{dR}^+ (and hence on its fraction field B_{dR}) plays an important role in the study of finer period rings; it can be shown that there is no G_K-equivariant lifting of the *entire* residue field $\mathbf{C}_K$ into B_{dR}^+ (whereas such an abstract lifting exists by commutative algebra and is not useful).

Another property of B_{dR} that is hard to see directly from the construction is the determination of its subfield of G_K-invariants. As we have just seen, there is a canonical G_K-equivariant embedding $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$, whence $K \subseteq B_{\mathrm{dR}}^{G_K}$. (Nothing like this holds for $W(R)[1/p]$ if $K \neq K_0$.) This inclusion is an equality, due to the Tate–Sen theorem:

Theorem 4.4.10. *The inclusion $K \subseteq B_{\mathrm{dR}}^{G_K}$ is an equality.*

Proof. Since the G_K-actions respect the (exhaustive and separated) filtration, the field extension $B_{\mathrm{dR}}^{G_K}$ of K with the subspace filtration has associated graded K-algebra that injects into $(\mathrm{gr}^\bullet(B_{\mathrm{dR}}))^{G_K} = B_{\mathrm{HT}}^{G_K}$. But by the Tate–Sen theorem this latter space of invariants is K. We conclude that $\mathrm{gr}^\bullet(B_{\mathrm{dR}}^{G_K})$ is 1-dimensional over K, so the same holds for $B_{\mathrm{dR}}^{G_K}$. ∎

The final property of B_{dR} that we record is its dependence on K. An inspection of the construction shows that B_{dR}^+ depends solely on $\mathscr{O}_{\mathbf{C}_K}$ and not on the particular p-adic field $K \subseteq \mathscr{O}_{\mathbf{C}_K}[1/p] = \mathbf{C}_K$ whose algebraic closure is dense in $\mathbf{C}_K$. More specifically, B_{dR}^+ depends *functorially* on $\mathscr{O}_{\mathbf{C}_K}$ (this requires reviewing the construction of R and θ), and the action of $\mathrm{Aut}(\mathscr{O}_{\mathbf{C}_K})$ on B_{dR}^+ via functoriality induces the action of G_K (via the natural inclusion of G_K into $\mathrm{Aut}(\mathscr{O}_{\mathbf{C}_K})$). Hence, if $K \to K'$ is a map of p-adic fields and we pick a compatible embedding $\overline{K} \to \overline{K'}$ of algebraic closures then the induced map $\mathscr{O}_{\mathbf{C}_K} \to \mathscr{O}_{\mathbf{C}_{K'}}$ induces a map $B_{\mathrm{dR},K}^+ \to B_{\mathrm{dR},K'}^+$ that is equivariant relative to the corresponding map of Galois groups $G_{K'} \to G_K$. In particular, if the induced map $\mathbf{C}_K \to \mathbf{C}_{K'}$ is an isomorphism then we have $B_{\mathrm{dR},K}^+ = B_{\mathrm{dR},K'}^+$ (compatibly with the inclusion $G_{K'} \hookrightarrow G_K$) and likewise for the fraction fields. This applies in two important cases: K'/K a finite extension and $K' = \widehat{K^{\mathrm{un}}}$. In other words, B_{dR}^+ and B_{dR} are naturally insensitive to replacing K with a finite extension or with a completed maximal unramified extension. The invariance of B_{dR}^+ and B_{dR} under these two kinds of changes in K is important in practice when replacing G_K with an open subgroup or with I_K in the context of studying deRham representations in §6. We will return to this issue in more detail in Proposition 6.3.8 and the discussion immediately preceding it.

5. Formalism of admissible representations

Now that we have developed some experience with various functors between Galois representations and semilinear algebra categories via suitable rings with structure, we wish to axiomatize this kind of situation for constructing and analyzing functors defined via "period rings" in order that we do not have to repeat the same kinds of arguments every time we introduce a new period ring. In §6 we shall use the following formalism.

5.1. **Definitions and examples.**

Let F be a field and G be a group. Let B be an F-algebra domain equipped with a G-action (as an F-algebra), and assume that the invariant F-subalgebra $E = B^G$ is a field. We do not impose any topological structure on B or F or G. Our goal is to use B to construct an interesting functor from finite-dimensional F-linear representations of G to finite-dimensional E-vector spaces (endowed with extra structure, depending on B).

We let $C = \mathrm{Frac}(B)$, and observe that G also acts on C in a natural way.

Definition 5.1.1. We say B is (F, G)*-regular* if $C^G = B^G$ and if every nonzero $b \in B$ whose F-linear span Fb is G-stable is a unit in B.

Note that if B is a field then the conditions in the definition are obviously satisfied. The cases of most interest will be rather far from fields. We now show how the Tate–Sen theorem (Theorem 2.1.5) provides two interesting examples of (F, G)-regular domains.

Example 5.1.2. Let K be a p-adic field with a fixed algebraic closure $\overline{K}$, and let $\mathbf{C}_K$ denote the completion of $\overline{K}$. Let $G = G_K = \mathrm{Gal}(\overline{K}/K)$. Let $B = B_{\mathrm{HT}} = \oplus_{n\in\mathbf{Z}}\mathbf{C}_K(n)$ endowed with its natural G-action. Non-canonically, $B = \mathbf{C}_K[T, 1/T]$ with G acting through the p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ via $g(\sum a_n T^n) = \sum g(a_n)\chi(g)^n T^n$. Obviously in this case $C = \mathbf{C}_K(T)$. We claim that B is $(\mathbf{Q}_p, G)$-regular (with $B^G = K$).

By the Tate–Sen theorem, $B^G = \oplus \mathbf{C}_K(n)^G = K$. To compute that C^G is also equal to K, consider the G_K-equivariant inclusion of $C = \mathbf{C}_K(T)$ into the formal Laurent series field $\mathbf{C}_K((T))$ equipped with its evident G-action. It suffices to show that $\mathbf{C}_K((T))^G = K$. The action of $g \in G$ on a formal Laurent series $\sum c_n T^n$ is given by $\sum c_n T^n \mapsto \sum g(c_n)\chi(g)^n T^n$, so G-invariance amounts to the condition $c_n \in \mathbf{C}_K(n)^G$ for all $n \in \mathbf{Z}$. Hence, by the Tate–Sen theorem we get $c_n = 0$ for $n \neq 0$ and $c_0 \in K$, as desired.

Verifying the second property in $(\mathbf{Q}_p, G_K)$-regularity goes by a similar method, as follows: if $b \in B - \{0\}$ spans a G_K-stable $\mathbf{Q}_p$-line then G_K acts on the line $\mathbf{Q}_p b$ by some character $\psi : G_K \to \mathbf{Q}_p^\times$. It is a crucial fact (immediate from the continuity of the G_K-action on each direct summand $\mathbf{C}_K(n)$ of $B = B_{\mathrm{HT}}$) that ψ must be continuous (so it takes values in $\mathbf{Z}_p^\times$). Writing the Laurent polynomial b as $b = \sum c_j T^j$, we have $\psi(g)b = g(b) = \sum g(c_j)\chi(g)^j T^j$, so for each j we have $(\psi^{-1}\chi^j)(g) \cdot g(c_j) = c_j$ for all $g \in G_K$. That is, each c_j is G_K-invariant in $\mathbf{C}_K(\psi^{-1}\chi^j)$. But by the Tate–Sen theorem, for a $\mathbf{Z}_p^\times$-valued continuous character η of G_K, if $\mathbf{C}_K(\eta)$ has a nonzero G_K-invariant element then $\eta|_{I_K}$ has finite order. Hence, $(\psi^{-1}\chi^j)|_{I_K}$ has finite order whenever $c_j \neq 0$. It follows that we cannot have $c_j, c_{j'} \neq 0$ for some $j \neq j'$, for otherwise taking the ratio of the associated finite-order characters would give that $\chi^{j-j'}|_{I_K}$ has finite order, so $\chi|_{I_K}$ has finite order (as $j - j' \neq 0$), but this is a contradiction since χ cuts out an infinitely ramified extension of K. It follows that there is at most one j such that $c_j \neq 0$, and there is a nonzero c_j since $b \neq 0$. Hence, $b = cT^j$ for some j and some $c \in \mathbf{C}_K^\times$, so obviously $b \in B^\times$.

Example 5.1.3. Consider $B = B_{\mathrm{dR}}^+$ equipped with its natural action by $G = G_K$. This is a complete discrete valuation ring with uniformizer t on which G acts through χ and with fraction field $C = B_{\mathrm{dR}} = B[1/t]$. We have seen in Theorem 4.4.10 (using that the associated graded ring to B_{dR} is B_{HT}) that $C^G = K$, so obviously $B^G = K$ too. Since B_{dR} is a field, it follows trivially that B_{dR} is $(\mathbf{Q}_p, G)$-regular. Let us consider whether $B = B_{\mathrm{dR}}^+$ is also $(\mathbf{Q}_p, G)$-regular. The first requirement in the definition of $(\mathbf{Q}_p, G)$-regularity for B is satisfied in this case, as we have just seen. But the second requirement in $(\mathbf{Q}_p, G)$-regularity fails: $t \in B$ spans a G-stable $\mathbf{Q}_p$-line but $t \notin B^\times$.

The most interesting examples of $(\mathbf{Q}_p, G_K)$-regular rings are Fontaine's rings B_{cris} and B_{st} (certain subrings of B_{dR} with "more structure"), which turn out (ultimately by reducing to the study of B_{HT}) to be $(\mathbf{Q}_p, G_K)$-regular with subring of G_K-invariants equal to $K_0 = \mathrm{Frac}(W(k)) = W(k)[1/p]$ and K respectively.

In the general axiomatic setting, if B is an (F, G)-regular domain and E denotes the field $C^G = B^G$ then for any object V in the category $\mathrm{Rep}_F(G)$ of finite-dimensional F-linear representations of G we define

$$D_B(V) = (B \otimes_F V)^G,$$

so $D_B(V)$ is an E-vector space equipped with a canonical map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_E (B \otimes_F V) = (B \otimes_E B) \otimes_F V \to B \otimes_F V.$$

This is a B-linear G-equivariant map (where G acts trivially on $D_B(V)$ in the right tensor factor of the source), by inspection.

As a simple example, for $V = F$ with trivial G-action we have $D_B(F) = B^G = E$ and the map $\alpha_V : B = B \otimes_E E \to B \otimes_F F = B$ is the identity map. It is not a priori obvious if $D_B(V)$ always lies in the category Vec_E of finite-dimensional vector spaces over E, but we shall now see that this and much more is true.

5.2. **Properties of admissible representations.** The aim of this section is to prove the following theorem which shows (among other things) that $\dim_E D_B(V) \leq \dim_F V$; in case equality holds we call V a B-*admissible* representation. For example, $V = F$ is always B-admissible. In case we fix a p-adic field K and let $F = \mathbf{Q}_p$ and $G = G_K$ then for $B = B_{\mathrm{HT}}$ this coincides with the concept of being a *Hodge-Tate* representation. For the ring B_{dR} and Fontaine's finer period rings B_{cris}, and B_{st} the corresponding notions are called being a *deRham*, *crystalline*, and *semi-stable* representation respectively.

Theorem 5.2.1. *Fix V as above.*

(1) *The map α_V is always injective and $\dim_E D_B(V) \leq \dim_F V$, with equality if and only if α_V is an isomorphism.*
(2) *Let $\mathrm{Rep}_F^B(G) \subseteq \mathrm{Rep}_F(G)$ be the full subcategory of B-admissible representations. The covariant functor $D_B : \mathrm{Rep}_F^B(G) \to \mathrm{Vec}_E$ to the category of finite-dimensional E-vector spaces is exact and faithful, and any subrepresentation or quotient of a B-admissible representation is B-admissible.*
(3) *If $V_1, V_2 \in \mathrm{Rep}_F^B(G)$ then there is a natural isomorphism*

$$D_B(V_1) \otimes_E D_B(V_2) \simeq D_B(V_1 \otimes_F V_2),$$

so $V_1 \otimes_F V_2 \in \mathrm{Rep}_F^B(G)$. If $V \in \mathrm{Rep}_F^B(G)$ then its dual representation $V^\vee$ lies in $\mathrm{Rep}_F^B(G)$ and the natural map

$$D_B(V) \otimes_E D_B(V^\vee) \simeq D_B(V \otimes_F V^\vee) \to D_B(F) = E$$

is a perfect duality between $D_B(V)$ and $D_B(V^\vee)$.

In particular, $\mathrm{Rep}_F^B(G)$ is stable under the formation of duals and tensor products in $\mathrm{Rep}_F(G)$, and D_B naturally commutes with the formation of these constructions in $\mathrm{Rep}_F^B(G)$ and in Vec_E.

Moreover, B-admissibility is preserved under the formation of exterior and symmetric powers, and D_B naturally commutes with both such constructions.

Before proving the theorem, we make some remarks.

Remark 5.2.2. In practice $F = \mathbf{Q}_p$, $G = G_K$ for a p-adic field K, and $E = K$ or $E = K_0$ (the maximal unramified subfield, $W(k)[1/p]$), and the ring B has more structure (related to a Frobenius operator, filtration, monodromy operator, etc.). Corresponding to this extra structure on B, the functor D_B takes values in a category of finite-dimensional E-vector spaces equipped with "more structure", with morphisms being those E-linear maps which "respect the extra structure".

By viewing D_B with values in such a category, it can fail to be fully faithful (such as for $B = B_{\mathrm{HT}}$ or $B = B_{\mathrm{dR}}$ using categories of graded or filtered vector spaces respectively), but for more subtle period rings such as B_{cris} and B_{st} one does get full faithfulness into a suitably enriched category of linear algebra objects. One of the key results in recent years in p-adic Hodge theory is a purely linear algebraic description of the essential image of the fully faithful functor D_B for such better period rings (with the D_B viewed as taking values in a suitably enriched subcategory of Vec_E).

Remark 5.2.3. Once the theorem is proved, there is an alternative description of the B-admissibility condition on V: it says that $B \otimes_F V$ with its B-module structure and G-action is isomorphic to a direct sum $B^{\oplus r}$ (for some r) respecting the B-structure and G-action. Indeed, since α_V is G-equivariant and B-linear, we get the necessity of this alternative description by choosing an E-basis of $D_B(V)$. As for sufficiency, if $B \otimes_F V \simeq B^{\oplus r}$ as B-modules and respecting the G-action then necessarily $r = d := \dim_F V$ (as $B \otimes_F V$ is finite free of rank d over B), and taking G-invariants gives $D_B(V) \simeq (B^G)^{\oplus d} = E^{\oplus d}$ as modules over $B^G = E$. This says $\dim_E D_B(V) = d = \dim_F V$, which is the dimension equality definition of B-admissibility.

Proof. First we prove (1). Granting for a moment that α_V is injective, let us show the rest of (1). Extending scalars from B to $C := \mathrm{Frac}(B)$ preserves injectivity (by flatness of C over B), so $C \otimes_E D_B(V)$ is a C-subspace of $C \otimes_F V$. Comparing C-dimensions then gives $\dim_E D_B(V) \le \dim_F V$. Let us show that in case of equality of dimensions, say with common dimension d, the map α_V is an isomorphism (the converse now being obvious). Let $\{e_j\}$ be an E-basis of $D_B(V)$ and let $\{v_i\}$ be an F-basis of V, so relative to these bases we can express α_V using a $d \times d$ matrix (b_{ij}) over B (thanks to the assumed dimension equality). In other words, $e_j = \sum b_{ij} \otimes v_i$. The determinant $\det(\alpha_V) := \det(b_{ij}) \in B$ is nonzero due to the isomorphism property over $C = \mathrm{Frac}(B)$ (as $C \otimes_B \alpha_V$ is a C-linear injection between C-vector spaces with the same finite dimension d, so it must be an isomorphism). We want $\det(\alpha_V) \in B^\times$, so then α_V is an isomorphism over B. Since B is an (F, G)-regular ring, to show the nonzero $\det(\alpha_V) \in B$ is a unit it suffices to show that it spans a G-stable F-line in B.

The vectors $e_j = \sum b_{ij} \otimes v_i \in D_B(V) \subseteq B \otimes_F V$ are G-invariant, so passing to dth exterior powers on α_V gives that

$$\wedge^d(\alpha_V)(e_1 \wedge \cdots \wedge e_d) = \det(b_{ij}) v_1 \wedge \cdots \wedge v_d$$

is a G-invariant vector in $B \otimes_F \wedge^d(V)$. But G acts on $v_1 \wedge \cdots \wedge v_d$ by some character $\eta : G \to F^\times$ (just the determinant of the given F-linear G-representation on V), so G must act on $\det(b_{ij}) \in B - \{0\}$ through the $F^\times$-valued η^{-1}.

This completes the reduction of (1) to the claim that α_V is injective. Since B is (F, G)-regular, we have that $E = B^G$ is equal to C^G. For $D_C(V) := (C \otimes_F V)^G$ we also have a commutative diagram

$$\begin{tikzcd} B \otimes_E D_B(V) \arrow[r, "\alpha_V"] \arrow[d] & B \otimes_F V \arrow[d] \\ C \otimes_E D_C(V) \arrow[r] & C \otimes_F V \end{tikzcd}$$

in which the sides are injective. To prove injectivity of the top it suffices to prove it for the bottom. Hence, we can replace B with C so as to reduce to the case when B is a field. In this case the injectivity amounts to the claim that α_V carries an E-basis of $D_B(V)$ to a B-linearly independent set in $B \otimes_F V$, so it suffices to show that if $x_1, \ldots, x_r \in B \otimes_F V$ are E-linearly independent *and* G-invariant then they are B-linearly independent. Assuming to the contrary that there is a nontrivial B-linear dependence relation among the x_i's, consider such a relation of minimal length. We may assume it to have the form

$$x_r = \sum_{i<r} b_i \cdot x_i$$

for some $r \geq 2$ since B is a field and all x_i are nonzero. Applying $g \in G$ gives

$$x_r = g(x_r) = \sum_{i<r} g(b_i) \cdot g(x_i) = \sum_{i<r} g(b_i) \cdot x_i.$$

Thus, minimal length for the relation forces equality of coefficients: $b_i = g(b_i)$ for all $i < r$, so $b_i \in B^G = E$ for all i. Hence, we have a nontrivial E-linear dependence relation among $x_1, \ldots, x_r$, a contradiction.

Now we prove (2). For any B-admissible V we have a natural isomorphism $B \otimes_E D_B(V) \simeq B \otimes_F V$, so it is clear that D_B is exact and faithful on the category of B-admissible V's (since a sequence of E-vector spaces is exact if and only if it becomes so after applying $B \otimes_E (\cdot)$, and similarly from F to B). To show that subrepresentations and quotients of a B-admissible V are B-admissible, consider a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of $F[G]$-modules with B-admissible V. We have to show that V' and V'' are B-admissible. From the definition it is clear that D_B is left-exact without any B-admissibility hypothesis, so we have a left-exact sequence of E-vector spaces

$$0 \to D_B(V') \to D_B(V) \to D_B(V'')$$

with $\dim_E D_B(V) = d$ by B-admissibility of V, so $d \leq \dim_E D_B(V') + \dim_E D_B(V'')$. By (1) we also know that the outer terms have respective E-dimensions at most $d' = \dim_F V'$ and $d'' = \dim_F V''$. But $d = d' + d''$ from the given short exact sequence of $F[G]$-modules, so these various inequalities are forced to be equalities, and in particular V' and V'' are B-admissible.

Finally, we consider (3). For B-admissible V_1 and V_2, say with $d_i = \dim_F V_i$, there is an evident natural map

$$D_B(V_1) \otimes_E D_B(V_2) \to (B \otimes_F V_1) \otimes_E (B \otimes_F V_2) \to B \otimes_F (V_1 \otimes V_2)$$

that is clearly seen to be invariant under the G-action on the target, so we obtain a natural E-linear map

$$t_{V_1,V_2} : D_B(V_1) \otimes_E D_B(V_2) \to D_B(V_1 \otimes_F V_2),$$

with source having E-dimension $d_1 d_2$ (by B-admissibility of the V_i's) and target having E-dimension at most $\dim_F(V_1 \otimes_F V_2) = d_1 d_2$ by applying (1) to $V_1 \otimes_F V_2$. Hence, as long as this map is an injection then it is forced to be an isomorphism and so $V_1 \otimes_F V_2$ is forced to be B-admissible. To show that t_{V_1,V_2} is injective it suffices to check injectivity after composing with the inclusion of $D_B(V_1 \otimes_F V_2)$ into $B \otimes_F (V_1 \otimes_F V_2)$, and by construction this composite is easily seen to coincide with the composition of the injective map

$$D_B(V_1) \otimes_E D_B(V_2) \to B \otimes_E (D_B(V_1) \otimes_E D_B(V_2)) = (B \otimes_E D_B(V_1)) \otimes_B (B \otimes_E D_B(V_2))$$

and the isomorphism $\alpha_{V_1} \otimes_B \alpha_{V_2}$ (using again that the V_i are B-admissible).

Having shown that B-admissibility is preserved under tensor products and that D_B naturally commutes with the formation of tensor products, as a special case we see that if V is B-admissible then so is $V^{\otimes r}$ for any $r \geq 1$, with $D_B(V)^{\otimes r} \simeq D_B(V^{\otimes r})$. The quotient $\wedge^r(V)$ of $V^{\otimes r}$ is also B-admissible (since $V^{\otimes r}$ is B-admissible), and there is an analogous map $\wedge^r(D_B(V)) \to D_B(\wedge^r V)$ that fits into a commutative diagram

$$\begin{tikzcd} D_B(V)^{\otimes r} \arrow[r, "\simeq"] \arrow[d] & D_B(V^{\otimes r}) \arrow[d] \\ \wedge^r(D_B(V)) \arrow[r] & D_B(\wedge^r V) \end{tikzcd} \tag{5.2.1}$$

in which the left side is the canonical surjection and the right side is surjective because it is D_B applied to a surjection between B-admissible representations. Thus, the bottom side is surjective. But the left and right terms on the bottom have the same dimension (since V and $\wedge^r V$ are B-admissible, with $\dim_F V = \dim_E D_B(V)$), so the bottom side is an isomorphism!

The same method works with symmetric powers in place of exterior powers. Note that the diagram (5.2.1) without an isomorphism across the top can be constructed for any $V \in \mathrm{Rep}_F(G)$, so for any such V there are natural E-linear maps $\wedge^r(D_B(V)) \to D_B(\wedge^r V)$ and likewise for rth symmetric powers, just as we have for tensor powers (and in the B-admissible case these are isomorphisms).

The case of duality is more subtle. Let V be a B-admissible representation of G over F. To show that $V^\vee$ is B-admissible and that the resulting natural pairing between $D_B(V)$ and $D_B(V^\vee)$ is perfect, we use a trick with tensor algebra. For any finite-dimensional vector space W over a field with $\dim W = d \geq 1$ there is a natural isomorphism

$$\det(W^\vee) \otimes \wedge^{d-1}(W) \simeq W^\vee$$

defined by

$$(\ell_1 \wedge \cdots \wedge \ell_d) \otimes (w_2 \wedge \cdots \wedge w_d) \mapsto (w_1 \mapsto \det(\ell_i(w_j))),$$

and this is equivariant for the naturally induced group actions in case W is a linear representation space for a group. Hence, to show that $V^\vee$ is a B-admissible F-linear representation space for G we are reduced to proving B-admissibility for $\det(V^\vee) = (\det V)^\vee$ (as then its tensor product against the B-admissible $\wedge^{d-1}(V)$ is B-admissible, as required). Since

$\det V$ is B-admissible, we are reduced to the 1-dimensional case (for proving preservation of B-admissibility under duality).

Now assume the B-admissible V satisfies $\dim_F V = 1$, and let v_0 be an F-basis of V, so B-admissibility gives that $D_B(V)$ is 1-dimensional (rather than 0). Hence, $D_B(V) = E(b \otimes v_0)$ for some nonzero $b \in B$. The isomorphism $\alpha_V : B \otimes_E D_B(V) \simeq B \otimes_F V = B(1 \otimes v_0)$ between free B-modules of rank 1 carries the B-basis $b \otimes v_0$ of the left side to $b \otimes v_0 = b \cdot (1 \otimes v_0)$ on the right side, so $b \in B^\times$. The G-invariance of $b \otimes v_0$ says $g(b) \otimes g(v_0) = b \otimes v_0$, and we have $g(v_0) = \eta(g)v_0$ for some $\eta(g) \in F^\times$ (as V is a 1-dimensional representation space of G over F, say with character η), so $\eta(g)g(b) = b$. Thus, $b/g(b) = \eta(g) \in F^\times$. Letting $v_0^\vee$ be the dual basis of $V^\vee$, it is easy to then compute that $D_B(V^\vee)$ contains the nonzero vector $b^{-1} \otimes v_0^\vee$, so it is a nonzero space. The 1-dimensional $V^\vee$ is therefore B-admissible, as required.

Now that we know duality preserves B-admissibility in general, we fix a B-admissible V and aim to prove the perfectness of the pairing defined by

$$\langle \cdot, \cdot \rangle_V : D_B(V) \otimes_E D_B(V^\vee) \simeq D_B(V \otimes_F V^\vee) \to D_B(F) = E.$$

For $\dim_F V = 1$ this is immediate from the above explicitly computed descriptions of $D_B(V)$ and $D_B(V^\vee)$ in terms of a basis of V and the corresponding dual basis of $V^\vee$. In the general case, since V and $V^\vee$ are both B-admissible, for any $r \geq 1$ we have natural isomorphisms $\wedge^r(D_B(V)) \simeq D_B(\wedge^r(V))$ and $\wedge^r(D_B(V^\vee)) \simeq D_B(\wedge^r(V^\vee)) \simeq D_B((\wedge^r V)^\vee)$ with respect to which the pairing

$$\wedge_E^r(D_B(V)) \otimes_E \wedge_E^r(D_B(V^\vee)) \to E$$

induced by $\langle \cdot, \cdot \rangle_V$ on rth exterior powers is identified with $\langle \cdot, \cdot \rangle_{\wedge^r V}$. Since perfectness of a bilinear pairing between finite-dimensional vector spaces of the same dimension is equivalent to perfectness of the induced bilinear pairing between their top exterior powers, by taking $r = \dim_F V$ we see that the perfectness of the pairing $\langle \cdot, \cdot \rangle_V$ for the B-admissible V is equivalent to perfectness of the pairing associated to the B-admissible 1-dimensional $\det V$. But the 1-dimensional case is settled, so we are done. ■

6. deRham representations

6.1. **Basic definitions.** Since B_{dR} is $(\mathbf{Q}_p, G_K)$-regular with $B_{\mathrm{dR}}^{G_K} = K$, the general formalism of admissible representations provides a good class of p-adic representations: the B_{dR}-admissible ones. More precisely, we define the covariant functor $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbf{Q}_p}(\mathbf{G}_K) \to \mathrm{Vec}_K$ valued in the category Vec_K of finite-dimensional K-vector spaces by

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K},$$

so $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbf{Q}_p} V$. In case this inequality is an equality we say that V is a *deRham representation* (i.e., V is B_{dR}-admissible). Let $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ denote the full subcategory of deRham representations.

By the general formalism from §5, for $V \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K)$ we have a B_{dR}-linear G_K-compatible comparison isomorphism

$$\alpha_V : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \to B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$$

and the subcategory $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \subseteq \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is stable under passage to subquotients, tensor products, and duals (and so also exterior and symmetric powers), and moreover the functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Vec}_K$ is faithful and exact and commutes with the formation of duals and tensor powers (and hence exterior and symmetric powers).

Since duality does not affect whether or not the deRham property holds, working with D_{dR} is equivalent to working with the contravariant functor

$$D^*_{\mathrm{dR}}(V) := D_{\mathrm{dR}}(V^\vee) \simeq \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\mathrm{dR}});$$

this alternative functor can be very useful. In general $D^*_{\mathrm{dR}}(V)$ is a finite-dimensional K-vector space, and its elements correspond to $\mathbf{Q}_p[G_K]$-linear maps from V into B_{dR}. In particular, for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ the collection of all such maps spans a finite-dimensional K-subspace of B_{dR}, generally called the space of *p-adic periods of* V (or of $V^\vee$, depending on one's point of view). This space of periods for V is the only piece of B_{dR} that is relevant in the formation of $D^*_{\mathrm{dR}}(V)$. As an example, if V is an irreducible $\mathbf{Q}_p[G_K]$-module then any nonzero map from V to B_{dR} is injective and so $D^*_{\mathrm{dR}}(V) \neq 0$ precisely when V occurs as a subrepresentation of B_{dR}. In general $\dim_K D^*_{\mathrm{dR}}(V) \leq \dim_{\mathbf{Q}_p}(V)$, so an irreducible V appears in B_{dR} with finite multiplicity at most $\dim_{\mathbf{Q}_p}(V)$, and this maximal multiplicity is attained precisely when V is deRham (as this is equivalent to $V^\vee$ being deRham).

Example 6.1.1. For $n \in \mathbf{Z}$, $D_{\mathrm{dR}}(\mathbf{Q}_p(n)) = Kt^{-n}$ if we view $\mathbf{Q}_p(n)$ as $\mathbf{Q}_p$ with G_K-action by χ^n. This is 1-dimensional over K, so $\mathbf{Q}_p(n)$ is deRham for all n.

The output of the functor D_{dR} has extra K-linear structure (arising from additional structure on the K-algebra B_{dR}), namely a K-linear filtration arising from the canonical K-linear filtration on the fraction field B_{dR} of the complete discrete valuation ring B^+_{dR} over K. Before we explain this in §6.3 and axiomatize the resulting finer target category of D_{dR} (as a subcategory of Vec_K), in §6.2 we review some terminology from linear algebra.

6.2. **Filtered vector spaces.** Let F be a field, and let Vec_F be the category of finite-dimensional F-vector spaces. In Definition 4.1.1 we defined the notion of a filtered vector space over F. In the finite-dimensional setting, if $(D, \{\mathrm{Fil}^i(D)\})$ is a filtered vector space over F with $\dim_F D < \infty$ then the filtration is exhaustive if and only if $\mathrm{Fil}^i(D) = D$ for $i \ll 0$ and it is separated if and only if $\mathrm{Fil}^i(D) = 0$ for $i \gg 0$. We let Fil_F denote the category of finite-dimensional filtered vector spaces $(D, \{\mathrm{Fil}^i(D)\})$ over F equipped with an exhaustive and separated filtration, where a *morphism* between such objects is a linear map $T : D' \to D$ that is filtration-compatible in the sense that $T(\mathrm{Fil}^i(D')) \subseteq \mathrm{Fil}^i(D)$ for all i.

In the category Fil_F there are good functorial notions of kernel and cokernel of a map $T : D' \to D$ between objects, namely the usual F-linear kernel and cokernel endowed respectively with the *subspace filtration*

$$\mathrm{Fil}^i(\ker T) := \ker(T) \cap \mathrm{Fil}^i(D') \subseteq \ker T$$

and the *quotient filtration*

$$\mathrm{Fil}^i(\mathrm{coker}\, T) := (\mathrm{Fil}^i(D) + T(D'))/T(D') \subseteq \mathrm{coker}(T).$$

These have the expected universal properties (for linear maps $D'_0 \to D'$ killed by T and linear maps $D \to D_0$ composing with T to give the zero map respectively), but beware that Fil_F is *not* abelian!!

More specifically, it can easily happen that $\ker T = \operatorname{coker} T = 0$ (i.e., T is an F-linear isomorphism) but T is not an isomorphism in Fil_F. The problem is that the even if T is an isomorphism when viewed in Vec_F, the filtration on D may be "finer" than on D' and so although $T(\mathrm{Fil}^i(D')) \subseteq \mathrm{Fil}^i(D))$ for all i, such inclusions may not always be equalities (so the linear inverse is not a filtration-compatible map). For example, we could take $D = D'$ as vector spaces and give D' the trivial filtration $\mathrm{Fil}^i(D') = D'$ for $i \leq 0$ and $\mathrm{Fil}^i(D') = 0$ for $i > 0$ whereas we define $\mathrm{Fil}^i(D) = D$ for $i \leq 4$ and $\mathrm{Fil}^i(D) = 0$ for $i > 4$. The identity map T is then such an example. Thus, the forgetful functor $\mathrm{Fil}_F \to \mathrm{Vec}_F$ loses too much information (though it is a faithful functor).

Despite the absence of a good abelian category structure on Fil_F, we can still define basic notions of linear algebra in the filtered setting, as follows.

Definition 6.2.1. For $D, D' \in \mathrm{Fil}_F$, the *tensor product* $D \otimes D'$ has underlying F-vector space $D \otimes_F D'$ and filtration

$$\mathrm{Fil}^n(D \otimes D') = \sum_{p+q=n} \mathrm{Fil}^p(D) \otimes_F \mathrm{Fil}^q(D')$$

that is easily checked to be exhaustive and separated. The *unit object* $F[0]$ is F as a vector space with $\mathrm{Fil}^i(F[0]) = F$ for $i \leq 0$ and $\mathrm{Fil}^i(F[0]) = 0$ for $i > 0$. (Canonically, $D \otimes F[0] \simeq F[0] \otimes D \simeq D$ in Fil_F for all D.)

The *dual* $D^\vee$ of $D \in \mathrm{Fil}_F$ has underlying F-vector space given by the F-linear dual $\mathrm{Hom}_F(D, F)$, and has the (exhaustive and separated) filtration

$$\mathrm{Fil}^i(D^\vee) = (\mathrm{Fil}^{1-i} D)^\perp := \{\ell \in D^\vee \mid \mathrm{Fil}^{1-i}(D) \subseteq \ker \ell\}.$$

The reason we use $\mathrm{Fil}^{1-i}(D)$ rather than $\mathrm{Fil}^{-i}(D)$ is to ensure that $F[0]^\vee = F[0]$ (check this identification!).

A *short exact sequence* in Fil_F is a diagram

$$0 \to D' \to D \to D'' \to 0$$

in Fil_F that is short exact as vector spaces with $D' = \ker(D \to D'')$ (i.e., D' has the subspace filtration from D) and $D'' = \operatorname{coker}(D \to D'')$ (i.e., D'' has the quotient filtration from D). Equivalently, for all i the diagram

$$0 \to \mathrm{Fil}^i(D') \to \mathrm{Fil}^i(D) \to \mathrm{Fil}^i(D'') \to 0 \tag{6.2.1}$$

is short exact as vector spaces.

Example 6.2.2. It is easy to check that the unit object $F[0]$ is naturally self-dual in Fil_F, and that there is a natural isomorphism $D^\vee \otimes D'^\vee \simeq (D \otimes D')^\vee$ in Fil_F induced by the usual F-linear isomorphism. Likewise we have the usual double-duality isomorphism $D \simeq D^{\vee\vee}$ in Fil_F, and the evaluation morphism $D \otimes D^\vee \to F[0]$ is clearly a map in Fil_F.

Example 6.2.3. There is a natural "shift" operation in Fil_F: for $D \in \mathrm{Fil}_F$ and $n \in \mathbf{Z}$, define $D[n] \in \mathrm{Fil}_F$ to have the same underlying F-vector space but $\mathrm{Fil}^i(D[n]) = \mathrm{Fil}^{i+n}(D)$ for all $i \in \mathbf{Z}$.

It is easy to check that $D[n]^\vee \simeq D^\vee[-n]$ in Fil_F in the evident manner, and that the shifting can be passed through either factor of a tensor product.

Observe that if $T : D' \to D$ is a map in Fil_F there are two notions of "image" that are generally distinct in Fil_F but have the same underlying space. We define the *image* of T to be $T(D') \subseteq D$ with the subspace filtration from D. We define the *coimage* of T to be $T(D')$ with the quotient filtration from D'. Equivalently, $\mathrm{coim}\, T = D'/\ker T$ with the quotient filtration and $\mathrm{im}\, T = \ker(D \to \mathrm{coker}\, T)$ with the subspace filtration. There is a canonical map $\mathrm{coim}\, T \to \mathrm{im}\, T$ in Fil_F that is a linear bijection, and it is generally not an isomorphism in Fil_F.

Definition 6.2.4. A morphism $T : D' \to D$ in Fil_F is *strict* if the canonical map $\mathrm{coim}\, T \to \mathrm{im}\, T$ is an isomorphism, which is to say that the quotient and subspace filtrations on $T(D')$ coincide.

Exercise 6.2.5. For $D, D' \in \mathrm{Fil}_F$ we can naturally endow $\mathrm{Hom}_F(D', D)$ with a structure in Fil_F (denoted $\mathrm{Hom}(D', D)$). This can be done in two equivalent ways. First of all, the usual linear isomorphism $D \otimes_F D'^\vee \simeq \mathrm{Hom}_F(D', D)$ imposes a Fil_F-structure by using the dual filtration on $D'^\vee$ and the tensor product filtration on $D \otimes_F D'^\vee$. However, this is too *ad hoc* to be useful, so the usefulness rests on the ability to describe this filtration in more direct terms in the language of Hom's: prove that this *ad hoc* definition yields

$$\mathrm{Fil}^i(\mathrm{Hom}_F(D', D)) = \{T \in \mathrm{Hom}_F(D', D) \,|\, T(\mathrm{Fil}^j(D')) \subseteq \mathrm{Fil}^{j+i}(D) \text{ for all } j\}.$$

In other words, $\mathrm{Fil}^i(\mathrm{Hom}_F(D', D)) = \mathrm{Hom}_{\mathrm{Fil}_F}(D', D[i])$ for all $i \in \mathbf{Z}$. (Hint: Compute using bases of D and D' adapted to the filtrations on these spaces.)

There is a natural functor $\mathrm{gr} = \mathrm{gr}^\bullet : \mathrm{Fil}_F \to \mathrm{Gr}_{F,f}$ to the category of finite-dimensional graded F-vector spaces via $\mathrm{gr}(D) = \oplus_i \mathrm{Fil}^i(D)/\mathrm{Fil}^{i+1}(D)$. This functor is dimension-preserving, and it is exact in the sense that if carries short exact sequences in Fil_F (see Definition 6.2.1, especially (6.2.1)) to short exact sequences in $\mathrm{Gr}_{F,f}$. By choosing bases compatible with filtrations we see that the functor gr is compatible with tensor products in the sense that there is a natural isomorphism

$$\mathrm{gr}(D) \otimes \mathrm{gr}(D') \simeq \mathrm{gr}(D \otimes D')$$

in $\mathrm{Gr}_{F,f}$ for any $D, D' \in \mathrm{Fil}_F$, using the tensor product grading on the left side and the tensor product filtration on $D \otimes D'$ on the right side.

6.3. **Filtration on D_{dR}.** For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, the K-vector space $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes V)^{G_K} \in \mathrm{Vec}_K$ has a natural structure of object in Fil_K: since B_{dR} has an exhaustive and separated G_K-stable K-linear filtration via $\mathrm{Fil}^i(B_{\mathrm{dR}}) = t^i B_{\mathrm{dR}}^+$, we get an evident K-linear G_K-stable filtration $\{\mathrm{Fil}^i(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V\}$ on $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$, so this induces an exhaustive and separated filtration on the *finite-dimensional* K-subspace $D_{\mathrm{dR}}(V)$ of G_K-invariant elements. Explicitly,

$$\mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = (t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K}.$$

The finite-dimensionality of $D_{\mathrm{dR}}(V)$ is what ensures that this filtration fills up all of $D_{\mathrm{dR}}(V)$ for sufficiently negative filtration degrees and vanishes for sufficiently positive filtration degrees.

Example 6.3.1. For $n \in \mathbf{Z}$, $D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ is 1-dimensional with its unique filtration jump in degree $-n$ (i.e., gr^{-n} is nonzero).

Proposition 6.3.2. *If V is deRham then V is Hodge–Tate and* $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ *as graded K-vector spaces. In general there is an injection* $\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\mathrm{HT}}(V)$ *and it is an equality of $\mathbf{C}_K$-vector spaces when V is deRham.*

The inclusion in the proposition can be an equality in some cases with V not deRham, such as when $D_{\mathrm{HT}}(V) = 0$ and $V \neq 0$.

Proof. By left exactness of the formation of G_K-invariants, we get a natural K-linear injection

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\mathrm{HT}}(V)$$

for all $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ because $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ as graded $\mathbf{C}_K$-algebras with G_K-action. Thus,

$$\dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}(D_{\mathrm{dR}}(V)) \leq \dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbf{Q}_p}(V)$$

for all V. In the deRham case the outer terms are equal, so the inequalities are all equalities. ∎

In the spirit of the Hodge–Tate case, we say that the *Hodge–Tate weights* of a deRham representation V are those i for which the filtration on $D_{\mathrm{dR}}(V)$ "jumps" from degree i to degree $i+1$, which is to say $\mathrm{gr}^i(D_{\mathrm{dR}}(V)) \neq 0$. This says exactly that the graded vector space $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ has a nonzero term in degree i, which is the old notion of $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ having i as a Hodge–Tate weight. The multiplicity of such an i as a Hodge–Tate weight is the K-dimension of the filtration jump, which is to say $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$.

Since $D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ is a line with nontrivial gr^{-n}, we have that $\mathbf{Q}_p(n)$ has Hodge–Tate weight $-n$ (with multiplicity 1). Thus, sometimes it is more convenient to define Hodge–Tate weights using the same filtration condition ($\mathrm{gr}^i \neq 0$) applied to the contravariant functor $D^*_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V^\vee) = \mathrm{Hom}_{\mathbf{Q}_p[G_K]}(V, B_{\mathrm{dR}})$ so as to negate things (so that $\mathbf{Q}_p(n)$ acquires Hodge–Tate weight n instead).

The general formalism of §5 tells us that D_{dR} on the full subcategory $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}}(G_K)$ is exact and respects tensor products and duals when viewed with values in Vec_K, but it is a stronger property to ask if the same is true as a functor valued in Fil_K. For example, it is clear that when D_{dR} on $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$ is viewed with values in Fil_K then it is a faithful functor, since the forgetful functor $\mathrm{Fil}_K \to \mathrm{Vec}_K$ is faithful and D_{dR} is faithful when valued in Vec_K. However, it is less mechanical to check if the general isomorphism

$$D_{\mathrm{dR}}(V') \otimes_K D_{\mathrm{dR}}(V) \simeq D_{\mathrm{dR}}(V' \otimes_{\mathbf{Q}_p} V)$$

in Vec_K for deRham representations V and V' is actually an isomorphism in Fil_K (using the tensor product filtration on the left side). Fortunately, such good behavior of isomorphisms relative to filtrations does hold:

Proposition 6.3.3. *The faithful functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ carries short exact sequences to short exact sequences and is compatible with the formation of tensor products and duals. In particular, if V is a deRham representation and*

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ (so V' and V'' are deRham) then $D_{\mathrm{dR}}(V') \subseteq D_{\mathrm{dR}}(V)$ has the subspace filtration and the linear quotient $D_{\mathrm{dR}}(V'')$ of $D_{\mathrm{dR}}(V)$ has the quotient filtration.

Once this proposition is proved, it follows that D_{dR} with its filtration structure is compatible with the formation of exterior and symmetric powers (endowed with their natural quotient filtrations as operations on Fil_K).

Proof. For any short exact sequence

$$0 \to V' \to V \to V'' \to 0 \tag{6.3.1}$$

in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ the sequence

$$0 \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V')) \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) \to \mathrm{Fil}^i(D_{\mathrm{dR}}(V'')) \tag{6.3.2}$$

is always left-exact, but surjectivity may fail on the right. However, when V is deRham all terms in (6.3.1) are Hodge–Tate and so the functor D_{HT} applied to (6.3.1) yields an exact sequence. Passing to separate graded degrees gives that the sequence of $\mathrm{gr}^i(D_{\mathrm{HT}}(\cdot))$'s is short exact, but this is the same as the $\mathrm{gr}^i(D_{\mathrm{dR}}(\cdot))$'s since V', V, and V'' are deRham (by Theorem 5.2.1(2)). Hence, adding up dimensions of gr^j's for $j \leq i$ gives

$$\dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V)) = \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V')) + \dim_K \mathrm{Fil}^i(D_{\mathrm{dR}}(V'')),$$

so the left-exact sequence (6.3.2) is also right-exact in the deRham case. This settles the exactness properties for the Fil_K-valued D_{dR}, as well as the subspace and quotient filtration claims.

Now consider the claims concerning the behavior of D_{dR} with respect to tensor product and dual filtrations. By the general formalism of §5 we have K-linear isomorphisms

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V') \simeq D_{\mathrm{dR}}(V \otimes_{\mathbf{Q}_p} V'),\ \ D_{\mathrm{dR}}(V)^\vee \simeq D_{\mathrm{dR}}(V^\vee)$$

for $V, V' \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. The second of these isomorphisms is induced by the mapping

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V^\vee) \simeq D_{\mathrm{dR}}(V \otimes_{\mathbf{Q}_p} V^\vee) \to D_{\mathrm{dR}}(\mathbf{Q}_p) = K[0],$$

and so if the tensor-compatibility is settled then at least the duality comparison isomorphism in Vec_K is a morphism in Fil_K.

The construction of the tensor comparison isomorphism for the Vec_K-valued D_{dR} rests on the multiplicative structure of B_{dR}, so since B_{dR} is a filtered ring it is immediate that the tensor comparison isomorphism in Vec_K for D_{dR} is at least a morphism in Fil_K. In view of the finite-dimensionality and the exhaustiveness and separatedness of the filtrations, this morphism in Fil_K that is known to be an isomorphism in Vec_K is an isomorphism in Fil_K precisely when the induced map on associated graded spaces is an isomorphism. But $\mathrm{gr}(D_{\mathrm{dR}}) = D_{\mathrm{HT}}$ on deRham representations and $\mathrm{gr} : \mathrm{Fil}_K \to \mathrm{Gr}_{K,f}$ is compatible with the formation of tensor products, so our problem is reduced to the Hodge–Tate tensor comparison isomorphism being an isomorphism in $\mathrm{Gr}_{K,f}$ (and not just in Vec_K). But this final assertion is part of Theorem 2.3.9. The same mechanism works for the case of dualities. ■

The following corollary is very useful, and is often invoked without comment.

Corollary 6.3.4. *For $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ and $n \in \mathbf{Z}$, V is deRham if and only if $V(n)$ is deRham.*

Proof. By Example 6.3.1, this follows from the tensor compatibility in Proposition 6.3.3 and the isomorphism $V \simeq (V(n))(-n)$. ■

Example 6.3.5. We now give an example of a Hodge–Tate representation that is not deRham. Consider a *non-split* short exact sequence

$$0 \to \mathbf{Q}_p \to V \to \mathbf{Q}_p(1) \to 0 \tag{6.3.3}$$

in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. The existence of such a non-split extension amounts to the non-vanishing of $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(-1))$, and at least when k is finite such non-vanishing is a consequence of the Euler characteristic formula for H^1's in the $\mathbf{Q}_p$-version of Tate local duality.

We now show that any such extension V is Hodge–Tate. Applying $\mathbf{C}_K \otimes_{\mathbf{Q}_p} (\cdot)$ to (6.3.3) gives an extension of $\mathbf{C}_K(1)$ by $\mathbf{C}_K$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(-1)) = 0$ by the Tate–Sen theorem. Thus, our extension structure on $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V$ is split in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$, so implies $\mathbf{C}_K \otimes_{\mathbf{Q}_p} V \simeq \mathbf{C}_K \oplus \mathbf{C}_K(-1)$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. The Hodge–Tate property for V is therefore clear. However, we claim that such a non-split extension V is *never* deRham!

There is no known elementary proof of this fact. The only known proof rests on very deep results, namely that deRham representations must be *potentially* semistable in the sense of being $B_{\mathrm{st},K'}$-admissible after restriction to $G_{K'}$ for a suitable finite extension K'/K inside of $\overline{K}$, where $B_{\mathrm{st},K'} \subseteq B_{\mathrm{dR},K'} = B_{\mathrm{dR},K}$ is Fontaine's semistable period ring. It is an important fact that the category of $B_{\mathrm{st},K'}$-admissble p-adic representations of $G_{K'}$ admits a fully faithful functor $D_{\mathrm{st},K'}$ into a concrete abelian semilinear algebra category (of weakly admissible filtered (ϕ, N)-modules over K'), and that the Ext-group for $D_{\mathrm{st},K'}(\mathbf{Q}_p(1))$ by $D_{\mathrm{st},K'}(\mathbf{Q}_p)$ in this abelian category can be shown to vanish via an easy calculation in linear algebra. By full faithfulness of $D_{\mathrm{st},K'}$, this would force the original extension structure (6.3.3) on V to be $\mathbf{Q}_p[G_{K'}]$-linearly split. But the restriction map $\mathrm{H}^1(G_K, \mathbf{Q}_p(-1)) \to \mathrm{H}^1(G_{K'}, \mathbf{Q}_p(-1))$ is injective due to $[K' : K]$ being a unit in the coefficient ring $\mathbf{Q}_p$, so the original extension structure (6.3.3) on V in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ would then have to split, contrary to how V was chosen.

Example 6.3.6. To compensate for the incomplete justification (at the present time) of the preceding example, we now prove that for any extension structure

$$0 \to \mathbf{Q}_p(1) \to V \to \mathbf{Q}_p \to 0 \tag{6.3.4}$$

the representation V is always deRham. First we make a side remark that will not be used. If k is finite then by Kummer theory the space of such extension structures has $\mathbf{Q}_p$-dimension $1 + [K : \mathbf{Q}_p]$, and a calculation with weakly admissible filtered (ϕ, N)-modules shows that there is only a 1-dimensional space of such extensions for which V is semistable (i.e., B_{st}-admissible), namely those V's that arise from "Tate curves" over K. Hence, these examples exhibit the difference between the deRham property and the much finer admissibility property with respect to the finer period ring $B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$.

The deRham property for such V is the statement that $\dim_K D_{\mathrm{dR}}(V) = 2$. We have a left exact sequence

$$0 \to D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \to D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(\mathbf{Q}_p)$$

in Fil_K with $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$ and $D_{\mathrm{dR}}(\mathbf{Q}_p)$ each 1-dimensional over K with nonzero gr^{-1} and gr^0 respectively. Hence, our problem is to prove surjectivity on the right, for which it is necessary and sufficient to have surjectivity with Fil^0's, which is to say that we need to prove that the natural map $(B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K} \to (B_{\mathrm{dR}}^+)^{G_K} = K$ is surjective.

Applying $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} (\cdot)$ to the initial short exact sequence (6.3.4) gives a G_K-equivariant short exact sequence of finite free B_{dR}^+-modules, so it admits a B_{dR}^+-linear splitting. The problem is to give such a splitting that is G_K-equivariant, and the obstruction is a *continuous* 1-cocycle on G_K valued in the topological module tB_{dR}^+ endowed with its subspace topology from B_{dR}^+; see Exercise 4.4.7. Hence, it suffices to prove $\mathrm{H}^1_{\mathrm{cont}}(G_K, tB_{\mathrm{dR}}^+) = 0$. Since t is a unit multiple of $\xi = \xi_\pi$ as in Exercise 4.4.7, it follows from that exercise that the multiplication map $t : B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+$ is a closed embedding. Hence, the subspace topology on tB_{dR}^+ coincides with its topology as a free module of rank 1 over the topological ring B_{dR}^+. Likewise, the subspace topology on $t^n B_{\mathrm{dR}}^+$ from B_{dR}^+ for any $n \geq 1$ coincides with its topology as a free module of rank 1, so the G_K-equivariant exact sequence

$$0 \to t^{n+1} B_{\mathrm{dR}}^+ \to t^n B_{\mathrm{dR}}^+ \to \mathbf{C}_K(n) \to 0$$

is topologically exact for $n \geq 1$. Since $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(n)) = 0$ for all $n \geq 1$ by the Tate–Sen theorem, we can use a successive approximation argument with continuous 1-cocycles and the topological identification $tB_{\mathrm{dR}}^+ = \varprojlim tB_{\mathrm{dR}}^+/t^n B_{\mathrm{dR}}^+$ to deduce that $\mathrm{H}^1_{\mathrm{cont}}(G_K, tB_{\mathrm{dR}}^+) = 0$. (Concretely, by successive approximation we exhibit each continuous 1-cocycle as a 1-coboundary.)

An important refinement of Proposition 6.3.3 is that the deRham comparison isomorphism is also filtration-compatible:

Proposition 6.3.7. *For $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$, the G_K-equivariant B_{dR}-linear comparison isomorphism*

$$\alpha : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \simeq B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$$

respects the filtrations and its inverse does too.

Proof. By construction it is clear that α is filtration-compatible, so the problem is to prove that its inverse is as well. It is equivalent to show that the induced B_{HT}-linear map $\mathrm{gr}(\alpha)$ on associated graded objects is an isomorphism. On the right side the associated graded object is naturally identified with $B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$. For the left side, we first recall that (by a calculation with filtration-adapted bases) the formation of the associated graded space of an arbitrary filtered K-vector space (of possibly infinite dimension) is naturally compatible with the formation of tensor products (in the graded and filtered senses), so the associated graded object for the left side is naturally identified with $B_{\mathrm{HT}} \otimes_K \mathrm{gr}(D_{\mathrm{dR}}(V))$.

By Proposition 6.3.2, the deRham representation V is Hodge–Tate and there is a natural graded isomorphism $\mathrm{gr}(D_{\mathrm{dR}}(V)) \simeq D_{\mathrm{HT}}(V)$. In this manner, $\mathrm{gr}(\alpha)$ is naturally identified with the graded comparison morphism

$$\alpha_{\mathrm{HT}} : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$$

that is a graded isomorphism because V is Hodge–Tate. ■

Recall that the construction of B^+_{dR} as a topological ring with G_K-action only depends on $\mathscr{O}_{\mathbf{C}_K}$ endowed with its G_K-action. Thus, replacing K with a discretely-valued complete subfield $K' \subseteq \mathbf{C}_K$ has no effect on the construction (aside from replacing G_K with the closed subgroup $G_{K'}$ within the isometric automorphism group of $\mathbf{C}_K$). It therefore makes sense to ask if the property of $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ being deRham is insensitive to replacing K with such a K', in the sense that this problem involves the same period ring B_{dR} throughout (but with action by various subgroups of the initial G_K).

For accuracy, we now write $D_{\mathrm{dR},K}(V) := (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$, so for a discretely-valued complete extension K'/K inside of $\mathbf{C}_K$ we have $D_{\mathrm{dR},K'}(V) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_{K'}}$. There is an evident map

$$K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$$

in $\mathrm{Fil}_{K'}$ for all $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ via the canonical compatible embeddings of K and K' into the same B^+_{dR} (determined by the embedding of $W(\overline{k})[1/p]$ into B^+_{dR} and considerations with Hensel's Lemma and the residue field $\mathbf{C}_K$).

Proposition 6.3.8. *For any complete discretely-valued extension K'/K inside of $\mathbf{C}_K$ and any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, the natural map $K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$ is an isomorphism in $\mathrm{Fil}_{K'}$. In particular, V is deRham as a G_K-representation if and only if V is deRham as a $G_{K'}$-representation.*

As special cases, the deRham property for G_K can be checked on $I_K = G_{\widehat{K^{\mathrm{un}}}}$ and it is insensitive to replacing K with a finite extension inside of $\mathbf{C}_K$.

Proof. The fields $\widehat{K'^{\mathrm{un}}}$ and $\widehat{K^{\mathrm{un}}}$ have the same residue field $\overline{k}$, so by finiteness of the absolute ramification we see that the resulting extension $\widehat{K^{\mathrm{un}}} \to \widehat{K'^{\mathrm{un}}}$ of completed maximal unramified extensions is of finite degree. Hence, it suffices to separately treat two special cases: K'/K of finite degree and $K' = \widehat{K^{\mathrm{un}}}$. In the case of finite-degree extensions a transitivity argument easily reduces us to the case when K'/K is finite Galois. It is clear from the definitions that for all $i \in \mathbf{Z}$, the finite-dimensional K'-vector space $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ has a natural semilinear action by $\mathrm{Gal}(K'/K)$ whose K-subspace of invariants is $\mathrm{Fil}^i(D_{\mathrm{dR},K}(V))$. Thus, classical Galois descent for vector spaces as in (2.3.3) (applied to K'/K) gives the desired isomorphism result in $\mathrm{Fil}_{K'}$ in this case.

To adapt this argument to work in the case $K' = \widehat{K^{\mathrm{un}}}$, we wish to apply the "completed unramified descent" argument for vector spaces as in the proof of Theorem 2.3.5. It is clear from the definitions that for all $i \in \mathbf{Z}$, the finite-dimensional K'-vector space $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ has a natural semilinear action by $G_K/I_K = G_k$ and the K-subspace of invariants is $\mathrm{Fil}^i(D_{\mathrm{dR},K}(V))$. Hence, to apply the completed unramified descent result we just have to check that the G_k-action on each $\mathrm{Fil}^i(D_{\mathrm{dR},K'}(V))$ is continuous for the natural topology on this finite-dimensional K'-vector space. More generally, consider the G_K-action on $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$. We view this as a free module of finite rank over the topological ring B_{dR}^+ (using the topology from Exercise 4.4.7). It suffices to prove two things: (i) the G_K-action on $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ relative to the finite free module topology is continuous, and (ii) any finite-dimensional K'-subspace of $t^i B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ inherits as its subspace topology the natural topology as such a finite-dimensional vector space (over the p-adic field K'). Note that for the proof of (ii) we may rename K' as K since this does not affect the formation of B_{dR}^+, so it suffices for both claims to consider a common but arbitrary p-adic field K.

For (i), we can use multiplication by t^{-i} and replacement of V by $V(i)$ to reduce to checking continuity of the G_K-action on $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ for any $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Continuity of the G_K-action on V and on B_{dR}^+ then easily gives the continuity of the G_K-action on $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ by computing relative to a $\mathbf{Q}_p$-basis of V. To prove (ii) with $K' = K$, we may again replace V with $V(i)$ to reduce to the case $i = 0$. It is harmless to replace the given finite-dimensional K-subspace of $B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$ with a larger one, so by considering elementary tensor expansions relative to a choice of $\mathbf{Q}_p$-basis of V we reduce to the case when the given finite-dimensional K-vector space has the form $W \otimes_{\mathbf{Q}_p} V$ for a finite-dimensional K-subspace of B_{dR}^+. We may therefore immediately reduce to showing that if $W \subseteq B_{\mathrm{dR}}^+$ is a finite-dimensional K-subspace then its subspace topology from B_{dR}^+ is its natural topology as a finite-dimensional K-vector space.

Pick a K-basis $\{w_1, \dots, w_n\}$ of W, so $w_j = u_j t^{e_j}$ with $e_j \geq 0$ and u_j a unit in B_{dR}^+. Since B_{dR}^+ is a topological K-algebra (see Exercise 4.4.7) and it is Hausdorff with a countable base of opens around the origin, the subspace topology on W is a Hausdorff topological vector space structure with a countable base of opens around the origin. Hence, the problem is just to show that if $x_m = \sum a_{j,m} w_j$ is a sequence in W tending to 0 for the subspace topology as $m \to \infty$ then $a_{j,m} \to 0$ in K for each j. By working in the successive topological quotients $t^i B_{\mathrm{dR}}^+ / t^{i+1} B_{\mathrm{dR}}^+$ for $0 \leq i \leq \max\{e_1, \dots, e_n\}$ (with each $t^i B_{\mathrm{dR}}^+$ given its subspace topology from B_{dR}^+) we reduce to solving the analogous problem for finite-dimensional subspaces of each such quotient space. In Exercise 6.3.6 we saw that multiplication by t^i topologically identifies the subspace topology on $t^i B_{\mathrm{dR}}^+$ with its topology as a free module of rank 1. Hence, $t^i B_{\mathrm{dR}}^+ / t^{i+1} B_{\mathrm{dR}}^+ \simeq \mathbf{C}_K(i)$ *topologically* for all $i \geq 0$.

Our topological problem is now reduced to the classical setting of finite-dimensional K-subspaces W of $\mathbf{C}_K$ (since the Tate twist has no effect on the topological aspects of the problem), as follows. The open subset $L = W \cap \mathscr{O}_{\mathbf{C}_K}$ in W is an $\mathscr{O}_K$-submodule of W that contains a K-basis of W and has no infinitely p-divisible elements. The injective hull of k as an $\mathscr{O}_K$-module is $K/\mathscr{O}_K$, so by the duality properties of injective hulls over local noetherian rings as in pages 146–149 of Matsumura's *Commutative Ring Theory* (or some alternative elementary argument?) it follows that L is finitely generated over the complete discrete valuation ring $\mathscr{O}_K$. The problem is to show that L has its $\mathscr{O}_K$-module topology (i.e., p-adic topology) as its subspace topology from the p-adic topology of $\mathscr{O}_{\mathbf{C}_K}$.

Let $L_n = L \cap p^n \mathscr{O}_{\mathbf{C}_K}$, so clearly L/L_n has finite length (it is killed by p^n since $L \subseteq \mathscr{O}_{\mathbf{C}_K}$) and $\cap L_n = 0$. It is a classical (elementary but non-obvious) result of Chevalley that for any finite module M over any complete local noetherian ring $(A, \mathfrak{m})$ whatsoever, if $\{M_i\}_{i \geq 1}$ is a descending *sequence* of submodules such that each M/M_i has finite length and $\cap M_i = 0$, then the M_i's define the module topology of M (i.e., each M_i contains $\mathfrak{m}^{j(i)}M$ for sufficiently large $j(i)$); presumably this fact admits a short easy proof in the relevant case that A is a complete discrete valuation ring. (Chevalley's result is Exercise 8.7 in Matsumura's book for $M = A$, with a solution in the back of the book that easily adapts to handle general M.) Chevalley's result ensures that the subspace topology on L has no additional open sets beyond those from the finite $\mathscr{O}_K$-module topology. ∎

Example 6.3.9. In the 1-dimensional case, the Hodge–Tate and deRham properties are equivalent. Indeed, we have seen in general that deRham representations are always Hodge–Tate (in any dimension), and for the converse suppose that V is a 1-dimensional Hodge–Tate representation. Thus, it has some Hodge–Tate weight i, so if we replace V with $V(-i)$ (as we may without loss of generality since every $\mathbf{Q}_p(n)$ is deRham) we may reduce to the case when the continuous character $\psi : G_K \to \mathbf{Z}_p^\times$ of V is Hodge–Tate with Hodge–Tate weight 0. Hence, $\mathbf{C}_K(\psi)^{G_K} \neq 0$, so by the Tate–Sen theorem $\psi(I_K)$ is finite. By choosing a sufficiently ramified finite extension K'/K we can thereby arrange that $\psi(I_{K'}) = 1$. Since the deRham property is insensitive to replacing K with $\widehat{K'^{\mathrm{un}}}$, we thereby reduce to the case of the trivial character, which is deRham.

The same argument shows that any finite-dimensional p-adic representation W of G_K with open kernel on I_K is deRham with 0 as the only Hodge–Tate weight, and that $D_{\mathrm{dR}}(W)$ is then a direct sum of copies of the unit object $K[0]$ in Fil_K.

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