LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

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Lecture 1: Deformations of Representations of pro-finite groups

(1.1) Throughout these notes p will be a rational prime and $\mathbb{F}$ a finite field of characteristic p. We will denote by G be a pro-finite group and $V_{\mathbb{F}}$ a finite dimensional $\mathbb{F}$-vector space equipped with a continuous action of G. We write $n = \dim_{\mathbb{F}} V_{\mathbb{F}}$.

For the first three lectures G will usually be an arbitrary pro-finite group sometimes satisfying a certain finiteness condition. Later we will consider the case where G is the Galois group of a number field or of a finite extension of $\mathbb{Q}_p$.

(1.1.1) Let $\mathfrak{AR}_{W(\mathbb{F})}$ denote the category of finite local, Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. If A is in $\mathfrak{AR}_{W(\mathbb{F})}$ then by a *deformation* of $V_{\mathbb{F}}$ to A we mean a finite free A-module V_A equipped with a continuous action of G, and a G-equivariant isomorphism $V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$.

We define a functor $D_{V_{\mathbb{F}}}$ on $\mathfrak{AR}_{W(\mathbb{F})}$ by setting

$$D_{V_{\mathbb{F}}}(A) = \{\text{isomorphisms classes of deformations of } V_{\mathbb{F}} \text{ to A}\}.$$

Fix an $\mathbb{F}$-basis β of $V_{\mathbb{F}}$. A framed deformation of $V_{\mathbb{F}}$ to A is a deformation V_A of $V_{\mathbb{F}}$ to A together with a lift of the chosen $\mathbb{F}$-basis of $V_{\mathbb{F}}$ to an A-basis β_A of V_A. We define

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\text{isomorphism classes of framed deformations of } V_{\mathbb{F}} \text{ to } A\}$$

Remarks (1.1.2): (1) The fixed basis β allows us to view $V_{\mathbb{F}}$ as a representation

$$\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$$

Then $D^{\square}_{V_{\mathbb{F}}}(A)$ is the set of representations

$$\rho_A : G \to \mathrm{GL}_n(A)$$

lifting $\bar{\rho}$, while $D_{V_{\mathbb{F}}}(A)$ is the set of such representations modulo the action of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ acting by conjugation.

(2) As the definition suggests there are *categories* underlying these functors. Although we postpone the explanation of this, it will be important.

(1.2) A finiteness condition: We say that G satisfies the condition Φ_p if for all finite index subgroups $G' \subset G$ the maximal pro-p quotient of G' is topologically finitely generated. This is equivalent to asking that $\mathrm{Hom}(G, \mathbb{F}_p)$ is finite dimensional.

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Examples of groups satisfying this condition include the absolute Galois group of a finite extension $K/\mathbb{Q}_p$, and the Galois group $\mathrm{Gal}(F_S/F)$ where F is a number field and F_S is a maximal extension of F unramified outside a finite set of primes S of F. Both these groups are frequently encountered in arithmetic applications.

The following proposition is due to Mazur [Ma].

Proposition (1.2.1). *Assume G satisfies Φ_p. Then*

(1) *$D^{\square}_{V_{\mathbb{F}}}$ is pro-representable by a complete local (Noetherian) $W(\mathbb{F})$-algebra $R^{\square}_{V_{\mathbb{F}}}$.*
(2) *If $\mathrm{End}_{\mathbb{F}[G]} V_{\mathbb{F}} = \mathbb{F}$ then $D_{V_{\mathbb{F}}}$ is a pro-representable by a complete local $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}$ called the universal deformation ring of $V_{\mathbb{F}}$.*

Remarks (1.2.2) (1) The condition Φ_p is not really necessary for the proposition by its suppression leads to non-Noetherian rings.

(2) Recall that (pro-)representability (for example for $D^{\square}_{V_{\mathbb{F}}}$ means that there exists an isomorphism

$$D^{\square}_{V_{\mathbb{F}}}(A) \xrightarrow{\sim} \mathrm{Hom}_{W(\mathbb{F})}(R^{\square}_{V_{\mathbb{F}}}, A)$$

which is functorial in A.

In particular, the identity map in $\mathrm{Hom}(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}})$ gives rise to a universal framed deformation over $R^{\square}_{V_{\mathbb{F}}}$.

(3) Note that when the conditions in (2) of the Proposition hold the map of local $W(\mathbb{F})$-algebras $R_{V_{\mathbb{F}}} \to R^{\square}_{V_{\mathbb{F}}}$ is formally smooth, thus the singularities of these two rings are in some sense equivalent.

In general even if $D_{V_{\mathbb{F}}}$ is not representable there is a sense in which it has an intrinsic geometry. However this is best formulated in terms of groupoids and we postpone it till later.

Proof. Proof of (i). Let $G' = \ker(\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F}))$. For any lift $\rho_A : G \to \mathrm{GL}_n(A)$ of $\bar{\rho}$, $\rho_A|_{G'}$ factors through $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ which is a pro-p group, and hence $\rho_A|_{G'}$ factors through the maximal pro$-p$ quotient of G'. Write this quotient as G'/H for H a normal subgroup of G', which is also normal in G.

By assumption, G'/H is topologically finitely generated and hence so is G/H. Let $\gamma_1, \dots, \gamma_s$ be topological generators of G/H. Then $R^{\square}_{V_{\mathbb{F}}}$ is a quotient of $W(\mathbb{F})[\![X_i^{j,k}]\!]$ where $i = 1, \dots, s$ and $k, j = 1, \dots, n^2$. and the universal deformation is given by

$$\gamma_i \mapsto (X_i^{j,k}) + [\rho](\gamma_i)$$

where $[\rho](\gamma_i) \in \mathrm{GL}_n(W(\mathbb{F}))$ denotes the matrix whose entries are the Teichmïler representatives of the entries of $\rho(\gamma_i)$.

The second part of the proposition will be proved later. We will give a somewhat different proof to the original one of Mazur which uses Schlessinger's representability criterion. To explain the idea let $\widehat{\mathrm{PGL}}_n$ denote the completion of the group PGL_n over $W(\mathbb{F})$ along its identity section. Then $\widehat{\mathrm{PGL}}_n$ acts on the functor $D^{\square}_{V_{\mathbb{F}}}$ by conjugation and hence it acts on the formal scheme $\mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}}$. The condition $\mathrm{End}_{\mathbb{F}[G]} V_{\mathbb{F}} = \mathbb{F}$ implies that this action is free, and the idea is to define

$$\mathrm{Spf}\, R_{V_{\mathbb{F}}} = \mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}}/\widehat{\mathrm{PGL}}_n$$

(1.3) Tangent Spaces: Let $\mathbb{F}[\epsilon] = \mathbb{F}[X]/X^2$ denote the dual numbers.

Lemma (1.3.1).

(1) *There is a canonical isomorphism*

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \xrightarrow{\sim} H^1(G, \mathrm{ad}V_{\mathbb{F}})$$

where $\mathrm{ad}V_{\mathbb{F}}$ *denotes the* G*-representation* $\mathrm{End}_{\mathbb{F}}V_{\mathbb{F}}$.

(2) *If* G *satisfies* Φ_p *then* $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ *is a finite dimensional* $\mathbb{F}$*-vector space and satisfies*

$$\begin{aligned}\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) &= \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) + n^2 - \dim_{\mathbb{F}}(\mathrm{ad}V_{\mathbb{F}})^G \\ &= n^2 + H^1(G, \mathrm{ad}V_{\mathbb{F}}) - H^0(G, \mathrm{ad}V_{\mathbb{F}}).\end{aligned}$$

Proof. An element of $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ $V_{\mathbb{F}[\epsilon]}$ gives rise to an extension

$$0 \to V_{\mathbb{F}} \to V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}} \to 0$$

where we have identified $\epsilon \cdot V_{\mathbb{F}[\epsilon]}$ with $V_{\mathbb{F}}$. Thus we obtain a class in

$$\mathrm{Ext}^1_G(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\sim} H^1(G, \mathrm{ad}V_{\mathbb{F}}).$$

Conversely an element of $H^1(G, \mathrm{ad}V_{\mathbb{F}})$ gives rise to an extension of one copy of $V_{\mathbb{F}}$ by another $V_{\mathbb{F}}$, and such an extension can be viewed as an $F[\epsilon]$-module, with multiplication by ϵ identifying the two copies of $V_{\mathbb{F}}$.

To prove the second part of the lemma, fix a deformation of $V_{\mathbb{F}}$ to $\mathbb{F}[\epsilon]$ $V_{\mathbb{F}[\epsilon]}$. The set of $\mathbb{F}[\epsilon]$-bases of $V_{\mathbb{F}[\epsilon]}$ lifting a fixed basis of $V_{\mathbb{F}}$ is an $\mathbb{F}$-vector space of dimension n^2. Let β', β'' be two such lifted bases. Then there is an isomorphism of framed deformations

$$(V_{\mathbb{F}[\epsilon]}, \beta') \xrightarrow{\sim} (V_{\mathbb{F}[\epsilon]}, \beta'')$$

if and only if there is an automorphism of $V_{\mathbb{F}[\epsilon]}$ which is the identity mod ϵ and takes β' to β''. That is, the fibres of

$$D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \to D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$$

are $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^G$ torsors. The lemma follows. □

(1.4) Traces: Absolutely irreducible representations of finite groups are determined by their trace functions. A result of Carayol [Ca] and Mazur [Ma] says that the analogous result holds also for deformations:

Theorem (1.4.1). *(Mazur, Carayol): Suppose that* $V_{\mathbb{F}}$ *is absolutely irreducible. If* A *is in* $\mathfrak{AR}_{W(\mathbb{F})}$ *and* $V_A, V'_A \in D_{V_{\mathbb{F}}}(A)$ *are deformations such that* $\mathrm{tr}(\sigma|V_A) = \mathrm{tr}(\sigma|V'_A)$ *for all* $\sigma \in G$*, then* V_A *and* V'_A *are isomorphic deformations.*

Proof. We give (one of) Carayol's argument(s).

Fix bases for V_A and V'_A and extend the resulting representations to A-linear maps

$$\rho_A, \rho'_A : A[G] \to M_n(A).$$

We have to show that the bases can be chosen so that $\rho_A = \rho'_A$.

Let $\mathfrak{m}_A$ be the radical of A, and $I \subset A$ an ideal such that $I \cdot \mathfrak{m}_A = 0$. By induction on the length of A, we may assume that $\rho_A = \rho'_A$ modulo I, and write $\rho_A = \rho'_A + \delta$ where for $\sigma \in A[G]$, $\delta(\sigma) \in M_n(I)$ has trace 0.

As ρ_A, ρ'_A are multiplicative we find that for $\sigma_1, \sigma_2 \in A[G]$

$$\delta(\sigma_1\sigma_2) = \bar{\rho}(\sigma_1)\delta(\sigma_2) + \delta(\sigma_1)\bar{\rho}(\sigma_2).$$

If $\sigma_2 \in \ker \bar{\rho}$, we get $\delta(\sigma_1\sigma_2) = \bar{\rho}(\sigma_1)\delta(\sigma_2)$ for all $\sigma \in A[G]$, so $\mathrm{tr}(\bar{\rho}(\sigma_1)\delta(\sigma_2)) = 0$ for all $\sigma_1 \in A[G]$. But by Burnside's theorem $\bar{\rho}(\mathbb{F}[G]) = M_n(\mathbb{F})$ as $\bar{\rho}$ is absolutely irreducible. Hence $\mathrm{tr}(X\delta(\sigma_2)) = 0$ for any $X \in M_n(\mathbb{F})$, so $\delta(\sigma_2) = 0$.

It follows that δ may be regarded as a *derivation*

$$\delta : M_n(\mathbb{F}) \to M_n(I) := M_n(\mathbb{F}) \otimes_{\mathbb{F}} I$$

on $M_n(\mathbb{F})$. Such a derivation is always *inner*:

$$\exists U \in M_n(I) \text{such that} \delta(\sigma) = \bar{\rho}(\sigma)U - U\bar{\rho}(\sigma).$$

Hence $\rho'_A = (1 - U)\rho_A(1 + U)$. $\square$

(1.5) Exercises:

Exercise 1: Show that the following are equivalent:

(1) For all open subgroups $G' \subset G$ the maximal pro-p quotient of G' is topologically finitely generated
(2) For all $G' \subset G$ as above $\mathrm{Hom}(G', \mathbb{F}_p)$ is finite dimensional over $\mathbb{F}_p$.
(3) For all $G' \subset G$ as above, and all continuous representations of G' on a finite dimensional $\mathbb{F}$-vector space W, $H^1(G', W)$ is finite dimensional

Exercise 2: Show that

$$\mathrm{Ext}^1_G(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\sim} H^1(G, \mathrm{ad}V_{\mathbb{F}}).$$

Exercise 3:

(1) Give an example where $V_{\mathbb{F}}$ is not absolutely irreducible and there exist non-isomorphic deformations $V_A, V'_A \in D_{V_{\mathbb{F}}}(A)$ with the same traces. (Hint: Consider two character $\chi_1, \chi_2 : G \to F^\times$ with $\dim_{\mathbb{F}}\mathrm{Ext}^1(\chi_1, \chi_2) > 1$.)
(2) Show that if $\chi_1, \chi_2 : G \to \mathbb{F}^\times$ are distinct characters with such that $\mathrm{Ext}^1(\chi_1, \chi_2)$ are 1-dimensional, and $V_{\mathbb{F}}$ is an extension of chi_1 by χ_2 then the analogue of Carayol's theorem hold for $V_{\mathbb{F}}$: any deformation of $V_{\mathbb{F}}$ is determined by its trace.

Exercise 4: Suppose $R_{V_{\mathbb{F}}}$ pro-represents $D_{V_{\mathbb{F}}}$. Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : R_{V_{\mathbb{F}}}[1/p] \to E$ am E-valued point such that the ideal $\ker x$ has residue field E. Specializing the universal representation over $R_{V_{\mathbb{F}}}$ by x produces a representation of G on a finite dimensional E-vector space V_x.

Let $\widehat{R}_x$ denote the complete local ring at the point $\ker x \in \mathrm{Spec}\, R_{V_{\mathbb{F}}}[1/p]$. Show that $\widehat{R}_x$ is the universal deformation ring of V_x.

Formulate and prove the analogous statement for $R^{\square}_{V_{\mathbb{F}}}$.

References

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Lecture 2: Pseudo-representations

(2.1) we saw in the previous lecture that if $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible then its deformations are determined by their traces.

The idea of pseudo-representations, introduced by Wiles [Wi] for odd two dimensional representations and by Taylor [Ta] for an arbitrary group, is to try to characterize those functions on G which are traces and to study deformation theory via deformations of the trace functions.

Definition (2.1.2). *Let R be any (topological) ring. An E-valued (continuous) pseudo-representation of dimension $d \in \mathbb{N}^+$ is a continuous function $T : G \to R$ such that*

1. *$T(1) = d$.*
2. *$T(g_1 g_2) = T(g_1)T(g_2)$ for $g_1, g_2 \in G$.*
3. *If S_{d+1} denotes the symmetric group on $d+1$ letters and $\epsilon : S_{d+1} \to \{\pm 1\}$ denotes its sign character then*

$$\sum_{\sigma \in S_{d+1}} \varepsilon(\sigma) T_\sigma(g_1, \dots, g_{d+1}) = 0$$

for $g_1, \dots, g_{d+1} \in G$. Here if $\sigma \in S_{d+1}$ has cycle decomposition

$$\sigma = (i_1^{(1)}, \dots, i_{i_{r_1}}^{(1)}) \dots (i_1^{(s)}, \dots, i_{r_s}^{(1)})$$

then $T_\sigma : G^{d+1} \to R$ is the function

$$(g_1, \dots, g_{d+1}) \mapsto T(g_{i_1^{(1)}} \dots g_{r_1}^{(1)}) \dots T(g_{i_1}^{(s)} \dots g_{i_{r_s}}^{(s)}).$$

It will often be convenient to form the R-linear extension of a pseudo-representation $T : R[G] \to R$. The relations in the definition are then satisfied for $g_1, \dots, g_{d+1} \in R[G]$.

Theorem (2.1.3). *(Taylor)*

1. *If $\rho : G \to \mathrm{GL}_d(R)$ is a representation then $\mathrm{tr}\rho$ is a pseudo-representation of dimension d.*
2. *If R is an algebraically closed field of characteristic 0 and T is a pseudo-representation of dimension d, then there exists a unique semi-simple representation $\rho : G \to \mathrm{GL}_d(R)$ with $\mathrm{tr}\rho = T$.*

(3) *If G is (topologically) finitely generated then for every integer $d \geq 1$, there is a finite subset $S \subset G$, such that a pseudo-representation $T : G \to R$ of dimension d valued in an $\mathbb{Z}[1/d]$-algebra R is determined by its restriction to S.*

Proof. Taylor proves this using results of Procesi on invariant theory [Pr]. A self contained proof was given by Rouquier [Ro]. We sketch some of the arguments for (1) here.

For $g_1, \dots, g_{d+1} \in M_d(R)$ let

$$\Theta(g_1, \dots, g_{d+1}) = \sum_{\sigma \in S_{d+1}} \varepsilon(\sigma) T_\sigma(g_1, \dots, g_{d+1}).$$

We want to show $\Theta \equiv 0$. Writing R as a quotient of a domain of characteristic 0, one sees it is enough to consider such rings, and then that it suffices to consider the case of R a field of characteristic 0. Thus we assume this from now on.

Let $V = R^d$ and $W = V \otimes V^* = \operatorname{End} V$. Note that Θ is invariant by the action of S_{d+1}. Hence if we extend Θ to a multi-linear map $\Theta : W^{\otimes d+1} \to R$, it is determined by its value on $\mathrm{Sym}^{d+1} W \subset W^{\otimes d+1}$. Since $\mathrm{Sym}^{d+1} W$ is spanned by the image of the diagonal map

$$\Delta : W \to W^{\otimes d+1}; \quad w \mapsto w \otimes w \otimes \cdots \otimes w$$

it suffices to show that $\Theta(\Delta(w)) = 0$ for all $w \in W$.

As the semi-simple elements in $\mathrm{Aut} V$ are Zariski dense in W, it is enough to show $\Theta(\Delta(w)) = 0$ for w semi-simple.

Choose a basis for V in which w is diagonal. Then a simple computation shows that $w \sum_{\sigma \in S_{d+1}} \varepsilon(\sigma)\sigma$ acting on $V^{\otimes d+1}$ has trace $\Theta(\Delta(w))$. Here S_{d+1} acts on $V^{\otimes d+1}$ by permutation of the factors in the tensor product, while w acts as $\Delta(w)$. Since

$$(\sum_{\sigma \in S_{d+1}} \varepsilon(\sigma)\sigma) V^{\otimes d+1} \subset \wedge^{d+1} V = 0,$$

the proposition follows. □

(2.2) Deformations: Let $\tau : G \to \mathbb{F}$ be a pseudo-representation. For A in $\mathfrak{AR}_W(\mathbb{F})$ define

$$D_\tau(A) = \{\text{pseudo-representations } \tau_A : G \to A \text{ lifting } \tau\}.$$

We suppose from now on that $p \nmid d!$.

Proposition (2.2.1). *Suppose G satisfies Φ_p. Then D_τ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra R_τ.*

This would be immediate of G were topologically finitely generated. In general we need some preparation.

Definition (2.2.2). *For any pseudo-representation $T : G \to R$ define*

$$\ker T = \{h \in G : T(gh) = T(g) \text{for all } g \in G\}.$$

If we view T as an R-linear map $\tilde{T} : R[G] \to R$ then we set

$$\ker \tilde{T} = \{h \in R[G] : T(gh) = 0 \text{for all } g \in G.\}$$

Lemma (2.2.3). *Let A be in $\mathfrak{AR}_{W(\mathbb{F})}$ and $\tau_A \in D_\tau(A)$. If $G' = \ker\tau$, and G'/H is the maximal pro-p quotient of G', then $\ker\tau_A$ contains H.*

Proof. If $\ker\tau_A$ does not contain H, then the finite group $G'/\ker(\tau_A|_{G'})$ contains a non-trivial element of order coprime to p $\gamma \in G'/\ker(\tau_A|_{G'})$.

Suppose $g \in A[G']$ satisfies $\tau_A(g^i)^2 = 0$ for $i \geq 1$. Then taking $g_1 = g_2 = \cdots = g$ in (2.1.2)(3), we find $tau_A(g^{d+1}) = 0$. Taking $g_1 = h$, $g_2 = g_3 = \cdots = g^{d+1}$ in (2.1.2)(3) we get $\tau_A(hg^{(d+1)^2}) = 0$ for all $h \in A[G']$. Hence $g^{(d+1)^2} \in \ker\tilde{\tau}_A$.

By induction on the length of A, we may assume that $\tilde{\tau}_A((\gamma-1)^i)\mathfrak{m}_A = 0$ for $i \geq 1$. Hence $(\gamma-1)^{(d+1)^2} \in \ker\tilde{\tau}_A$. But since γ has prime to p order, $(\gamma-1)A[\langle\gamma\rangle]$ is an idempotent ideal, so $\gamma - 1 \in \ker\tilde{\tau}_A$ and $\gamma \in \ker\tau_A$. □

Proof. By Lemma (2.2.3), we map consider τ_A as a pseudo-representation of G/H. This is a topologically finitely generated group so the proposition follows from (2.1.3)(3). □

(2.3) Relationship between pseudo-representations and representations:

For absolutely irreducible representations we have the following

Theorem (2.3.1). *(Nyssen-Rouquier) Suppose that G satisfies Φ_p and that $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}})$ is absolutely irreducible. IF $\bar{\tau} = \mathrm{tr}(\bar{\rho})$, then there is an isomorphism of functors on $\mathfrak{AR}_{W(\mathbb{F})}$, $D_{V_{\mathbb{F}}} \xrightarrow{\sim} D_{\bar{\tau}}$.*

(2.3.2) For any $\bar{\rho} : G \to \mathrm{GL}(V_{\mathbb{F}})$ and $\bar{\tau} - \mathrm{tr}(\bar{\rho})$, one has, of course the morphism of functors $D^{\Box}_{V_{\mathbb{F}}} \to D_{\bar{\tau}}$, however there is something more interesting:

Suppose for example that $\chi_1, \chi_2 : G \to \mathbb{F}^times$ are characters and $c_1, c_2 \in \mathrm{Ext}^1(\chi_2, \chi_1)$. Then $\begin{pmatrix} \chi_1 & c+1+Tc_2 \\ 0 & \chi_2 \end{pmatrix}$ is a representation $G \to \mathrm{GL}_n(\mathbb{F}[T])$. More naturally, one obtains a family of representations of G over $\mathbb{P}(\mathrm{Ext}^1(\chi_1, \chi_2))$, the projectivization of $\mathrm{Ext}^1(\chi_2, \chi_1)$, all have the pseudo-character $\chi_1 + \chi_2$.

(2.3.3) To fully express the relationship between representations and pseudo-representations it will be convenient to work with groupoids see the appendix of [Ki] for a summary of what is needed. Briefly, if $\mathcal{C}$ is a category, a groupoid over $\mathcal{C}$ is a morphism of categories $\pi : \mathcal{F} \to \mathcal{C}$. The morphism π is required to satisfy certain axioms, the most important of which is that a morphism in $\mathcal{F}$ covering an identity map in $\mathcal{C}$ is an isomorphism.

We will often specify groupoids by specifying the categories $\pi^{-1}(c)$ for $c \in \mathcal{C}$. For example, let $\mathcal{C} = \mathfrak{AR}_{W(\mathbb{F})}$, and $D_{V_{\mathbb{F}}}(A)$ the *category* of deformations of $V_{\mathbb{F}}$ to a finite free A-module V_A. Then the $D_{V_{\mathbb{F}}}(A)$ form a groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$; for $\psi : A \to A'$ a map in $\mathfrak{AR}_{W(\mathbb{F})}$ a morphism in $D_{V_{\mathbb{F}}}$ covering ψ is an isomorphism $V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ where $V_A, V_{A'}$ are deformations of $V_{\mathbb{F}}$ to A and A' respectively.

Previously we also denoted by $D_{V_{\mathbb{F}}}$ the functor of isomorphism classes of the groupoid, which assigns to A in $\mathfrak{AR}_{W(\mathbb{F})}$ the set of isomorphism classes of $D_{V_{\mathbb{F}}}$. When $V_{\mathbb{F}}$ has non-trivial automorphisms, then so do the objects in the categories $D_{V_{\mathbb{F}}}(A)$. In this situation and groupoid $D_{V_{\mathbb{F}}}$ captures the geometry of the deformation theory of $V_{\mathbb{F}}$ more accurately than its functor of isomorphism classes.

(2.3.4) For a pseudo-representation $\bar{\tau} : G \to \mathbb{F}$.

Let $\mathfrak{Aug}_{W(\mathbb{F})}$ denote the category of pairs (A, B) where A is in $\mathfrak{AR}_{W(\mathbb{F})}$ and B is an A-algebra. We consider such a B with the topology induced by the radical $\mathfrak{m}_A$ of A.

Define a groupoid $\mathrm{Rep}_{\bar{\tau}}$ on $\mathfrak{Aug}_{W(\mathbb{F})}$ by setting $\mathrm{Rep}_{\bar{\tau}}(B)$ equal to the category of finite free B-modules V_B equipped with a continuous action of G such that $\mathrm{tr}(\sigma | V_B \otimes_A A/\mathfrak{m}_A) = \bar{\tau}(\sigma)$ for $\sigma \in G$.

Similarly we define $\mathrm{Rep}^{\square}_{\bar{\tau}}(B)$ to be the category are pairs (V_B, β) where V_B is in $\mathrm{Rep}_{\bar{\tau}}(B)$ and β is a basis for V_B.

Finally we extend $D_{\bar{\tau}}$ to a groupoid on $\mathfrak{Aug}_{W(\mathbb{F})}$ by setting

$$D_{\bar{\tau}}(A, B) = \lim_{A' \subset B} D_{\bar{\tau}}(A')$$

where the limit runs over subalgebras $A' \subset B$ which are local, Artinian, A-algebras. This limit is also equal to the set of continuous pseudo-representations $\tau_B : G \to B$ such that $\tau_B \otimes_B B/\mathfrak{m}_A B = \bar{\tau}$.

Lemma (2.3.5). *If G satisfies Φ_p, then $\mathrm{Rep}^{\square}_{\bar{\tau}}$ is representable by a formal scheme over $\mathrm{Spf}\, R_{\bar{\tau}}$ which is formally of finite type.*

Proof. This is an exercise. See the exercises below for the definition of representability of a groupoid. $\square$

(2.4) Exercises:

Exercise 1: Check carefully all the steps in the proof of Proposition (2.1.3)(1).

Exercise 2: If $\pi : \mathcal{F} \to \mathcal{C}$is a groupoid and $\xi \in \mathrm{Ob}(\mathcal{F})$ define a groupoid $\tilde{\xi} \to \mathcal{C}$ as follows: An object of $\tilde{\xi}$ is a morphism $\xi \to \eta$ in $\mathcal{F}$, and $\tilde{\xi} \to \mathcal{C}$ is given by $(\xi \to \eta) \mapsto \pi(\eta)$.

We say that ξ represents $\mathcal{F}$ if

$$\tilde{\xi} \to \mathcal{F}; \quad (\xi \to \eta) \mapsto \eta$$

is an equivalence of categories. If such a ξ exists we say that $\mathcal{F}$ is representable.

Show that if $\mathcal{F}$ is representable then $\mathrm{Aut}(\eta) = \mathrm{id}$ for all η in $\mathrm{Ob}(\mathcal{F})$.

Exercise 3: If $\epsilon' : \mathcal{F}' \to \mathcal{F}$ and $\epsilon'' : \mathcal{F}'' \to \mathcal{F}$ are morphisms of categories, let $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}''$ be the category whose objects are triples (η', η'', θ) where $\eta' \in \mathrm{Ob}(\mathcal{F}')$, $\eta'' \in \mathrm{Ob}(\mathcal{F}'')$ and θ is an isomorphism $\theta : \epsilon'(\eta') \xrightarrow{\sim} \epsilon''(\eta'')$.

For examples if $\mathcal{F}' \to \mathcal{F}$ is a morphism of groupoids over $\mathcal{C}$ and $\xi \in \mathcal{F}$ we can form $\mathcal{F}'_{\xi} = \mathcal{F}' \times_{\mathcal{F}} \tilde{\xi}$.

Let S be a scheme. Then we may consider an S-scheme as a groupoid over S-schemes, using the construction in Exercise 2. If $X \to Y$ and $X' \to Y'$ be morphisms of S-schemes, show that there is an isomorphism $\tilde{X} \times_{\tilde{Y}} \tilde{X}' \xrightarrow{\sim} (X \times_Y X')^{\sim}.xs$

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LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

Mark Kisin

Lecture 3: Representability

(3.1) Quotients by finite (formal) group actions are often representable, and indeed there are general results which guarantee this in certain situations.

In this section we assume that G satisfies the condition Φ_p. We begin with a result from Lecture 1, whose proof had been postponed.

Theorem (3.1.1). *Suppose* $\mathrm{End}_{\mathbb{F}[G]} V_{\mathbb{F}} = \mathbb{F}$. *Then* $D_{V_{\mathbb{F}}}$ *is representable.*

Proof. We already saw that $D^{\square}_{V_{\mathbb{F}}}$ is representable by $\mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}} =: X_{V_{\mathbb{F}}}$ where $R^{\square}_{V_{\mathbb{F}}}$ is a complete local $W(\mathbb{F})$-algebra.

Let $\widehat{\mathrm{PGL}}_d$ denote the formal completion of the $W(\mathbb{F})$-group scheme PGL_d along its identity section. Then $\widehat{\mathrm{PGL}}_d$ acts on $X_{V_{\mathbb{F}}}$ and we have

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}; \quad (x, g) \mapsto (x, gx).$$

The action of $\widehat{\mathrm{PGL}}_d$ on $X_{V_{\mathbb{F}}}$ is *free* which means that the induced map

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \to X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}}.$$

is a closed immersion.

We would like to take the quotient of $X_{V_{\mathbb{F}}}$ by this action. To do this we need a little preparation.

(3.1.2) Let $\widehat{\mathfrak{AR}}_W$ denote the category of complete local Noetherian $W(\mathbb{F})$-algebras, so the opposite category $(\widehat{\mathfrak{AR}}_W)^{\circ}$ is equivalent to the category of formal spectra of such $W(\mathbb{F})$-algebras.

An *equivalence relation* $R \rightrightarrows X$ in $(\widehat{\mathfrak{AR}}_W)^{\circ}$ is a pair of morphisms such that

(1) $R \to X \times X$ is a closed embedding.
(2) For all T in $(\widehat{\mathfrak{AR}}_W)^{\circ}$ $R(T) \subset (X \times X)(T)$ is an equivalence relation.

For example let G be a group object in $(\widehat{\mathfrak{AR}}_W)^{\circ}$, and $G \times X \to X$ a free action. Then the map

$$G \times X \rightrightarrows X; \quad (g, x) \mapsto (x, gx)$$

is an equivalence relation.

A flat morphism $X \to Y$ in $(\widehat{\mathfrak{AR}}_W)^{\circ}$ is said to be a quotient of X by R, if the embedding $R \to X \times X$ induces an isomorphism $R \xrightarrow{\sim} X \times_Y X$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem (3.1.3). *(SGA 3, VIIb, Thm. 1.4): Let* $R \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ *be an equivalence relation in* $(\widehat{\mathfrak{AR}_W})^\circ$ *such that the first projection* $R \to X$ *is flat. Then the quotient of* X *by* R *exists. If* $X = \operatorname{Spf} B$ *and* $R = \operatorname{Spf} C$*, then* $X/R = \operatorname{Spf} A$*, where*

$$A = \{b \in B : p_0^*(b) = p_1^*(b)\}.$$

We can now complete the proof of Theorem (3.1.1) by applying (3.1.3) to the equivalence relation $X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}$. $\square$

(3.2) Now fix a pseudo-representation $\bar\tau : G \to \bar\tau$. We now want to construct some representable subgroupoids of $\mathrm{Rep}_{\bar\tau} \to D_{\bar\tau}$.

Suppose that for $i = 1, \dots, s$ $\bar\rho_i : G \to \mathrm{GL}_{d_i}(\mathbb{F})$ are pairwise distinct absolutely irreducible representation of G, such that $\bar\tau = \sum_{i=1}^s \bar\rho_i$.

Let $\mathrm{Rep}'_{\bar\tau} \subset \mathrm{Rep}_{\bar\tau}$ be the full subgroupoid of $\mathrm{Rep}_{\bar\tau}$ such that $\mathrm{Rep}'_{\bar\tau}(A, B)$ consists of the objects V_B in $\mathrm{Rep}_{\bar\tau}(A, B)$ such that

$$V_B \otimes_B B/\mathfrak{m}_A B \sim \begin{pmatrix} \bar\rho_1 & c_1 & \cdots & \cdots \\ & \bar\rho_2 & c_2 & \cdots \\ & & \cdots & \cdots \\ & & & \bar\rho_s \end{pmatrix}. \tag{3.2.1}$$

where for $i = 1, \dots, s-1$ c_i is a *non-trivial* extension of $\bar\rho_{i+1}$ by $\bar\rho_i$.

Note that the isomorphism in (3.2.1) is uniquely determined as out conditions imply that the representation on the right has no non-trivial automorphisms.

Theorem (3.2.2). *The groupoid* $\mathrm{Rep}'_{\bar\tau} \to D_{\bar\tau}$ *is representable by a proper formal scheme over* $\operatorname{Spf} R_{\bar\tau}$.

Proof. Let

$$\mathrm{Rep}^{\square,\prime}_{\bar\tau} = \mathrm{Rep}'_{\bar\tau} \times_{\mathrm{Rep}_{\bar\tau}} \mathrm{Rep}^{\square}_{\bar\tau} \subset \mathrm{Rep}^{\square}_{\bar\tau}.$$

This is a locally closed subspace. The group PGL_d acts freely on $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ as $V_B \in \mathrm{Rep}'_{\bar\tau}(A, B)$ has no non-trivial automorphisms. To take the quotient we need the following

Theorem (3.2.3). *Let* $S = \operatorname{Spec} A$*, with* A *a local Artin ring, and let* X/S *be a finite type* S*-scheme equipped with a free action by a reductive group* G/S.[1] *Suppose that every* $x \in X$ *is contained in an affine,* G*-stable open subset of* X*. Then the quotient* X/G *exists.*

x When S is a field this is explained in Mumford's book [GIT, Ch 1, §4, Prop 1.9]. The general case will be explained by Brian Conrad in another lecture.

We want to apply the theorem to the quotient $\mathrm{Rep}^{\square,\prime}_{\bar\tau}/\mathrm{PGL}_d$.

For simplicity we will consider only the case when $d = 2$ and $\bar\rho_1, \bar\rho_2$ are characters, which we will denote by χ_1 and χ_2 respectively. We have to check the condition that every point of $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ has an affine PGL_d-stable neighbourhood.

Let $U \subset \mathrm{Ext}^1(\chi_1, \chi_2)\backslash\{0\}$ be affine open and define $\mathrm{Rep}^{\square,U}_{\bar\tau} \subset \mathrm{Rep}^{\square,\prime}_{\bar\tau}$ the subgroupoid consisting of those (V_B, β) in $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ such that $V_B \xrightarrow{\sim} \begin{pmatrix} \chi_1 & c \\ 0 & \chi_2 \end{pmatrix}$ with $c \in U$. Then $\mathrm{Rep}^{\square,U}_{\bar\tau}$ is stable by PGL_2, and its fibre over the closed point of $\operatorname{Spf} R_{\bar\tau}$

[1] A reductive group over a base S is a smooth group scheme with reductive fibres.

is isomorphic to U. In particular $\mathrm{Rep}_{\bar{\tau}}^{\Box,U}$ is affine, and we may apply (3.2.3) with $A = R_{\bar{\tau}}/\mathfrak{m}_{R_{\bar{\tau}}}^n$ for $n = 1, \dots$.

We obtain an (a priori non-separated) scheme $\mathcal{E}_{\bar{\tau},n}$ over $R_{\bar{\tau}}/\mathfrak{m}_{R_{\bar{\tau}}}^n$, and hence a formal scheme $\mathcal{E}_{\bar{\tau}} = \lim_n \mathcal{E}_{\bar{\tau},n}$. Since $\mathcal{E}_{\bar{\tau},1} \xrightarrow{\sim} \mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$ is proper, we see also that $\mathcal{E}_{\bar{\tau}}$ is proper. $\Box$

Remark (3.2.4) I expect that the representing formal scheme is projective and therefore arises from a *scheme* of finite type over $Spec R_{\bar{\tau}}$.

Corollary (3.2.5). *Let $x \in \mathrm{Rep}_{\bar{\tau}}^{\Box}(\mathbb{F})$ and $V_{\mathbb{F}}$ the corresponding representation of G. Then the complete local ring at of $\mathrm{Rep}_{\bar{\tau}}^{\Box}$ at x pro-represents $D_{V_{\mathbb{F}}}^{\Box}$.*

On the situation of (3.2), if $x \in \mathrm{Rep}_{\bar{\tau}}^{\Box}(\mathbb{F})$ then the complete local ring at of $\mathrm{Rep}_{\bar{\tau}}^{\Box}$ at x is a quotient of $R_{V_{\mathbb{F}}}^{\Box}$.

Proof. This follows immediately from the definitions. $\Box$

Corollary (3.2.6). *Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : Rep'_{\bar{\tau}} \to E$ a point such that the corresponding E-valued pseudo-representation τ_x is absolutely irreducible. Then the map*

$$\mathrm{Rep}'_{\bar{\tau}} \to \mathrm{Spf}\, R_{\bar{\tau}}$$

is a closed embedding over a formal neighbourhood of τ_x.[2]

Proof. First we remark that x is the only point of $\mathrm{Rep}'_{\bar{\tau}}$ lying over τ_x. To see this suppose x' is another such point and let denote by V_x and $V_{x'}$ the corresponding G-representations. Then, by the properness of $\mathrm{Rep}'_{\bar{\tau}}$ x and x' arise from $\mathcal{O}_E$ valued points, which in turn correspond to G-stable lattice $L_x \subset V_x$ and $L'_x \subset V_{x'}$. Since V_x and $V_{x'}$ are absolutely irreducible with the same trace, they are isomorphic. We may choose this isomorphism so that it induces a non-zero map $L_x \to L_{x'}$, whose reduction modulo the radical (π_E) of $\mathcal{O}_E$ is non-zero. Now as $L_x/\pi_E L_x$ and $L_{x'}/\pi_E L_{x'}$ are both non-zero described in (3.2.1), the only non-zero maps between them are isomorphisms. Hence we find that $L_x \xrightarrow{\sim} L_{x'}$.

Next let $\widehat{R}_{\tau_x}$ be the complete local ring at τ_x. By (3.2.4) $\widehat{R}_{\tau_x}$ is the universal deformation ring of τ_x. Similarly the complete local ring at x, $\widehat{R}_x$ is a quotient of the universal deformation ring of V_x, by Lecture 1, Exercise 4 and (3.2.6). Hence the map $\widehat{R}_{\tau_x} \to \widehat{R}_x$ is a surjection by the Theorem of Nyssen-Rouquier (2.3.1). $\Box$

Exercises:

Exercise 1: Verify that the quotient produced in the proof of (3.1.1) does represent $R_{V_{\mathbb{F}}}$.

Exercise 2: Complete the proof of (3.2.2) for arbitrary d.

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[2] Note that when we refer to E-valued points of formal schemes over $W(\mathbb{F})$, and complete local rings at such points, what we really mean is the complete local ring at the corresponding point of the p-adic analytic space

LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

Mark Kisin

Lecture 4: Presenting global deformation rings over local ones

(4.1) Let F be a number field and S a finite set of primes of F containing the primes dividing p. Fix an algebraic closure $\bar{F}$ of F and denote by $F_S \subset \bar{F}$ the maximal extension of F unramified outside S. Write $G_{F,S} = \mathrm{Gal}(F_S/F)$.

Let $\Sigma \subset S$ and fix an algebraic closure $\bar{F}_v$ of F_v for each $v \in \Sigma$, as well as an embedding $\bar{F} \hookrightarrow \bar{F}_v$. We write $G_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$.

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and uniformizer $\pi_{\mathcal{O}}$. We fix a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ equipped with a continuous action of $G_{F,S}$, and a continuous character $\psi : G_{F,S} \to \mathcal{O}^\times$ such that $\det_{\mathbb{F}} V_{\mathbb{F}} \sim \psi$.

For simplicity we assume in the following that $p \nmid \dim V_{\mathbb{F}}$, although this is not really necessary (see [Ki 2]).

(4.1.1) For each $v \in \Sigma$ fix a basis β_v of $V_{\mathbb{F}}$. For A in $\mathfrak{AR}_W(\mathbb{F})$ denote by $D_v^{\square,\psi}(A)$ the category of framed deformations of $(V_{\mathbb{F}}|_{G_{F_v}}, \beta)$ to A, with determinant ψ. This is pro-representable by a complete local $\mathcal{O}$-algebra $R_v^{\square,\psi}$.

Similarly we denote by $D_{F,S}^{\square,\psi}(A)$ the category whose objects consist of a deformation of $V_{\mathbb{F}}$ to V_A together with a lifting of *each* basis β_v to an A-basis of V_A.

We also have the analogous functors D_v^ψ and $D_{F,S}^\psi$ for unframed deformations and the universal $\mathcal{O}$-algebras R_v^ψ and $R_{F,S}^\psi$ when these functors are representable.

We set

$$R_\Sigma^{\square,\psi} = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^{\square,\psi} \text{ and } R_\Sigma^\psi = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^\psi$$

when the latter ring exists.

Finally we denote by $\mathfrak{m}_\Sigma^\square$ and $\mathfrak{m}_{F,S}^\square$ the radicals of $R_\Sigma^{\square,\psi}$ and $R_{F,S}^{\square,\psi}$ respectively, and similarly for $\mathfrak{m}_\Sigma$ and $\mathfrak{m}F, S$.

Proposition (4.1.2). *For $i \geq 1$ let h_Σ^i and c_Σ^i denote respectively the dimension of the kernel and cokernel of*

$$\theta^i : H^i(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \to \prod_{v\in\Sigma} H^i(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

where $\mathrm{ad}^0 V_{\mathbb{F}} \subset \mathrm{ad} V_{\mathbb{F}}$ denotes the space of endomorphisms with trace 0. If R_Σ^ψ exists then $R_{F,S}^\psi$ is a quotient of $R_\Sigma^\psi[\![x_1, \dots, x_{h_\Sigma^1}]\!]$ by $c_\Sigma^1 + h_\Sigma^2$ relations.

In general let

$$\eta : \mathfrak{m}_\Sigma^\square/(\mathfrak{m}_\Sigma^{\square,2}, \pi_{\mathcal{O}}) \to \mathfrak{m}_{F,S}^\square/(\mathfrak{m}_{F,S}^{\square,2}, \pi_{\mathcal{O}}).$$

Typeset by $\mathcal{AMS}$-TEX

Then $R^{\psi,\square}_{F,S}$ *is a quotient of a power series ring over* $R^{\square}_{\Sigma}$ *in* $\dim\mathrm{coker}\eta$ *variables by* $h^2_{\Sigma} + \dim\ker\eta$ *relations.*

Proof. We prove only the first statement. Choose a surjection

$$\tilde{R} := R^{\psi}_{\Sigma}[\![x_1, \dots, x_{h^1_{\Sigma}}]\!] \to R^{\psi}_{F,S} \tag{4.1.3}$$

which induces a surjection on reduced tangent spaces and denote by J its kernel. We denote by $\tilde{\mathfrak{m}}$ the radical of $\tilde{R}$.

The universal representation $\rho_{R^{\psi}_{F,S}} : G_{F,S} \to \mathrm{GL}_d(R^{\psi}_{F,S})$ has a set theoretic lifting $\tilde{\rho} : G_{F,S} \to G_{F,S} \to \mathrm{GL}_d(\tilde{R}/\tilde{\mathfrak{m}}J)$ such that $\det\tilde{\rho}(\gamma) = \psi(\gamma)$ for all $\gamma \in G_{F,S}$. Such a lifting exists as the fibres of $\det : \mathrm{GL}_d \to \mathrm{GL}_1$are torsors over SL_d, and in particular smooth.

Define

$$c : G^2_{F,S} \to J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}}; \quad c(g_1, g_2) = \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_2)^{-1}\tilde{\rho}(g_1)^{-1}$$

where we regard

$$J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}} \xrightarrow{\sim} \ker\left(\mathrm{GL}_d(\tilde{R}/\tilde{m}J) \to \mathrm{GL}_d(\tilde{R}/J)\right)$$

Then $[c] \in H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ depends only on $\rho_{R^{\psi}_{F,S}}$ and not on $\tilde{\rho}$, and $[c] \sim 0$ if and only if $\tilde{\rho}$ can be chosen to be a homomorphism (see Exercise 2 below).

As $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ is induced by the universal representation over R^{ψ}_v, $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ lifts to $\tilde{R}/\tilde{\mathfrak{m}}J$ and hence $[c|_{G_{F_v}}] \sim 0$. Hence $[c] \in H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ where

$$H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) := \ker\left(H^2(G_{F,S}, \mathrm{ad}^- V_{\mathbb{F}}) \to \prod_{v \in \Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})\right),$$

and we obtain a map

$$(J/\tilde{\mathfrak{m}}J)^* \to H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}); \quad u \mapsto \langle [c], u \rangle. \tag{4.1.4}$$

Let

$$\begin{array}{ccc} I = \ker\left(\mathfrak{m}_{\Sigma}/(\mathfrak{m}^2_{\Sigma}, \pi_{\mathcal{O}})\right. & \longrightarrow & \left.\mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}})\right) \\ \downarrow\wr & & \downarrow\wr \\ \oplus D_{V_{\mathbb{F}}|_{G_{F_v}}}(F[\epsilon])^* & \longrightarrow & D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \end{array}$$

Note that $I \xrightarrow{\sim} \ker\left(\tilde{\mathfrak{m}}/(\tilde{\mathfrak{m}}^2, \pi_{\mathcal{O}}) \to \mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}})\right)$, so reducing mod $\tilde{\mathfrak{m}}$ we get a surjection $J/\tilde{\mathfrak{m}}J \to I$ and an injection $I^* \hookrightarrow (J/\tilde{\mathfrak{m}}J)^*$.

We claim that I^* contains the kernel of (4.1.4). If $0 \neq u \in (J/\tilde{\mathfrak{m}}J)^*$ let $\tilde{R}_u$ be the pushout of $\tilde{R}/\tilde{\mathfrak{m}}J$ by u. Then $R^{\psi}_{F,S} = \tilde{R}_u/I_u$ where $I_u \subset \tilde{R}_u$ is an ideal with $I_u \cdot \tilde{\mathfrak{m}}$ and which is 1-dimensional as an $\mathbb{F}$-vector space. If $\langle [c], u \rangle = 0$ then $\rho^{\psi}_{F,S}$ lifts to a representation into $\mathrm{GL}_d(\tilde{R}_u)$ with determinant ψ. Hence $\tilde{R}_u \to R^{\psi}_{F,S}$ has a section and $\tilde{R}_u = R^{\psi}_{F,S} \oplus I_u$. This implies that $\tilde{R}_u \to R^{\psi}_{F,S}$ does not induce a bijection on reduced tangent spaces. In particular, the composite

$$\ker\left(J/\tilde{\mathfrak{m}}J \to I\right) \to J/\tilde{\mathfrak{m}}J \to I_u$$

is not surjective, and is therefore 0. This means that u factors through I, which proves our claim.

It follows that

$$\dim_{\mathbb{F}}(J/\tilde{\mathfrak{m}}J)^* \leqslant \dim_{\mathbb{F}} I + h^2_{\Sigma} = c^1_{\Sigma} + h^2_{\Sigma}.$$

□

Theorem (4.2). *Suppose $\{v|p\} \subset \Sigma$, $\{v|\infty\} \subset S$, and $S\backslash\Sigma$ contains a finite prime. Then*

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_{r+s})$$

for some $r \geq 0$, and $s = \sum_{v|\infty, v\notin\Sigma} \dim_{\mathbb{F}}(\mathrm{ad}^0 V_{\mathbb{F}})^{G_{F_v}}$, provided the rings $R^{\psi}_{F,S}$ and R^{ψ}_{Σ} exist.

Moreover,

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1, \dots, x_{r+\Sigma-1}]\!]/(f_1, \dots, f_{r+s})$$

(4.2.1) To prove the theorem, we need the Poitou-Tate sequence. Late X be a finite abelian p-group equipped with an action of $G_{F,S}$. We denote by $X^{\vee}$ the Pontryagin dual of X, and by $X^* = X^{\vee}(1)$ its Tate dual. Then there is an exact sequence

(PT(X))

$$\begin{aligned}0 \to H^0(G_{F,S}, X) \to \prod_{v|\infty} \widehat{H}^0(G_{F_v}, X) \times \prod_{v\in S_f} H^0(G_v, X) \to H^0(G_{F,S}, X^*)^{\vee} \\ \to H^1(G_{F,S}.X) \to \prod_{v\in S} H^1(G_{F_v}, X) \to H^1(G_{F,S}, X^*)^{\vee} \\ \to H^2(G_{F,S}, X) \to \prod_{v\in S} H^2(G_{F_v}, X) \to H^0(G_{F,S}, X^*)^{\vee} \to 0\end{aligned}$$

Here $\widehat{H}^0(G_{F_v}, X)$ denotes $H^0(G_{F_v}, X)$ modulo the subgroup of norms in X.

Local Tate duality provides an isomorphism

$$H^i(G_{F_v}, X)^{\vee} \xrightarrow{\sim} H^{2-i}(G_{F_v}, X^*)$$

for v a finite prime and $i = 0, 1, 2$. Using this, one can identify the Pontryagin dual of the sequence $PT(X)$ with $PT(X^*)$.

Proof of (4.2). We will prove only the first statement since the proof of the second statement requires only some extra book keeping.

We apply the above sequence with $X = \mathrm{ad}^0 V_{\mathbb{F}}$. First note that, using the remark on the duality of $PT(X)$ and $PT(X^*)$ one sees that the map

$$\prod_{S\backslash\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) \to H^0(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}(1))^{\vee}$$

induced by the final map of $PT(X)$ is surjective, as $S\backslash\Sigma$ contains a finite prime. Hence the map

$$H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \to \prod_{v\in\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

is surjective and

$$h^2_{\Sigma} = h^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}).$$

Here we use the convention that $h^i = \dim H^i$.

By (4.1.2) we have $R^{\psi}_{F,S}[\![x_1,\dots,x_r]\!]/(f_1,\dots f_{r+s})$. with

$$(4.2.2)\quad -s = h^1_\Sigma - c^1_\Sigma - h^2_\Sigma$$
$$= h^1(G_{F,S}.\mathrm{ad}^0V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^1(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}) - h^2(G_{F,S}.\mathrm{ad}^0V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}})$$
$$= -\chi(G_{F,S},\mathrm{ad}^0V_{\mathbb{F}}) + \sum_{v\in\Sigma}\chi(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}).$$

Here we have used that fact that the existence of $R^{\psi}_{G_{F,S}}$ and $\mathcal{R}^{\psi}_{\Sigma}$ implies that $\mathrm{ad}^0V_{\mathbb{F}}$ has no $G_{F,S}$ or G_{F_v} invariants for $v\in\Sigma$.

Now we use Tate's global Euler characteristic formula which says that

$$(4.2.3)\quad \chi(G_{F,S},\mathrm{ad}^0V_{\mathbb{F}}) = \sum_{v|\infty}(h^0(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}) - [F_v:R]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0V_{\mathbb{F}})$$
$$= (\sum_{v|\infty} h^0(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}})) - [F;\mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0V_{\mathbb{F}}.$$

The local Euler characteristic $\chi(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}})$ is 0 if $v\nmid p$ is a finite prime. Hence the contributions of the local terms in (4.2.2) is

$$(4.2.4)\quad \sum_{v|\infty,v\in\Sigma} h^0(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}) - \sum_{v|p}[F_v:\mathbb{Q}_p]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0V_{\mathbb{F}}$$
$$= \sum_{v|\infty,v\in\Sigma} h^0(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}) - [F:\mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0V_{\mathbb{F}}.$$

Subtracting (4.2.4) from (4.2.3) one finds

$$s = \sum_{v|\infty,v\notin S} h^0(G_{F_v},\mathrm{ad}^0V_{\mathbb{F}}).$$

□

Exercises:

Exercise 1: Prove the second statement in Proposition (4.1.2).

Exercise 2: Check the statements about the cocycle c in Proposition: That $[c]$ does not depend on $\tilde\rho$ and is trivial if and only if $\tilde\rho$ can be chosen to be a homomorphism.

Exercise 3: Prove the second statement in Theorem (4.2)

Exercise 4: (This is more difficult.) Formulate and prove Theorem (4.2) without assuming $p\nmid \mathrm{dim}V_{\mathbb{F}}$.

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LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

Mark Kisin

Lecture 5: Flat deformations

(5.1) Flat deformations: Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. Let $W = W(k)$ and $K_0 = \mathrm{Fr}W$. We consider a finite dimensional $\mathbb{F}$-vector space equipped with a continuous action of G_K. Fix an algebraic closure $\bar{K}$ of K, and let $G_K = \mathrm{Gal}(\bar{K}/K)$.

Recall that a representation of G_K on a finite abelian p-group is called *flat* if it arises from a finite flat group scheme over $\mathcal{O}_K$.

The following result is due to Ramakrishna [Ra]:

Proposition (5.1.1). *Let $A \in \mathfrak{AR}_W$ and V_A in $D_{V_\mathbb{F}}(A)$. There exists a quotient A^{fl} of A such that for any morphism $A \to A'$ in $\mathfrak{AR}_W$, $V_{A'} =: V_A \otimes_A A'$ is flat if and only if $A \to A'$ factors through A^{fl}.*

Proof. First note that if V is a finite abelian p-group equipped with a continuous action of G_K and V is flat, then any G_K-stable subgroup $V' \subset V$ is flat. Indeed, suppose $V = \mathcal{G}(\mathcal{O}_{\bar{K}})$ think of V, V' as finite étale group schemes over K. If $\mathcal{G}'$ is the closure of V' in $\mathcal{G}$, then $V' = \mathcal{G}'(\mathcal{O}_{\bar{K}})$.

This remark shows that if $\theta : A \to A'$ is a morphism in $\mathfrak{AR}_W$ then $V_{A'}$ is flat if and only if $V_{\theta(A)}$ is flat. Similarly, if $I, J \subset A$ are ideals and $V_{A/I}$ and $V_{A/J}$ is flat then $V_{A/I\cap J} \subset V_{A/I} \oplus V_{A/J}$ is flat. $\square$

Corollary (5.1.2). *Let $D^{\mathrm{fl}}_{V_\mathbb{F}} \subset D_{V_\mathbb{F}}$ denote the sub-functor corresponding to flat deformations. Then $D^{\mathrm{fl}}_{V_\mathbb{F}} \subset D_{V_\mathbb{F}}$ is relatively representable.*

Proof. In the language of groupoids this just means that if ξ in $D_{V_\mathbb{F}}$, then $(D^{\mathrm{fl}}_{V_\mathbb{F}})_\xi$ is representable, and this follows from (5.1.1). $\square$

(5.2) Weakly admissible modules and Smoothness of the generic fibre:

Proposition (5.2.1). *Suppose that $D_{V_\mathbb{F}}$ is pro-represented by $R_{V_\mathbb{F}}$ and let $R^{\mathrm{fl}}_{V_\mathbb{F}}$ be the quotient of $R_{V_\mathbb{F}}$ which pro-represents $D^{\mathrm{fl}}_{V_\mathbb{F}}$. Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : R^{\mathrm{fl}}_{V_\mathbb{F}}[1/p] \to E$ be a point such that $\ker x$ has residue field E. Write $\widehat{R}^{\mathrm{fl}}_x$ (resp. $\widehat{R}_x$) for the completion of $R^{\mathrm{fl}}_{V_\mathbb{F}}[1/p]$ (resp. $R_{V_\mathbb{F}}[1/p]$) at x.*

For any Artinian quotient $\epsilon : \widehat{R}_x \to B$ denote the specialization of the universal deformation by $R_{V_\mathbb{F}} \to B$. Then ϵ factors through $R^{\mathrm{fl}}_{V_\mathbb{F}}$ if and only if V_B factors through $\widehat{R}^{\mathrm{fl}}_x$ if and only if V_B arises from a p-divisible group. Moreover this condition holds if and only if V_B is crystalline.

Proof. Let B be in $\mathfrak{AR}_E$ and denote by Int_B the set of finite $\mathcal{O}_E$-subalgebras $A \subset B$. Then $R_{V_{\mathbb{F}}} \to B$ factors through some A in Int_B. Denote by V_A the induced G_K representation.

Then V_B arises from a p-divisible group if and only if V_A does. By a result of Raynaud [Ray, 2.3.1], V_A arises from a p-divisible group if and only if V_A/p^nV_A is flat for $n \geq 1$. This is equivalent to asking that $V_A \otimes_A A/\mathfrak{m}_A^n$ be flat for $n \geq 1$, or that $V_A \otimes_A A/\mathfrak{m}_A^n$ is in $D^{\mathrm{fl}}(A/\mathfrak{m}_A^n)$.

For the final statement we use Breuil's result that a crystalline representation with all Hodge-Tate weights equal to $0, 1$ arises from a p-divisible group [Br, Thm. 5.3.2], [Ki 4, 2.2.6]. □

(5.2.2) The deformation theoretic description of $\widehat{R}_x^{\mathrm{fl}}$ allows us to show that this ring is always formally smooth over E, and hence that $R_{V_{\mathbb{F}}}^{\mathrm{fl}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$. To prove this we need a little preparation (see [Ki 3]).

For any weakly admissible filtered φ-module D over K, denote by $C^\bullet(D)$ the complex $C^\bullet(D)$ the complex

$$D \overset{(1-\varphi,\mathrm{id})}{\longrightarrow} D \oplus D_K/\mathrm{Fil}^0 D_K$$

concentrated in degrees $0, 1$.

Lemma (5.2.3). *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D)).$$

where $\mathbf{1} = K_0$ *denotes the unit object in the category of weakly admissible module.*

Proof. Let

$$0 \to D \to \tilde{D} \to \mathbf{1} \to 0$$

be an extension. Let $\tilde{d} \in \mathrm{Fil}^0\tilde{D}$ be a lift of $1 \in \mathbf{1}$. Note that $D_K/\mathrm{Fil}^0 D_K \xrightarrow{\sim} \tilde{D}_K/\mathrm{Fil}^0\tilde{D}_K$, so we may regard $\tilde{d} \in D_K/\mathrm{Fil}^0 D_K$. Moreover $(1-\varphi)(\tilde{d}) \in D$. We associate the class

$$((1-\varphi)\tilde{d}, \tilde{d}) \in H^1(C^\bullet(D))$$

to the given extension.

If $(d_0, d_1) \in D \oplus D_K/\mathrm{Fil}^0 D_K$ we construct an extension of $\mathbf{1}$ by D by setting $\tilde{D} = D \oplus 1$ on underlying K_0-vector spaces and defining φ on $\tilde{D}$ by $\varphi(1) = 1 + d_0$ and the filtration by

$$\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K + K \cdot d_1 \quad i \leqslant 0$$

and $\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K$ if $i > 0$.

One checks that these two constructions induce the required isomorphism and its inverse. □

(5.2.4) Let $E/\mathbb{Q}_p$ be a finite extension, and D_E a weakly admissible filtered φ-module over K, equipped with an action of E. For B in $\mathfrak{AR}_E$ we denote by $D_{D_E}(B)$ the category of weakly admissible filtered filtered φ-modules D_B, equipped with an action of B, and an isomorphism $D_B \otimes_B E \xrightarrow{\sim} D$, such that D and $\mathrm{gr}^\bullet D_K$ are free B-modules.

For V_E a crystalline representation on a finite dimensional E-vector space, and B in $\mathfrak{AR}_E$ we denote by $D_{\mathrm{cris}}(V_B)$ the category of crystalline deformations of V_E to B.

Lemma (5.2.5). *Let V_E be as above and $D_E = D_{\text{cris}}(V_E)$. Then D_{cris} induces an equivalences of groupoids over $\mathfrak{AR}_E$ $D_{V_E} \xrightarrow{\sim} D_{D_E}$. Moreover each of these groupoids is formally smooth.*

Proof. The proof of the first statement is formally similar to (5.3.5) below, and left as an exercise to the reader. The formal smoothness can be proved by a deformation theoretic argument using (5.2.3) and the fact that $H^2(C^\bullet(\text{ad}D_E)) = 0$. □

Proposition (5.2.6). *In the notation of (5.2.1), let $D_E = D_{\text{cris}}(V_E)$. The E-algebra $\widehat{R}_x^{\text{fl}}$ is formally smooth of dimension*

$$\dim_E H^1(\mathcal{C}^\bullet(\text{ad}D_x)) = 1 + \dim_E D_K/\text{Fil}^0 D_K.$$

Proof. By (5.2.1) $\widehat{R}_x^{\text{fl}}$ pro-represents D_{V_E}, which is equivalent to D_{D_E} and hence formally smooth by (5.2.5). The dimension of $\widehat{R}_x^{\text{fl}}$ is equal to

$$\dim_E \text{Ext}^1_{\text{cris}}(V_E, V_E) = \dim_E \text{Ext}^1_{w.adm}(D_E, D_E) = \\ \dim_E \text{Ext}^1_{w.adm}(\mathbf{1}, \text{ad}D_x) = \dim_E H^1(C^\bullet(\text{ad}D)) = 1 + \dim_E D_K/\text{Fil}^0 D_K.$$

Here the first term means crystalline self extensions of V_E, as a representation of $E[G_K]$, the second last equality follows from (5.2.3), and the final one from the fact that the Euler characteristic of a finite complex is equal to that of its cohomology. □

(5.3) The Fontaine-Laffaille functor and Smoothness when $e = 1$: Recall the Fontaine-Laffaille category MF^1_{tor} whose objects consist of a finite, torsion W-module M, together with a submodule $M^1 \subset M$, and Frobenius semi-linear maps

$$\varphi : M \to M \text{ and } \varphi^1 : M^1 \to M$$

such that

(1) $\varphi|_{M^1} = p\varphi^1$.
(2) $\varphi(M) + \varphi^1(M^1) = M$.

MF^1_{tor} is an abelian subcategory of the category of filtered W-modules of finite length [FL, 1.9.1.10]. In particular, any morphism on MF^1_{tor} is strict for filtrations.

Note also that if $p \cdot M = 0$, then $\varphi(M^1) = 0$, and so comparing the lengths of the two sides of (2) above shows that φ^1 is injective and

$$\varphi(M) \oplus \varphi^1(M^1) \xrightarrow{\sim} M.$$

Theorem (5.3.1). *(Fontaine-Laffaille, Raynaud) Suppose that $K = K_0$ and $p > 2$. Then there exist equivalences of abelian categories*

$$\text{MF}^1_{\text{tor}} \xrightarrow[\text{FL}]{\sim} \{\text{f.flat group schemes}/W\} \xrightarrow{\sim} \{\text{flat repns. of } G_K\}.$$

Proof. The first equivalence is obtained by composing the anti-equivalence [FL, 9,11] with Cartier duality. The second follows from Raynaud's result [Ray, 3.3.6] that when $e(K/K_0) < p - 1$ the functor $\mathcal{G} \mapsto \mathcal{G}(\mathcal{O}_{\bar{K}})$ is fully faithful and the category of finite flat group schemes over $\mathcal{O}_K$ is abelian. □

(5.3.2) We will need a little more information about the functor FL. For a finite flat group scheme $\mathcal{G}$ over W we denote by $t_{\mathcal{G}}$ the tangent space of $\mathcal{G} \otimes_W k$, and by $\mathcal{G}^*$ the Cartier dual of $\mathcal{G}$.

The contravariant version of the functor is constructed via the theory of Honda systems, which is an extension of Dieudonné theory (which classifies finite flat group schemes over k) [FL, 9.7].

In particular, if $\mathcal{G}$ killed by p, and $M = \mathrm{FL}(\mathcal{G})$, there is an exact sequence

$$0 \to (t_{\mathcal{G}^*}{}^\vee \to \sigma^{-1*}M \to t_{\mathcal{G}} \to 0$$

where σ denote the Frobenius on W. Moreover the linear map $1\otimes\varphi_1 : M^1 \to \sigma^{-1*}M$ identifies M^1 with with $t^\vee_{\mathcal{G}^*}$. In particular

$$\dim_k M^1 = \dim_k \varphi^1(M_1) = \dim_k t_{\mathcal{G}^*}.$$

Now suppose that $\mathcal{G}$ is a *p-divisible group*, $T_p\mathcal{G}$ its Tate module, and $\mathcal{G}^*$ its Cartier dual. Write $V_p\mathcal{G} = T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then

$$D(\mathcal{G}) := D_{\mathrm{cris}}(T_p\mathcal{G})(1) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(T_p\mathcal{G}^*, B_{\mathrm{cris}})$$

is a weakly admissible module whose associated graded is zero except in degrees $0, 1$. The lattice $T_p\mathcal{G} \subset V_p\mathcal{G}$ corresponds to a *strongly divisible lattice* $M \subset D_{\mathrm{cris}}(T_p\mathcal{G})$ with M/pM canonically isomorphic to $\mathrm{FL}(\mathcal{G}[p])$ as an object of $\mathrm{MF}^1_{\mathrm{tor}}$. In particular

$$\dim_{K_0}\mathrm{Fil}^1 D(\mathcal{G}) = \mathrm{rk}_{\mathcal{O}_{K_0}}\mathrm{Fil}^1 M = \dim_k M^1 = \dim_k t_{\mathcal{G}^*[p]}. \tag{5.3.3}$$

Theorem (5.3.4). *Suppose $K = K_0$ and $p > 2$. Then $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth. If $\mathcal{G}_{\mathbb{F}}$ denote the unique finite flat model of $V_{\mathbb{F}}$, $\mathcal{G}^*_{\mathbb{F}}$ denotes its Cartier dual, and $t_{\mathcal{G}_{\mathbb{F}}}$ denotes the tangent space of $\mathcal{G}_{\mathbb{F}}$, then*

$$\dim_{\mathbb{F}} D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}_{\mathbb{F}}} \dim_{\mathbb{F}} t_{\mathcal{G}^*_{\mathbb{F}}}.$$

Proof. Let $M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$ denote the object corresponding to $V_{\mathbb{F}}$. Then $M_{\mathbb{F}}$ is naturally an $\mathbb{F}$-vector space by the full faithfulness of (5.2.1). Let $D_{M_{\mathbb{F}}}$ denote the groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$ such that $D_{M_{\mathbb{F}}}(A)$ is the category of objects M_A in $\mathrm{MF}^1_{\mathrm{tor}}$ equipped with an action of A, such that such that M_A is a finite free A-module and M^1 is an A-module direct summand, and an isomorphism $M_A \otimes_A \mathbb{F} \xrightarrow{\sim} M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$.

Lemma (5.3.5). *The Fontaine-Laffaille functor of (5.2.1) induces an equivalence of categories*

$$\mathrm{FL} : D_{M_{\mathbb{F}}} \xrightarrow{\sim} D^{\mathrm{fl}}_{V_{\mathbb{F}}}.$$

Proof. If M_A is in $D_{M_{\mathbb{F}}}(A)$ let V_A be its image under FL. As FL is exact V_A is finite free over A. Indeed, for any finite A-module N we have

$$\mathrm{FL}(M_A) \otimes_A N \xrightarrow{\sim} \mathrm{FL}(M_A \otimes_A N). \tag{5.3.4}$$

This is obvious if N is free over A, and the general case follows by choosing a presentation of N by free modules. As the right hand side is an exact functor in N, so is the left hand side, which shows that V_A is a free A-module. Applying (5.3.4) with $N = \mathbb{F}$, one also sees that V_A is naturally a deformation of $V_{\mathbb{F}}$.

Conversely if V_A is in $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$, and $M_A \in \mathrm{MF}^1_{\mathrm{tor}}$ satisfies $\mathrm{FL}(M_A) \xrightarrow{\sim} V_A$, then M_A is an A-module by the full faithfulness of FL, and since $\mathrm{MF}^1_{\mathrm{tor}}$ is abelian, the same argument as above shows that M_A is free over A, and that $M^1_A \subset M_A$ is an A-module direct summand □

(5.3.6) We return to the proof of (5.3.4). By the lemma to prove the formal smoothness of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ it suffices to prove the formal smoothness of $D_{M_{\mathbb{F}}}$. Let A be in $\mathfrak{AR}_{W(\mathbb{F})}$, $I \subset A$ an ideal and $M_{A/I}$ in $D_{M_{\mathbb{F}}}(A/I)$. We have to show that $M_{A/I}$ lifts to an object of $D_{M_{\mathbb{F}}}(A)$.

First choose a lifting of the A/I-module $M_{A/I}$ to an A-module M_A, and a submodule $M^1_A \subset M_A$ which is a direct summand and lifts $M^1_{A/I}$. Next let $L_{A/I} = \varphi^1(M^1_{A/I})$, and choose a lift of $L_{A/I}$ to a direct summand $L_A \subset M_A$ and a lift of the composite

$$\varphi^*(M_A) \to \varphi^*(M_{A/I}) \stackrel{1\otimes\varphi^1}{\to} M_{A/I}$$

to L_A. Finally one checks that the map $1\otimes p\varphi^1 : \varphi^*(M^1_A) \to M_A$ admits an extension to a map $\varphi^*(M_A) \to M_A$ which induces the given map $\varphi^*(M_{A/I}) \to M_{A/I}$.

(5.3.7) It remains to check that the dimension of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ has the claimed dimension. We could do this directly by computing the dimension of $D_{M_{\mathbb{F}}}(\mathbb{F}[\epsilon])$, however it is simpler to use our computation of the dimension of the generic fibre of $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$.

Let $x : R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p] \to E$ be a surjective map, where E is a finite extension of $\mathbb{Q}_p$. Write V_x for the corresponding crystalline representation. Since we already know that $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is smooth, we need to compute the dimension of the tangent space of $\mathcal{R}^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ at x. Let $D_x = D_{\mathrm{cris}}(V_x)$.

Using (5.2.6) and (5.3.3)one sees that this dimension is[1]

$$1 + \dim_E(D_x/\mathrm{Fil}^1 D_x)\dim_E \mathrm{Fil}^1 D_x = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}} \dim_{\mathbb{F}} t_{\mathcal{G}^*}.$$

□

Exercises:

Exercise 1: Formulate and prove Proposition (5.2.1) for framed deformations.

Exercise 2: Check that the two constructions used to define the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D))$$

in (5.2.3) are well defined and inverse.

Exercise 3: Give an explicit description of the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, \mathrm{ad}D) \xrightarrow{\sim} \mathrm{Ext}^1_{w.adm}(D, D)$$

used in (5.2.6).

Exercise 4: Show that the functor D_{D_E} in (5.2.5) is formally smooth.

Exercise 5: Show that the category $\mathrm{MF}^1_{\mathrm{tor}}$ is equivalent to the category of *finite Honda systems* defined in Conrad's lectures. This is slightly tricky [FL, 9.4].

[1] The computation becomes slightly easier if $E = W(\mathbb{F})[1/p]$, which we may assume as $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth.

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