# Exponential and logarithmic Ax on semiabelian schemes.

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# Plan

**I .Tori (and constant groups)**

$\rightsquigarrow G = T/\mathbb{C}$ (or $G = G/\mathbb{C}$), base field $K \subset \mathbb{C}(t)^{alg}$

**II . Abelian schemes (and split extensions)**

$\rightsquigarrow G = A/K$ (or $G = T \times A$), constant parts, base fields $K^\sharp$.

**III . Semi-abelian schemes**

$\rightsquigarrow$ constant parts and images

In each case, we will study :

**Exp** : the exponential problem (Ax-Lindemann) : [B.-Pillay]
**Log** : the logarithmic problem (generalized periods) : [André], and [B.-Masser-Pillay-Zannier],

using $\partial = d/dt$ and

dim(diff. Galois group) = $\text{tr.deg.}_K$ [= $\text{tr.deg.}_\mathbb{C}$ - tr.deg.$(K/\mathbb{C})$ !]

Additive extensions will be forced onto us.

## I. Tori (and constant groups)

$S/\mathbb{C}$ an affine curve, $K = \mathbb{C}(S) \subset \mathbb{C}(t)^{alg}$, derivation $\partial \sim d/dt$.
$T = (\mathbb{G}_m)^n \times S$; $LT := Lie(T/S)$ (or over $K$).
$x = \{x_1, ..., x_n\}$ a local analytic section of $LT/S$,
$y = exp_T(x) = \{e^{x_1}, ..., e^{x_n}\}$, $x =$ "$ln_T(y)$".

We say that $x$ is *non-degenerate* (or *not weakly special*) if $\forall (m_1, .., m_n) \in \mathbb{Z}^n \setminus 0, m_1x_1 + ... + m_nx_n \notin \mathbb{C}$, or equivalently : *for any proper algebraic subgroup $H$ of $T$,*

$$x \notin LH + LT(\mathbb{C}).$$

Non-degeneracy of $y \in T$, i.e. $\forall H \neq T, y \notin H$, is an a priori stronger hypothesis, but is here equivalent. Then (**Ax-Schanuel**) $tr.deg.K(x, y)/K \geq n$, implying in its two extreme cases $x \in LT(K)$ (Ax-Lindemann), resp. $y \in T(K)$ ("logarithmic Ax") :

**Theorem (Ostrowski, Kolchin, Ax)**

(**Exp**) $x \in LT(K), \forall H \neq T, x \notin LH + LT(\mathbb{C}), y = exp_T(x) \Rightarrow$
$$tr.deg.K(y)/K = dim(T).$$
(**Log**) $y \in T(K), \forall H \neq T, y \notin H + T(\mathbb{C}), x = ln_T(y) \Rightarrow$
$$tr.deg.K(x)/K = dim(LT).$$

Proof : Picard-Vessiot theory $\rightsquigarrow$

- By rigidity of tori, **Exp** reduces to $\partial ln(K^*) \hookrightarrow K/\partial(K)$.
- **Log** is more involved, requiring : $\mathbb{C} \otimes \partial ln(K^*) \hookrightarrow K/\partial(K)$

But to go from $\mathbb{C}$ to $\mathbb{Z}$, we can alternatively appeal to

*the lattice of periods* $\mathcal{P} = (2\pi i\mathbb{Z})^n \subset \mathbb{C}^n$.

Here, $y \in T(K)$ is the data, the logarithmic derivative $\partial ln_T(y)$ lies in $LT(K)$, and we view the unknown $x = ln_T(y)$ as a solution of

$$\partial x = \partial ln_T(y).$$

Let $N := Gal_\partial(K(x)/K)$. Via $\xi(\sigma) = \sigma(x) - x$, $N$ identifies with a $\mathbb{C}$-subspace of $(LT)^\partial = LT(\mathbb{C})$, and we must show that it fills it up.

$$\begin{array}{ccc} K(x) & & \xi \\ | & \}N & \hookrightarrow \quad (LT)^\partial \\ K & . & \end{array}$$

Now, $(LT)^\partial$ carries a $\mathbb{Z}$-structure, given by the constant local system

$$\mathcal{P} = Ker(exp_T) = (2\pi i\mathbb{Z})^n.$$

Monodromy (action of $\pi_1(S)$ on logs) and regular singularities imply that $N$ is defined over $\mathbb{Q}$, i.e. is an **algebraic** Lie subalgebra : $\exists H \subset T, N = (LH)^\partial$. Finally, apply to the torus $T/H$ the easy

**Theorem ("bi-algebraicity")**

*$y \in T(K)$ and $\partial ln_T(y) \in \partial(LT(K)) \Rightarrow y \in T(\mathbb{C})$.*

## II. Abelian varieties (and split extensions)

Replace $T$ by an abelian scheme $A/S$ of relative dimension $g$, with

$$A_0/\mathbb{C} = S/\mathbb{C}\text{-trace (constant part)}.$$

Then, same conclusion, by merely replacing $LT(\mathbb{C}), T(\mathbb{C})$ by $LA_0(\mathbb{C}), A_0(\mathbb{C})$ in the non-weakly special hypotheses.

The $\mathbb{Z}$-local system of periods $\mathcal{P} = Ker(exp_A)$ is now of rank $2g$ and not constant any longer. Let $\tilde{A}/S$ be the universal vectorial extension : $0 \to W_A \to \tilde{A} \to A \to 0$, of relative rank $2g$, whose Lie algebra $L\tilde{A}/S$ carries the Gauss-Manin connexion $\nabla : L\tilde{A} \to L\tilde{A}$.

The kernel $(L\tilde{A})^\nabla$ of $\nabla$ is generated over $\mathbb{C}$ by the $\mathbb{Z}$-local system $\tilde{\mathcal{P}} = Ker(exp_{\tilde{A}})$, which is **isomorphic** to $\mathcal{P}$. Set

$$K^\sharp_{L\tilde{A}} := K((L\tilde{A})^\nabla) = K(\tilde{\mathcal{P}})$$

$$K^\sharp_{\tilde{A}} = K(Ker(\partial ln_{\tilde{A}})),$$

where $\partial ln_{\tilde{A}} = \nabla \circ ln_{\tilde{A}} : \tilde{A} \to L\tilde{A}$.

## Theorem

(**Exp**) [B.-Pillay] *Let* $x \in LA(K)$ *s.t.* $\forall H \neq A$, $x \notin LH + LA_0(\mathbb{C})$. *Let* $\tilde{x} \in L\tilde{A}(K)$ *be any lift of* $x$, *and let* $\tilde{y} = exp_{\tilde{A}}(\tilde{x})$. *Then,*

$$tr.deg.(K^{\sharp}_{\tilde{A}}(\tilde{y})/K^{\sharp}_{\tilde{A}}) = dim(\tilde{A}) \; (\Rightarrow tr.deg.(K(y)/K) = g).$$

(**Log**) [André] *Let* $y \in A(K)$ *such that* $\forall H \neq A, y \notin H + A_0(\mathbb{C})$. *Let* $\tilde{y} \in \tilde{A}(K)$ *be any lift of* $y$, *and let* $\tilde{x} = ln_{\tilde{A}}(\tilde{y}) \in L\tilde{A}$. *Then,*

$$tr.deg.(K^{\sharp}_{L\tilde{A}}(\tilde{x})/K^{\sharp}_{L\tilde{A}}) = dim(L\tilde{A}) \; (\Rightarrow tr.deg.(K(x)/K) = g).$$

For (**Exp**), we view the unknown $\tilde{y} = exp_{\tilde{A}}(\tilde{x})$ as a solution of the logarithmic DE

$$\partial ln_{\tilde{A}}(\tilde{y}) = \nabla(\tilde{x}),$$

and apply Pillay's Galois theory over the base field $K^{\sharp}_{\tilde{A}}$. Reduce to bi-algebraicity via a **Galois descent**, using that

$$K^{\sharp}_{\tilde{A}} = K^{\sharp}_{U_A \subset W_A} \subset K^{\sharp}_{L\tilde{A}}$$

are Picard-Vessiot extensions of $K$ with semi-simple Galois group.

For (**Log**), a solution of $\nabla(\tilde{x}) = \partial ln_{\tilde{A}}(\tilde{y})$ gives

$$\begin{array}{lccccc} & K^{\sharp}_{L\tilde{A}}(\tilde{x}) & & \xi & \\ & | & \}N & \hookrightarrow & (L\tilde{A})^{\nabla} \\ \Gamma_y\{ & K^{\sharp}_{L\tilde{A}} & & \rho & \\ & | & \}J & \hookrightarrow & \mathrm{Sp}((L\tilde{A})^{\nabla}) \\ & K & , & & \end{array}$$

where $\xi(\sigma) = \sigma(\tilde{x}) - \tilde{x}$, and $J$ acts on $(L\tilde{A})^{\nabla} = \tilde{\mathcal{P}} \otimes \mathbb{C}$ by $\rho$.

**Claim** : $N = (L\tilde{H})^{\nabla}$, where $H$ is an **abelian subvar.** of $A$. Indeed :

(i) $\forall \sigma \in N, \tau \in J$ , $\xi(\tau\sigma\tau^{-1}) = \rho(\tau)\big(\xi(\sigma)\big) \Rightarrow N$ stable under $\nabla$;

(ii) Fuchs theory as in the toric case $\Rightarrow N = (N \cap \tilde{\mathcal{P}}) \otimes \mathbb{C}$.

(iii) Hodge theory $\Rightarrow N \cap \tilde{\mathcal{P}}$ is a sub-VHS of the VHS $\tilde{\mathcal{P}} \simeq \mathcal{P} \Leftrightarrow$ it is stable under the action on $\tilde{\mathcal{P}}$ of the generic Mumford-Tate group $MT(\mathcal{P})$ of $\mathcal{P}$. Indeed :

Let $\Gamma_y^{mono}$ be the $\mathbb{Q}$-Zariski closure of the image of the action of $\pi_1(S)$ on the $\mathbb{Z}$-local system $\mathcal{P}_y$ formed by all logarithms of all multiples of $\tilde{y}$,

**Theorem (André's normality theorem)**

*$\Gamma_y^{mono}$ is a normal subgroup of the generic Mumford-Tate group $MT(\mathcal{P}_y)$ of the variation of mixed Hodge structures $\mathcal{P}_y$.*

This implies that $N \cap \tilde{\mathcal{P}} = Unip(\Gamma_y^{mono}) \subset Unip(\mathcal{P}_y)$ is stable under the action of the quotient $MT(\mathcal{P})$ of $MT(\mathcal{P}_y)$ by its unipotent radical $Unip(\mathcal{P}_y)$.

In fact, $\Gamma_y^{mono}$ is even contained in the derived group of $MT(\mathcal{P}_y)$. More on this later.

Going to the quotient $A/H$, we conclude by **Manin's theorem** :

**Theorem ("bi-algebraicity")**

$\tilde{y} \in \tilde{A}(K)$ *and* $\partial ln_{\tilde{A}}(\tilde{y}) \in \nabla_{L\tilde{A}}(L\tilde{A}(K)) \Rightarrow$ *y lies in* $A_{tor} + A_0(\mathbb{C})$

i.e., $A_{tor} + A_0(\mathbb{C})$ is the kernel of the "classical" Manin map $M_{A,K} : A(K) \to L\tilde{A}(K)/\nabla(L\tilde{A}(K))$, which sends a point $y \in A(K)$, arbitrarily lifted to $\tilde{y} \in \tilde{A}(K)$, to the image of $\partial ln_{\tilde{A}}(\tilde{y})$ in the cokernel of $\nabla$ (over $K$-points).

This is also the set of $K^{diff}$-points of the kernel of the "differential algebraic" Manin map $\mu_A : A \to (\mathbb{G}_a)^g$, which sends a point $y \in A$, arbitrarily lifted to $\tilde{y} \in \tilde{A}$, to the image of $\partial ln_{\tilde{A}}(\tilde{y})$ in $L\tilde{A}/\nabla(W_A)$ $\simeq (\mathbb{G}_a)^g$ (where $W_A =$ vectorial subgroup of $\tilde{A}$).

**Remark** : let $A/K$ be an abelian variety with $A_0 = 0$. Almost by definition, *the* $\mathbb{C}$*-linear extension* $M_{A,K} \otimes 1$ *of* $M_{A,K}$ *is injective*. This is in general **not** true of $\mu_A \otimes 1$ (counterexample with a simple abelian scheme of type IV).

## Conclusion for split extensions $G = T \times A$

In this case, we have semi-simplicity (Poincaré reducibility). Taking the constant parts $G \rightsquigarrow G_0$ is an **exact** functor and the minimal algebraic subgroups in the following statements make sense. We deduce from the previous results :

### Theorem

(**Exp**) [B.-Pillay] *Let $x \in LG(K)$ and let $H$ be the minimal algebraic subgroup of $G$ such that $x \in LH + LG_0(\mathbb{C})$. Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of $x$, and let $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$. Then,*

$$Gal_\partial(K^\sharp_{\tilde{G}}(\tilde{y})/K^\sharp_{\tilde{G}}) = (\tilde{H})^{\partial ln_{\tilde{H}}}.$$

(**Log**) [André] *Let $y \in A(K)$, and let $H$ be the minimal algebraic subgroup of $G$ such that for some positive $N$, $Ny \in H + G_0(\mathbb{C})$. Let $\tilde{y} \in \tilde{G}(K)$ be any lift of $y$, and let $\tilde{x} = ln_{\tilde{G}}(\tilde{y}) \in L\tilde{G}$. Then,*

$$Gal_\partial(K^\sharp_{L\tilde{G}}(\tilde{x})/K^\sharp_{L\tilde{G}}) = (L\tilde{H})^{\nabla_{L\tilde{H}}}.$$

## III. Semi-abelian varieties (well, surfaces)

We will restrict to an $S$-extension of an elliptic scheme $E/S$

$$1 \to \mathbb{G}_m \to G \to E \to 0$$

by $\mathbb{G}_m$, parametrized by a section $q \in Pic^0(E/S) \simeq E(S)$. The universal extension $\tilde{G} = G \times_E \tilde{E}$ of $G$ has relative dimension $2 \times 1 + 1 = 3$, and carries $\nabla : L\tilde{G} \to L\tilde{G}, \partial ln_{\tilde{G}} : \tilde{G} \to L\tilde{G}$. Set :

$$K^{\sharp}_{L\tilde{G}} := K(L\tilde{G})^{\nabla}) = K(\tilde{\mathcal{P}}_q)$$
$$K^{\sharp}_{\tilde{G}} = K(\tilde{G}^{\partial ln_{\tilde{G}}}),$$

$K^{\sharp}_{\tilde{G}}/K^{\sharp}_{\tilde{E}}$ is a compositum of Picard-Vessiot extensions of $\mathbb{G}_m$-type, usually *not* contained in $K^{\sharp}_{L\tilde{G}}$.

Let $E_0, G_0$ be the constant parts of $E, G$. The $S/\mathbb{C}$-image (maximal constant quotient) $G^0$ of $G$ usually does not coincide with $G_0$, i.e. $G \rightsquigarrow G_0$ is **not** exact anymore.

## Theorem

(**Exp**) [B.-Pillay] *Let* $x \in LG(K)$, *projecting to* $\overline{x} \in LE$, *such that* $\forall H \neq G, x \notin LH + LG_0(\mathbb{C})$. *Let* $\tilde{x} \in L\tilde{G}(K)$ *be any lift of* $x$, *and let* $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$. *Then,*

$$tr.deg.(K^{\sharp}_{\tilde{G}}(\tilde{y})/K^{\sharp}_{\tilde{G}}) = \begin{cases} 3 & \text{in general, except} \\ 1 & \text{if } \bar{x} \in LE_0(\mathbb{C}). \end{cases}$$

(**Log**) [B.-Masser-Pillay-Zannier] *Let* $y \in G(K)$, *projecting to* $\overline{y} \in E$, *such that* $\forall H \neq G, y \notin H + G_0(\mathbb{C})$. *Let* $\tilde{y} \in \tilde{G}(K)$ *be any lift of* $y$, *and let* $\tilde{x} = ln_{\tilde{G}}(\tilde{y}) \in L\tilde{G}$. *Then,*

$$tr.deg.(K^{\sharp}_{L\tilde{G}}(\tilde{x})/K^{\sharp}_{L\tilde{G}}) = \begin{cases} 3 & \text{in general, except} \\ 1 & \text{if } N\bar{y} \in End(E)q \ (mod.\ E_0(\mathbb{C})), \text{ except} \\ 0 & \text{if } y \text{ is Ribet } (mod.\ G_0(\mathbb{C})). \end{cases}$$

In view of the non-degeneracy of $x$, the second case of (**Exp**) forces (the non constant) $G$ to be semiconstant.

The last case of (**Log**) occurs when $G$ is semiconstant and $E_0$ has complex multiplications, and provides the only obstruction to the validity of the relative Manin-Mumford conjecture on $G/S$. This case has no (**Exp**) analogue, since for a non constant $G$ and $\overline{x} \in LE(K)$, the point $q \in E(K) \setminus E_0(\mathbb{C})$ cannot be linked to the constant or transcendental point $\overline{y} = exp_E(\overline{x})$.

For a "mixed Shimura" view-point on (**Exp**) and (**Log**) for semi-abelian schemes, see Ziyang Gao's recent papers on André-Oort and beyond. However, the algebraicity hypotheses on $x$ in the (**Exp**) results refer to different ambient spaces.

## The role of constant tensors

Compared with the study of their absolute versions (over number fields), one might infer from the above results that the constant parts and images suffice to control the functional degeneracies of transcendence degrees. In fact, other constant pieces intervene, but visible only in tensor constructions. A precise formulation in the (**Log**) case is provided by the following sharpening of André's normality theorem.

Let $M$ be a mixed motive over $K$, and let $< M >_0$ be the largest tannakian subcategory consisting of constant mixed motives in the tannakian category $< M >$ generated by $M$. Let $G_{mot}(< M >) \to G_{mot}(< M >_0)$ be the associated map between their motivic Galois groups. Then, $\pi_1(S)$ naturally maps into $G_{mot}(< M >)$, and we denote by $\Gamma^{mono}(M)$ the Zariski closure of its image. By regularity, the transcendence degree over $K$ of the field generated by the periods of $M$ is the dimension of $\Gamma^{mono}(M)$.

**Theorem (Nori, Ayoub)**

$\Gamma^{mono}(M)$ *is a normal subgroup of* $G_{mot}(< M >)$*, and the sequence* $1 \to \Gamma^{mono}(M) \to G_{mot}(< M >) \to G_{mot}(< M >_0) \to 1$ *is exact.*

As pointed out by Yves André, a similar exact sequence holds in the context of Mumford-Tate groups. These (and in particular their unipotent radicals) are more amenable to computations than motivic Galois groups (see work of Deligne, Bertolin, Jossen, in the case of 1-motives), and should help in extending the (**Log**) results above to all semi-abelian schemes.

Some open questions :

- (**Exp**) version ?

- differential proof of Pila's theorem on modular Ax-Lindemann (of course, with derivatives $j, j', j''$) ? See work of Scanlon on generalized Schwarzians.

- Ax-Schanuel for a general $G/K$ ?

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