

Enumerative geometry and geometric representation theory

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ABSTRACT. This is an introduction to: (1) the enumerative geometry of rational curves in equivariant symplectic resolutions, and (2) its relation to the structures of geometric representation theory. Written for the 2015 Algebraic Geometry Summer Institute.

1. Introduction

1.1. These notes are written to accompany my lectures given in Salt Lake City in 2015. With modern technology, one should be able to access the materials from those lectures from anywhere in the world, so a text transcribing them is not really needed. Instead, I will try to spend more time on points that perhaps require too much notation for a broadly aimed series of talks, but which should ease the transition to reading detailed lecture notes like [90].

The fields from the title are vast and they intersect in many different ways. Here we will talk about a particular meeting point, the progress at which in the time since Seattle I find exciting enough to report in Salt Lake City. The advances both in the subject matter itself, and in my personal understanding of it, owe a lot to M. Aganagic, R. Bezrukavnikov, P. Etingof, D. Maulik, N. Nekrasov, and others, as will be clear from the narrative.

1.2. The basic question in representation theory is to describe the homomorphisms

(1) $\text{some algebra } \mathcal{A} \rightarrow \text{matrices,}$

and the geometric representation theory aims to describe the source, the target, the map itself, or all of the above, geometrically.

For example, matrices may be replaced by correspondences, by which we mean cycles in $X \times X$, where X is an algebraic variety, or K-theory classes on $X \times X$ et cetera. These form algebras with respect to convolution and act on cycles in X and $K(X)$, respectively. Indeed, $\text{Mat}(n, R)$, where R is a ring, is nothing but $K(X^{\times 2}) \otimes_{\mathbb{Z}} R$ where X is a finite set of cardinality n , and the usual rules of linear algebra, when written in terms of pullback, product, and pushforward, apply universally. (This is also how e.g. integral operators act.)

Most of the time, these can and should be upgraded to Fourier-Mukai kernels [57], although doing this in the enumerative context should be weighted against

the cost of losing deformation invariance — a highly prized and constantly used property of pragmatically defined geometric counts.

1.3. There is some base ring implicit in (1) and, in real life, this ring is usually $H_G^\bullet(\text{pt})$ or $K_G(\text{pt})$, where G is a reductive group acting on X although, of course, generalizations are possible.

In what follows, it will be natural and important to work with the maximal equivariance allowed by the problem. Among their many advantages, equivariant counts are often defined when nonequivariant aren't. Concretely, the character of an infinite-dimensional module V may be well-defined as a rational function on G , but $1 \in G$ will typically be a pole of this function.

1.4. Enumerative geometry is an endless source of interesting correspondences of the following kind. The easiest way for two distant points $x_1, x_2 \in X$ to interact is to lie on a curve $C \subset X$ of some degree and genus and, perhaps, additionally constrained by e.g. incidence to a fixed cycle in X . Leaving the exact notion of a curve vague for a moment, one can contemplate a moduli space \mathcal{M} of two-pointed curves in X with an evaluation map

$$\text{ev} : \mathcal{M} \rightarrow X \times X .$$

One can use this evaluation map to construct correspondences given, informally, by pairs of points $(x_1, x_2) \in X^{\times 2}$ that lie on such and such curve. For this, one needs geometrically natural cycles or K-classes on \mathcal{M} to push forward and, indeed, these are available in some generality.

Deformation theory provides a local description of the moduli space \mathcal{M} near a given curve C in X . Counting the parameters for deformations minus the number of equations they have to satisfy, one computes

$$(2) \quad \text{expected dim } \mathcal{M} = (3 - \dim X)(g - 1) + (d, c_1(X)) + n ,$$

where g is the genus of a curve $C \in \mathcal{M}$, $d \in H_2(X, \mathbb{Z})$ is its degree, and $n = 2$ is the number of marked points. This is only a lower bound for the actual dimension of \mathcal{M} and it is seldom correct. Enumerative geometers treat this as they would treat any excess intersection problem: redundant equations still cut out a canonical cycle class $[\mathcal{M}]_{\text{vir}}$ of the correct dimension, known as the *virtual fundamental cycle*, see [8]. There is a parallel construction of the virtual structure sheaf $\mathcal{O}_{\mathcal{M}, \text{vir}} \in K_{\text{Aut}(X)}(\mathcal{M})$, see [32, 42]. As we deform X , the moduli spaces \mathcal{M} may jump wildly, but the curve counts constructed using $[\mathcal{M}]_{\text{vir}}$ and $\mathcal{O}_{\mathcal{M}, \text{vir}}$ do not change.

It will seem like a small detail now, but a certain symmetrized version

$$(3) \quad \widehat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathcal{K}_{\mathcal{M}, \text{vir}}^{1/2} \otimes \dots$$

of the virtual structure sheaf has improved self-duality properties and links better with both representation theory and mathematical physics. The square-root factor in (3) is the square root of the virtual canonical bundle, which exists in special circumstances [87]. The importance of such twist in enumerative K-theory was emphasized by Nekrasov [86], the rationale being that for Kähler manifolds the twist by a square root of the canonical bundle turns the Dolbeault operator into the Dirac operator. We will always use (3), where the terms concealed by the dots will be specified after we specify the exact nature of \mathcal{M} .

1.5. One can talk about a representation-theoretic answer to an enumerative problem if the correspondences $\text{ev}_* [\cdot]_{\text{vir}}$ or $\text{ev}_* \widehat{\mathcal{O}}_{\text{vir}}$ are identified as elements of some sufficiently rich algebra acting by correspondences on X . Here rich may be defined pragmatically as allowing for computations or proofs.

This is exactly the same as being able to place the evolution operator of a quantum-mechanical or field-theoretic problem into a rich algebra of operators acting on its Hilbert space. A mathematical physicist would call this phenomenon *integrability*. Nekrasov and Shatashvili [88, 89] were to first to suggest, in the equivalent language of supersymmetric gauge theories, that the enumerative problems discussed here are integrable.

1.6. My personal intuition is that there are much fewer “rich” algebras than there are interesting algebraic varieties, which means there has to be something very special about X to have a real link between curve-counting in X and representation theory. In any case, the progress in enumerative geometry that will be described in these lectures is restricted to certain very special algebraic varieties.

Ten years ago in Seattle, Kaledin already spoke about *equivariant symplectic resolutions*, see [61] and Section 2.1, and the importance of this class of algebraic varieties has been only growing since. In particular, it was understood by Bezrukavnikov and his collaborators that the geometry of rational curves in an equivariant symplectic resolution X is tightly intertwined with the geometric structures described in [61] and, specifically, with the derived autoequivalences of X and the representation theory of its quantization.

I find it remarkable that here one makes contact with a very *different* interpretation of what it means for (1) to be geometric. A noncommutative algebra \mathcal{X} in the source of (1) may be constructed geometrically as a *quantization* of a symplectic algebraic variety X and this very much ties \mathcal{X} -modules with coherent sheaves on X . We denote this noncommutative algebra \mathcal{X} partly to avoid confusion with some algebra \mathcal{A} acting on X by correspondences but, in fact, in this subject, algebras of both kinds often trade places! This means that \mathcal{A} turns out to be related to a quantization of some X^\vee , and vice versa.

Dualities of this kind originate in supersymmetric gauge theories and are known under various names there, see e.g. [17, 27, 28, 58], mathematicians call them *symplectic duality* following [20], see also [21–23, 80]. Perhaps at the next summer institute someone will present a definite treatment of these dualities.

1.7. The relation between rational curves in X and representation theory of \mathcal{X} will be explained in due course below; for now, one can note that perhaps a link between enumerative geometry and representation theory exists for a class of varieties which is at least as large and about as special as equivariant symplectic resolutions.

Prototypical examples of equivariant symplectic resolutions are the cotangent bundles T^*G/P of projective homogeneous spaces¹. Perhaps equivariant symplectic resolutions form the right generalization of semisimple Lie algebras for the needs of today’s mathematics?

¹Quantum cohomology of T^*G/P was sorted out in [24, 103]. Arguably, it is even simpler than the beautiful story we have for G/P itself.

While equivariant symplectic resolutions await Cartan and Killing of the present day to classify them, the largest and richest class of equivariant symplectic resolutions known to date is formed by Nakajima quiver varieties [81, 82]. These are associated to a quiver, which means a finite graph with possibly loops and multiple edges. Quivers generalize Dynkin diagrams (but the meaning of multiple edges is different). From their very beginning, Nakajima varieties played an important role in geometric representation theory; that role has been only growing since.

1.8. Geometers particularly like Nakajima varieties associated to affine ADE quivers, because they are moduli of framed sheaves on the corresponding ADE surfaces. In particular, Hilbert schemes of points of ADE surfaces are Nakajima varieties. By definition, a map

$$(4) \quad f : C \rightarrow \text{Hilb}(S, \text{points}),$$

where C is a curve and S is a surface, is the same as a 1-dimensional subscheme of $Y = C \times S$, flat over C , and similarly for other moduli of sheaves on S . This is the prosaic basic link between curves in affine ADE Nakajima varieties and the enumerative geometry of sheaves on threefolds, known as the Donaldson-Thomas theory.² We are primarily interested in 1-dimensional sheaves on Y , and we will often refer to them as curves.³

Other Nakajima varieties resemble moduli spaces of sheaves on a symplectic surface — they can be interpreted as moduli of stable objects in certain 2-dimensional Calabi-Yau categories. Correspondingly, enumerative geometry of curves in Nakajima varieties has to do with counting stable objects in 3-dimensional categories, and so belongs to the Donaldson-Thomas theory in the broad sense.

1.9. To avoid a misunderstanding, a threefold Y does not need to be Calabi-Yau to have an interesting enumerative geometry or enumerative K-theory of sheaves. Calabi-Yau threefolds are certainly distinguished from many points of view, but from the perspective of DT counts, either in cohomology or in K-theory, the geometry of curves in the good old projective space is at least as interesting. In fact, certain levels of complexity in the theory collapse under the Calabi-Yau assumption.

When Y is fibered over a curve in ADE surfaces, its DT theory may be directly linked to curves in the corresponding Nakajima variety. While this is certainly a very narrow class of threefolds, it captures the essential information about DT counts in all threefolds in the following sense.

Any threefold, together with curve counts in it, may be glued out of certain model pieces, and these model pieces are captured by the ADE fibrations (in fact, it is enough to take only A_0 , A_1 , and A_2 surfaces). This may be compared with Chern-Simons theory of real threefolds: that theory also has pieces, described by

²It is not unusual for two very different moduli spaces to have isomorphic open subsets, and it doesn't prevent the enumerative information collected from the two from having little or nothing in common. It is important for S to be symplectic in order for the above correspondence to remain uncorrected enumeratively in cohomology. A further degree of precision is required in K-theory.

³There is the following clear and significant difference between the curves in $\text{Hilb}(S)$ and the corresponding curves in Y . The source C in (4) is a *fixed* curve, and from the enumerative point of view, it may be taken to be a union of rational components. The “curves” in Y being enumerated can be as arbitrary 1-dimensional sheaves as our moduli spaces allow.

the representation theory of the corresponding loop group or quantum group at a root of unity, which have to be glued to obtain invariants.⁴

The algebraic version of breaking up a threefold Y we need here is the following. Assume that Y may be degenerated, with a smooth total space of the deformation, to a transverse union $Y_1 \cup_D Y_2$ of two smooth threefolds along a smooth divisor D . Then, on the one hand, degeneration formula of Li and Wu [69] gives⁵ the curve counts in Y in terms of counts in Y_1 and Y_2 that also record the *relative* information at the divisor D . On the other hand, Levine and Pandharipande prove [68] that the relation

$$(5) \quad [Y] = [Y_1] + [Y_2] - [\mathbb{P}(\mathcal{O}_D \oplus N_{Y_1/D})]$$

generates all relations of the algebraic cobordism, and so, in particular, any projective Y may be linked to a product of projective spaces by a sequence of such degenerations.⁶

If Y is toric, e.g. a product of projective spaces, then equivariant localization may be used to break up curve counts into further pieces and those are all captured by $Y = \mathbb{P}^1 \times A_n$ geometries, see [73].

1.10. Nakajima varieties X are special even among equivariant symplectic resolutions, in the following sense. The algebras \mathcal{A} acting by correspondences on X are, in fact, *Hopf algebras*, which means that there is a coproduct, that is, an algebra homomorphism

$$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A},$$

a counit $\varepsilon : \mathcal{A} \rightarrow$ base ring, and an antipode, satisfying standard axioms (a certain completion may be needed for infinite-dimensional \mathcal{A}). More abstractly, these mean that the category $\mathcal{C} = \mathcal{A}\text{-mod}$ has

- a tensor product \otimes , which need *not* be commutative,
- a trivial representation, given by ε , which is identity for \otimes ,
- left and right duals, compatible with \otimes .

The notion of a tensor category is, of course, very familiar to algebraic geometers, except possibly for the part that allows for

$$(6) \quad V \otimes W \not\cong W \otimes V.$$

This noncommutativity is what separates representation theory of *quantum groups* from representation theory of usual groups. Representation-theorists know that a

⁴In fact, Donaldson-Thomas theory was originally developed as a complex analog of the Chern-Simons theory. It is interesting to note that while representation theory of quantum groups or affine Lie algebras is so-to-speak the DNA of Chern-Simons theory, the parallel role for Donaldson-Thomas theory is played by algebras which are quantum group deformations of *double* loops.

⁵In K-theory, there is a correction to the gluing formula as in the work of Givental [50].

⁶This does not trivialize DT counts, in particular, does not make them factor through algebraic cobordism, for the following reasons.

First, degeneration formula glues *nonrelative* (also known as absolute) DT counts out of *relative* counts in Y_1 and Y_2 and to use it again one needs a strategy for replacing relative conditions by absolute ones. Such strategy has been perfected by Pandharipande and Pixton [93, 94]. It involves certain universal substitution rules, which can be, again, studied in model geometries and are best understood in the language of geometric representation theory [99].

Second, one doesn't always get to read the equation (5) from left to right: sometimes, one needs to add a component to proceed. This means one will have to solve for curve counts in Y_1 in terms of those in Y and Y_2 , and a good strategy for this is yet to be developed.

mild noncommutativity makes a tensor category even more constrained and allows for an easier reconstruction of \mathcal{A} from \mathcal{C} , see e.g. [39, 40] for an excellent exposition.

The antipode, if it exists, is uniquely reconstructed from the rest of data (just like the inverse in a group). We won't spend time on it in these notes.

1.11. Specifically, in K-theoretic setting, we will have $\mathcal{A} = \mathcal{U}_{\hbar}(\widehat{\mathfrak{g}})$ for a certain Lie algebra \mathfrak{g} . This means that \mathcal{A} is a Hopf algebra deformation⁷ of $\mathcal{U}(\mathfrak{g}[t^{\pm 1}])$, where $\mathfrak{g}[t^{\pm 1}]$ denotes the Lie algebra of Laurent polynomials with values in \mathfrak{g} .

The Lie algebra \mathfrak{g} is by itself typically infinite-dimensional and even, strictly speaking, infinitely-generated. The action of \mathcal{A} will extend the action of $\mathcal{U}_{\hbar}(\widehat{\mathfrak{g}}_{\text{KM}})$ for a certain Kac-Moody Lie algebra $\mathfrak{g}_{\text{KM}} \subset \mathfrak{g}$ defined by Nakajima via an explicit assignment on generators [82].

The noncommutativity (6) will be mild and, in its small concentration, very beneficial for the development of the theory. Concretely, the “loop rotation” automorphism

$$(7) \quad t \mapsto u \cdot t, \quad u \in \mathbb{C}^{\times},$$

of $\mathcal{U}(\mathfrak{g}[t^{\pm 1}])$ deforms, and so every \mathcal{A} -module V gives rise to a family of modules $V(u)$ obtained by precomposing with this automorphism. As it will turn out, the commutativity in (6) is restored after a generic shift, that is, there exists an \mathcal{A} -intertwiner

$$R^{\vee}(u_1/u_2) : V_1(u_1) \otimes V_2(u_2) \xrightarrow{\sim} V_2(u_2) \otimes V_1(u_1)$$

which is a rational function of $u = u_1/u_2 \in \mathbb{C}^{\times}$. The poles and zeros of $\det R^{\vee}$ correspond to the values of the parameter for which the two tensor products are really not isomorphic.

1.12. Usually, one works with the so-called R -matrix

$$R(u) = (12) \cdot R^{\vee} \in \text{End}(V_1 \otimes V_2) \otimes \mathbb{Q}(u),$$

which intertwines the action defined via the coproduct Δ and via the opposite coproduct $\Delta_{\text{opp}} = (12)\Delta$. We denote by \otimes_{opp} this opposite way to take tensor products.

The shortest way to reconstruction of \mathcal{A} from \mathcal{C} is through the matrix coefficients of the R -matrix.

Concretely, since R is an operator in a tensor product, its matrix coefficients in V_1 give $\text{End}(V_2)$ -valued functions of u . The coefficients of the expansion of these functions around $u = 0$ or $u = \infty$ give us countably many operators in V_2 , which starts looking like an algebra of the right size to be a deformation of $\mathcal{U}(\mathfrak{g}[t^{\pm 1}])$. The Yang-Baxter equation satisfied by R

$$(8) \quad R_{V_1, V_2}(u_1/u_2) R_{V_1, V_3}(u_1/u_3) R_{V_2, V_3}(u_2/u_3) = \\ = R_{V_2, V_3}(u_2/u_3) R_{V_1, V_3}(u_1/u_3) R_{V_1, V_2}(u_1/u_2),$$

which is an equality of rational functions with values in $\text{End}(V_1 \otimes V_2 \otimes V_3)$, is a commutation relation for these matrix elements.

Thus, a geometric construction of an R -matrix gives a geometric construction of an action of a quantum group.

⁷with deformation parameter \hbar , which is the equivariant weight of the symplectic form on X in its geometric origin

1.13. Nakajima varieties, the definition of which will be recalled below, are indexed by a quiver and two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{\text{vertices}}$ and one sets

$$(9) \quad X(\mathbf{w}) = \bigsqcup_{\mathbf{v}} X(\mathbf{v}, \mathbf{w}).$$

For example, for framed sheaves on ADE surfaces, \mathbf{w} is the data at infinity of the surface (it includes the rank), while \mathbf{v} records possible values of Chern classes. In the eventual description of $K_{\mathbb{T}}(X(\mathbf{w}))$ as an module over a quantum group, the decomposition (9) will be the decomposition into the weight spaces for a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

To make the collection $\{K_{\mathbb{T}}(X(\mathbf{w}))\}$ into modules over a Hopf algebra, one needs tensor product. The seed from which it will sprout is very simple and formed by an inclusion

$$(10) \quad X(\mathbf{w}) \times X(\mathbf{w}') \hookrightarrow X(\mathbf{w} + \mathbf{w}')$$

as a fixed locus of a torus $\mathbb{A} \cong \mathbb{C}^{\times}$. For framed sheaves, this is the locus of direct sums. The pull-back in K-theory under (10), while certainly a worthwhile map, is not what we are looking for because it treats the factors essentially symmetrically.

Instead, we will use a certain canonical correspondence

$$\text{Stab}_{\mathfrak{C}, \dots} : K_{\mathbb{T}}(X^{\mathbb{A}}) \rightarrow K_{\mathbb{T}}(X),$$

called the *stable envelope*. It may be defined in some generality for a pair of tori $\mathbb{T} \supset \mathbb{A}$ acting on an algebraic symplectic manifold X so that \mathbb{A} preserves the symplectic form. As additional input, the correspondence takes a certain cone $\mathfrak{C} \subset \text{Lie } \mathbb{A}$, and other ingredients which will be discussed below. For $\mathbb{A} \cong \mathbb{C}^{\times}$, the choice of \mathfrak{C} is a choice of $\mathbb{R}_{\geq 0} \subset \mathbb{R}$ or, equivalently, the choice of \otimes versus \otimes_{opp} .

We demand that the two solid arrows in the diagram

$$(11) \quad \begin{array}{ccc} K_{\mathbb{T}}(X(\mathbf{w}))(a) \otimes K_{\mathbb{T}}(X(\mathbf{w}')) & & \\ \downarrow & \searrow^{\text{Stab}_+} & \\ R(a) \downarrow & & K_{\mathbb{T}}(X(\mathbf{w} + \mathbf{w}')) \\ \downarrow & \nearrow_{\text{Stab}_-} & \\ K_{\mathbb{T}}(X(\mathbf{w}))(a) \otimes_{\text{opp}} K_{\mathbb{T}}(X(\mathbf{w}')) & & \end{array}$$

are morphisms in our category. By $K_{\mathbb{T}}(\text{pt})$ -linearity, this forces the rational vertical map $R(a)$ because stable envelopes are isomorphisms after localization.

Here $a \in \mathbb{A} \cong \mathbb{C}^{\times}$ is an automorphism which does not act on $X^{\mathbb{A}}$ but does act on vector bundles on $X^{\mathbb{A}}$ through their linearization inherited from X . This twists the module structure by an automorphism as in Section 1.10 and identifies a with the parameter $u = u_1/u_2$ in the R-matrix.

As already discussed, the assignment of R -matrices completely reconstructs the Hopf algebra with its representations. The consistency of this procedure, in particular the Yang-Baxter equation (8), follow from basic properties of stable envelopes.

1.14. One may note here that even if one is primarily interested in Hilbert schemes of ADE surfaces, it is very beneficial to think about framed sheaves of

higher rank. In higher rank, we have nontrivial tori A acting by changing of framing and hence the associated R -matrices.

Similarly, it will prove easier to go through enumerative K-theory of curves in the moduli spaces of higher-rank sheaves even if one is only interested in curves in the Hilbert scheme of points.

At this point, we have introduces the basic curve-counting correspondences in X and certain algebras \mathcal{A} acting by correspondences in X . The main goal of these note is to discuss how one identifies the former inside the latter.

Acknowledgments.

This is a report about a rapidly developing field. This is certainly exciting, with the flip side that the field itself, let alone its image in the author's head, is very far from equilibrium. What I understand about the field it is very much a product of my interactions with M. Aganagic, R. Bezrukavnikov, P. Etingof, D. Maulik, I. Losev, H. Nakajima, N. Nekrasov, and others, both within the format of a joint work and outside of it. I am very grateful to all these people for what I have learned in the process.

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2. Basic concepts

2.1. Symplectic resolutions.

2.1.1. By definition [7, 61], a *symplectic resolution* is a smooth algebraic symplectic variety (X, ω) such that the canonical map

$$X \longrightarrow X_0 = \text{Spec } H^0(\mathcal{O}_X)$$

is projective and birational.

For us, the focus is the smooth variety X and the singular affine variety X_0 plays an auxiliary role, because, for example, there are no curves to count in X_0 . Of course, there exists also an exactly complementary point of view that singularities are essential and resolutions — auxiliary.

2.1.2. A symplectic resolution X is called *equivariant* or conical if there is a \mathbb{C}^\times action on X that scales ω and contracts X_0 to a point. In other words, there is a grading on $\mathbb{C}[X_0]$ such that

$$\mathbb{C}[X_0]_d = \begin{cases} \mathbb{C}, & d = 0, \\ 0, & d < 0. \end{cases}$$

The combination of the two requirements implies \mathbb{C}^\times must scale ω by a nontrivial character and hence

$$[\omega] = 0 \in H^2(X, \mathbb{C}),$$

since $H^*(X)$ is a trivial $\text{Aut}(X)_{\text{connected}}$ -module.

2.1.3. The prototypical example of an equivariant symplectic resolution is T^*G/P where $P \subset G$ is a parabolic subgroup of a semisimple Lie group. In particular,

$$(12) \quad T^*G/B \xrightarrow{\text{moment map}} \text{Nilcone} \subset \mathfrak{g}^* \cong \mathfrak{g}$$

is a famous resolution with many uses in geometric representation theory.

For an example with a less pronounced Lie-theoretic flavor one can take

$$\text{Hilb}(\mathbb{C}^2, n \text{ points}) \xrightarrow{\text{Hilbert-Chow}} S^n\mathbb{C}^2,$$

where the symplectic form comes from the standard $\omega_{\mathbb{C}^2} = dx_1 \wedge dx_2$. This is a basic example of a Nakajima variety.

2.2. Nakajima quiver varieties.

2.2.1. Nakajima quiver varieties [81, 82] are defined as algebraic symplectic reductions

$$(13) \quad \begin{aligned} X &= T^*M // G_v \\ &= \text{moment map}^{-1}(0) //_{\theta} G_v, \end{aligned}$$

where $G_v = \prod GL(V_i)$ and M is a linear representation of a special kind. Namely, M is a sum of the defining and so-called bifundamental representations, that is,

$$(14) \quad M = \bigoplus_i \text{Hom}(W_i, V_i) \oplus \bigoplus_{i \leq j} \text{Hom}(V_i, V_j) \otimes Q_{ij},$$

where W_i and Q_{ij} are certain multiplicity spaces on which G_v does not act. This gives an action

$$(15) \quad \prod GL(W_i) \times \prod GL(Q_{ij}) \times \mathbb{C}_\hbar^\times \rightarrow \text{Aut}(X),$$

where the last factor scales the cotangent directions with weight \hbar^{-1} . This inverse is conventional, it gives weight \hbar to the symplectic form on X .

2.2.2. In the GIT quotient in (13), the stability parameter θ is a character of G_v , up to proportionality. It has to avoid the walls of a certain finite central hyperplane arrangement for the quotient to be smooth.

What is special about representations (14) is that the stabilizer in G_v of any point in T^*M is cut out by linear equations on matrices and is the set of invertible elements in an associative algebra over \mathbb{C} . Therefore, it cannot be a nontrivial finite group. In general, algebraic symplectic reductions output many Poisson orbifolds, but it is difficult to produce a symplectic resolution in this way (or in any other way, for that matter).

2.2.3. The data of a group G_v and a representation (14) is encoded by a quiver Q and two dimension vectors. One sets

$$\begin{aligned} I &= \text{vertices of } Q \\ &= \text{factors in } G_v \end{aligned}$$

and joins two vertices $i, j \in I$ by $\dim Q_{ij}$ edges. The vectors

$$\mathbf{v} = (\dots, \dim V_i, \dots), \quad \mathbf{w} = (\dots, \dim W_i, \dots)$$

belong to \mathbb{N}^I which makes them dimension vectors in quiver terminology.

Nakajima varieties $X(\mathbf{v}, \mathbf{w})$ corresponding to the same quiver Q have a great deal in common and it is natural to group them together. In particular, it is natural to define $X(\mathbf{w})$ as in (9).

2.2.4. The apparent simplicity of the definition (13) is misleading. For example, let Q be the quiver with one vertex and one loop. Then $X(\mathbf{v}, \mathbf{w})$ is the moduli space of torsion-free sheaves \mathcal{F} on \mathbb{P}^2 , framed along a line, with

$$(\text{rk } \mathcal{F}, c_2(\mathcal{F})) = (\mathbf{w}, \mathbf{v}) \in \mathbb{N}^2.$$

This variety has deep geometry and plays a very important role in mathematical physics, in particular, in the study of 4-dimensional supersymmetric gauge theories and instantons.

Nakajima varieties have other uses in supersymmetric gauge theories as they all may be interpreted as certain moduli of *vacua*, which is distinct from their interpretation as *instanton* moduli for quivers of affine ADE type.

2.2.5. Let a rank n torus \mathbf{A} with coordinates $a_i \in \mathbb{C}^\times$ act on $X(\mathbf{w})$ via

$$W_k = \bigoplus_{i=1}^n a_i \otimes W_k^{(i)},$$

where $W_k^{(i)}$ are trivial \mathbf{A} -modules. The fixed points of this action correspond to analogous gradings

$$V_k = \bigoplus_{i=1}^n a_i \otimes V_k^{(i)},$$

and to arrows that preserve it. Therefore

$$X(\mathbf{w})^{\mathbf{A}} = \prod_{i=1}^n X(\mathbf{w}^{(i)}),$$

with the inherited stability condition. The case $n = 2$ with $(a_1, a_2) = (a, 1)$ appears in (10).

2.3. Basic facts about rational curves in X .

2.3.1. Following the outline of section 1.4, but still not sewing permanently was tacked there, consider a moduli space \mathcal{M} of rational curves with 2 marked points in an equivariant symplectic resolution X . From formula (2), the virtual dimension of \mathcal{M} equals

$$\text{vir dim } \mathcal{M} = \dim X - 1,$$

independent of the degree, because $c_1(X) = 0$. This will be the dimension of $\text{ev}_* [\mathcal{M}]_{\text{vir}}$ in $X \times X$.

2.3.2. Deformations of (X, ω) are described⁸ by the period map

$$[\omega] \in H^2(X, \mathbb{C}),$$

see [61]. In particular, a generic deformation X' will have no classes $\alpha \in H_2(X', \mathbb{Z})$ such that $\int_\alpha \omega_{X'} = 0$ and hence no algebraic curves of any kind (in fact, X' will be affine). Therefore any deformation-invariant curve counts in X must vanish.

To avoid such trivialities, we consider equivariant curve counts, where we must include the action of the \mathbb{C}_\hbar^\times factor that scales the symplectic form. It scales nontrivially the identification of $H^2(X, \mathbb{C})$ with the base of the deformation, hence there are no \hbar -equivariant deformations of X .

Reflecting this, we have

$$[\mathcal{M}]_{\text{vir}} = \hbar [\mathcal{M}]_{\text{vir, reduced}}$$

⁸For Nakajima varieties, these deformations may be described explicitly by changing the value of moment map in (13).

where $\hbar \in H^{\bullet}_{\text{Aut}(X)}(\text{pt})$ is the weight of the symplectic form and

$$\dim [\mathcal{M}]_{\text{vir, reduced}} = \dim X .$$

2.3.3. Clearly,

$$\text{ev}(\mathcal{M}) \subset X \times_{X_0} X$$

and by a fundamental property of symplectic resolutions, see [62, 83],

$$(16) \quad \mathbf{St} = X \times_{X_0} X \subset X \times X$$

is at most half-dimensional with Lagrangian half-dimensional components. In the example (12), the subscheme (16) is known as the *Steinberg* variety, and it is convenient to extend this usage to general X .

Putting two and two together, we have the following basic

LEMMA 2.1 ([24]). *For any degree, $\text{ev}_* [\mathcal{M}]_{\text{vir, reduced}}$ is a \mathbb{Q} -linear combination of Lagrangian components of the Steinberg variety.*

Rational coefficients appear here because $[\mathcal{M}]_{\text{vir}}$ is defined with \mathbb{Q} -coefficients, due to automorphisms of objects parametrized by \mathcal{M} . Intuitively, it seems unlikely that the coefficients in Lemma (2.1) really depend on the details of the construction of \mathcal{M} , as long as it has a perfect obstruction theory. In fact, for Nakajima varieties, all possible flavors of curve counting theories give the same answer in cohomology.

2.3.4. One can make a bit more progress on basic principles. If $d \in H_2(X, \mathbb{Z})_{\text{eff}}$ is a degree of a rational curve, we denote by z^d the corresponding element of the semigroup algebra of $H_2(X, \mathbb{Z})_{\text{eff}}$. The spectrum of this semigroup algebra is an affine toric variety; it is a toric chart corresponding to X is the so-called Kähler moduli space \mathfrak{K} , see e.g. [33] for a discussion of the basic terminology of quantum cohomology.

For a Nakajima variety X ,

$$\mathfrak{K} = \overline{H^2(X, \mathbb{C})} / \overline{H^2(X, \mathbb{Z})}$$

is a toric variety corresponding to the fan in $H^2(X, \mathbb{Z})$ formed by the ample cones of all flops of X . By surjectivity theorem of [79], these are the cones of nonsingular values of the stability parameter θ .

We consider the operator of *quantum multiplication* in $H^{\bullet}_+(X)$ by $\lambda \in H^2(X)$

$$(17) \quad \lambda \star \cdot = \lambda \cup \cdot + \hbar \sum_d (\lambda, d) z^d \text{ev}_* [\mathcal{M}_d]_{\text{vir, reduced}} ,$$

where \mathcal{M}_d is the moduli space of curves of degree d . Here $\lambda \cup \cdot$ is the operator of cup product by λ , it is supported on the diagonal of $X \times X$ as a correspondence. On very general grounds, the operators (17) commute and, moreover, the quantum, or Dubrovin, connection with operators

$$(18) \quad \nabla_{\lambda} = \varepsilon \frac{\partial}{\partial \lambda} - \lambda \star \cdot , \quad \frac{\partial}{\partial \lambda} z^d = (\lambda, d) z^d$$

is flat for any ε .

For equivariant symplectic resolutions, one conjectures that after a shift

$$(19) \quad z^d \mapsto (-1)^{(\kappa_X, d)} z^d$$

by a certain canonical element $\kappa_X \in H^2(X, \mathbb{Z}/2)$ called the canonical theta characteristic in [75], the quantum connection takes the following form:

$$(20) \quad \text{quantum part in (17)} \stackrel{?}{=} \hbar \sum_{\alpha} (\lambda, \alpha) \frac{z^{\alpha}}{1 - z^{\alpha}} L_{\alpha} + \dots, \quad L_{\alpha} \in H_{\text{top}}(\mathbf{St}),$$

where dots stand for a multiple of the diagonal component of \mathbf{St} , that is, for a scalar operator.

Such scalar ambiguity is always present in this subject and is resolved here by the fact that

$$\forall \gamma \in H^{\bullet}(X), \quad 1 \star \gamma = \gamma,$$

and so the operator (20) annihilates $1 \in H^0(X)$. The sum (20) is over a certain finite set of effective classes α .

It is clear from (20) that (18) is a connection on \mathfrak{K} with regular singularities. Conjecture (20) is known for all concrete X discussed in these notes⁹ and will be assumed in what follows.

Let \mathcal{A}_{min} be the algebra of endomorphisms of $H_{\top}^{\bullet}(X)$ generated by the operators of cup product and $H_{\text{top}}(\mathbf{St})$. Clearly, the operators (17) lie in $\mathcal{A}_{\text{min}}[[z]]$. In all analyzed examples, these have a simple spectrum for generic z and thus generate the full algebra of operators of quantum product by $\gamma \in H_{\top}^{\bullet}(X)$; one expects this to happen in general. It would mean (17) generate a family, parametrized by z , of maximal commutative subalgebras of \mathcal{A}_{min} which deforms the algebra of cup products for $z = 0$.

In examples, this matches very nicely with known ways to produce maximal commutative subalgebras. For instance, for Nakajima varieties, the operator (17) lie in what is known as *Baxter* maximal commutative subalgebra in a certain \mathcal{A} as a corollary of the main result of [75].

One can imagine the above structures to be extremely constraining for general equivariant symplectic resolutions.

2.3.5. Now, in K-theory, the only thing that carries over from the preceding discussion is that, for any group G ,

$$(21) \quad \text{ev}_{*} : K_G(\mathcal{M}) \rightarrow K_G(\mathbf{St}),$$

where there right-hand side denotes K-classes with support in $\mathbf{St} \subset X^{\times 2}$. This is an algebra under convolution.

The best case scenario is when one has a complete control over $K_G(\mathbf{St})$. For example, one of the main results of [30], see Chapter 7 there, shows

$$K_{G \times \mathbb{C}_h^{\times}}(\mathbf{St}) = \text{affine Hecke algebra},$$

for $X = T^*G/B$. This is a beautiful algebra with a beautiful presentation; perhaps it would be too much to expect an equally nice description in general. The approach explained in these notes doesn't assume any knowledge about $K_G(\mathbf{St})$ as a prerequisite.

⁹Concretely, it is known for Nakajima varieties by [75], for T^*G/P by [103], and for hyper-toric varieties by [77].

2.3.6. Even when the target of the map (21) is under control, it is not so easy to identify, say, $\text{ev}_* \mathcal{O}_{\mathcal{M}, \text{vir}}$ in it. The dimension argument, which identifies $\text{ev}_* [\mathcal{M}]_{\text{vir, reduced}}$ with linear combinations of component of \mathbf{St} obviously does not apply in K-theory, so new ideas are needed. There exists a body of work, notably by Givental and his collaborators [51–53], that aims to lift the general structures of quantum cohomology to quantum K-theory of an equally general algebraic variety X . This theory has been studied for G/B itself and also, for example, for toric varieties. It would be interesting to see what it outputs for T^*G/B .

A different strategy for dealing with the complexities of K-theory works only for special X and is based on a systematic use of rigidity and self-duality arguments, see for example [90] for an introduction. For them to be available, one needs the obstruction theory of \mathcal{M} to have a certain degree of self-duality in the first place and to work with symmetrized K-classes on \mathcal{M} such as (3).

2.3.7. Let

$$f : (C, p_1, p_2) \rightarrow X$$

be a map of a 2-pointed rational curve to X , which we assume is symplectic, and so its tangent bundle TX is self-dual up-to the weight of the symplectic form

$$(TX)^\vee = \hbar TX.$$

This makes the obstruction theory

$$\text{Def } f - \text{Obs } f = H^\bullet(C, f^*TX)$$

self-dual up to a certain correction

$$(22) \quad (\text{Def} - \text{Obs})^\vee + \hbar (\text{Def} - \text{Obs}) = \hbar H^\bullet(C, f^*TX \otimes (\mathcal{O}_C - \mathcal{K}_C)).$$

For an irreducible curve, we have $\mathcal{O}_C - \mathcal{K}_C = \mathcal{O}_{p_1} + \mathcal{O}_{p_2}$, equivariantly with respect to $\text{Aut}(C, p_1, p_2)$, and we can use marked points to compensate¹⁰ for the correction in (22). But this breaks down when C is allowed to break and we get

$$\mathcal{O}_C - \mathcal{K}_C \neq \mathcal{O}_{p_1} + \mathcal{O}_{p_2}$$

away from the locus of chains of rational curves joining p_1 and p_2 .

The moduli space $\overline{\mathcal{M}}_{0,2}(X)$ of stable rational maps to X allows arbitrary trees as domains of the map, and so it appears problematic to make its virtual structure sheaf self-dual. As an alternative, one can use the moduli space $\text{QM}(X)$ of stable quasimaps to X . This has its origin in supersymmetric gauge theories and is known to geometers in various flavors. The version best suited for our needs has been designed by Ciocan-Fontanine, Kim, and Maulik in [31]. It requires a GIT presentation

$$(23) \quad X = \tilde{X} // G$$

where G is reductive and \tilde{X} is an affine algebraic variety with at worst locally complete intersection singularities in the unstable locus. It remains to be seen which of the equivariant symplectic resolutions can be presented in this way; for Nakajima varieties, it is provided by their construction. When available, quasimaps spaces have marked advantages in enumerative K-theory.

¹⁰These contributions at the marked points are the dots in (3), see [90] for more details.

2.3.8. For example, for $X = \text{Hilb}(\mathbb{C}^2, \text{points})$, the quasimaps from a fixed curve C to X are naturally identified with stable pairs or points in Pandharipande-Thomas moduli spaces [95] for $Y = C \times \mathbb{C}^2$. A stable pair is a pure 1-dimensional sheaf \mathcal{F} on Y with a section

$$\mathcal{O}_Y \xrightarrow{s} \mathcal{F}$$

such that $\dim \text{Coker } s = 0$. These moduli spaces have several advantages over other DT moduli spaces, including the Hilbert schemes of curves, and are used very frequently. Marked points $p_1, p_2 \in C$ introduce a *relative* divisor

$$D = D_1 \sqcup D_2 \subset Y$$

where D_i is the fiber over p_i . To have an evaluation map at these marked points, the curve C is allowed to develop a chain of rational components at each p_i ; note that the whole curve C remains a chain in this process. To have a moduli space of maps from a nonrigidified rational curve, we must quotient the above by $\text{Aut}(C, p_1, p_2)$. These are known affectionately as *accordions* among the practitioners.

Quasimaps to a general Nakajima variety behave in a very similar way, see [90] for an elementary introduction.

2.3.9. In fact, for quasimaps from either a rigid 2-pointed curve C or from pure accordions to a Nakajima variety X , one gets the *same* answer

$$(24) \quad \mathbf{G} = \text{ev}_* \widehat{\mathcal{O}}_{\text{QM}(X)} \in K_{\text{Aut } X}(\mathbf{St})$$

known as the *glue* matrix, see Theorem 7.1.4 in [90]. There is also a q -difference connection generalizing (18). The geometric meaning of the increment q in this q -difference equation is

$$q \in \mathbb{C}^\times = \text{Aut}(C, p_1, p_2), \quad (C, p_1, p_2) \cong (\mathbb{P}^1, 0, \infty).$$

In fact, the variable ε in (18) belongs to $\text{Lie } \text{Aut}(C, p_1, p_2)$ from its geometric origin. In contrast to cohomology, the operator of the quantum q -difference equation is much more involved than (24). In particular, the matrix of the q -difference equation depends on q , whereas the glue matrix doesn't. The glue matrix (24) may be obtained from the quantum difference equation by a certain limit.

Quantum difference equations for all Nakajima varieties have been determined in [92]. The description is representation-theoretic and a certain language needs to be developed before we can state it.

3. Roots and braids

3.1. Kähler and equivariant roots.

3.1.1. If equivariant symplectic resolutions are indeed destined to generalize semisimple Lie algebras, one should be able to say what becomes of the classical root data in this more general setting.

For a semisimple Lie group G , we have:

- roots, which are the nonzero weights of a maximal torus $A \subset G$ in the adjoint representation,
- coroots, which have to do with special maps $SL(2) \rightarrow G$.

The resolution $X = T^*G/B$ by itself does not distinguish between locally isomorphic groups. It is the adjoint group G_{ad} that acts naturally by symplectic automorphisms of X . On the other hand,

$$H_2(X, \mathbb{Z}) = \text{coroot lattice of } G_{\text{sc}},$$

where G_{sc} is the simply-connected group and this lattice is spanned by the images

$$\mathbb{P}^1 = SL(2)/B_{SL(2)} \xrightarrow{\text{coroot}} G/B \subset X.$$

The equivariant and Kähler roots of symplectic resolutions will generalize the roots and coroots for $X = T^*G/B$, respectively. They capture the weights of torus action and special rational curves in X , respectively.

3.1.2. Let

$$A \subset \text{Aut}(X, \omega)$$

be a maximal torus in the group of symplectic automorphisms of an equivariant symplectic resolution X .

DEFINITION 3.1. The *equivariant* roots of X are the weights of a maximal torus A in the normal bundle N_{X/X^A} to its fixed locus X^A .

For $X = T^*G/B$, this gives the roots of the adjoint group. For $X = \text{Hilb}(\mathbb{C}^2, n)$, the torus A is the maximal torus in $SL(\mathbb{C}^2)$. We have

$$X^A = \{\text{monomial ideals}\}$$

and the weights of A in the tangent space to these points are classically computed in terms of the hook-lengths for the corresponding partitions of n . Therefore

$$\text{equivariant roots of } \text{Hilb}(\mathbb{C}^2, n) = \{\pm 1, \dots, \pm n\}.$$

We see that, in contrast to finite-dimensional Lie theory, roots may be proportional for symplectic resolutions.

3.1.3. As in the classical Lie theory, equivariant roots define root hyperplanes in $\text{Lie } A$ and these partition the real locus into a finite set of chambers \mathfrak{C} . Once a chamber is fixed, it divides equivariant roots into positive and negative.

Geometrically, this means splitting normal directions to X^A into attracting and repelling directions with respect to a generic 1-parameter subgroup in A .

In these notes, we will see arrangements of both linear and affine rational hyperplanes and the components into which the hyperplanes partition the real locus will play an important role. In both linear (or *central*) and affine situations, these components are often called chambers or regions. To distinguish between the two, we will call the regions of central and affine arrangements *cones* and *alcoves*, respectively.

We call a codimension 1 stratum a *wall*. A wall is a part of a hyperplane of the arrangement that separates two regions.

3.1.4. The best definition of Kähler root of X currently known to me is:

DEFINITION 3.2. An effective curve class α is a positive Kähler root of X if it appears in the sum (20).

It would be clearly desirable to have a more direct definition of a root. One may try to define

$$\bigcup_{\alpha > 0} N\alpha \subset H_2(X, \mathbb{Z})_{\text{eff}}$$

as those classes that remain effective in a codimension one deformation of X as in Section 2.3.2. There seems to be no simple way to pick out the roots among all of their multiples using this approach.

Roots are also related to hyperplanes in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ along which stable envelope jump, see below and in particular (42). This approach requires a torus action which may not exist in general, and so cannot be used as a definition.

3.1.5. For $X = T^*G/B$, the components of the Steinberg variety

$$\mathbf{St} = \bigcup_{w \in W(G)} L_w$$

are indexed by elements of the Weyl group of G . By the main result of [24], Kähler roots α of X are the coroots of G_{sc} with

$$L_\alpha = L_{s_\alpha},$$

where s_α is the corresponding reflection.

3.1.6. Let $X = X_Q(\mathbf{v}, \mathbf{w})$ be a Nakajima variety associated to a quiver Q with vertex set I . The construction of [75], the main points of which will be explained below, associates to X a certain Lie algebra with a Cartan decomposition

$$(25) \quad \mathfrak{g}_Q = \mathfrak{h}_Q \oplus \bigoplus_{\alpha} (\mathfrak{g}_Q)_{\alpha},$$

in which the root subspaces are finite-dimensional and are indexed by roots $\alpha \in \mathbb{Z}^I$. Additionally

$$(26) \quad (\mathfrak{g}_Q)_{-\alpha} = (\mathfrak{g}_Q)_{\alpha}^{\vee}$$

with respect to an invariant bilinear form on \mathfrak{g}_Q .

For example, for the quiver Q with one vertex and one loop one has $\mathfrak{g}_Q = \widehat{\mathfrak{gl}}(1)$ and the roots of this algebra are all nonzero integers $\mathbb{Z} \setminus \{0\}$.

As a corollary of the main result of [75], we have

$$(27) \quad \text{Kähler roots of } X_Q(\mathbf{v}, \mathbf{w}) = \text{roots } \alpha \text{ of } \mathfrak{g}_Q \text{ such that } \pm\alpha \leq \mathbf{v},$$

assuming $X_Q(\mathbf{v}, \mathbf{w}) \neq \emptyset$. Here $\alpha \leq \mathbf{v}$ means that $\mathbf{v} - \alpha \in \mathbb{N}^I$. In particular,

$$\text{Kähler roots of } \text{Hilb}(\mathbb{C}^2, n) = \{\pm 1, \dots, \pm n\}.$$

3.1.7. In exceptional situations, it may turn out that the presentation of the form (20) is not unique because of special linear dependences that can exist between L_α and the identity operator. An example is

$$(28) \quad \text{Hilb}(\mathbb{C}^2, 2) \cong T^*\mathbb{P}^1 \times \mathbb{C}^2,$$

in which an attentive reader will notice that the two results just quoted give $\{\pm 1, \pm 2\}$ and $\{\pm 1\}$ as roots, respectively.

Perhaps such nonuniqueness is always an indication of nonuniqueness of a lift to K-theory? For example, the two points of view in (28) lead to *different* lifts to K-theory, with different sets of singularities, matching the different sets of roots.

3.1.8. There is a remarkable partial duality on equivariant symplectic resolutions which extends

$$T^*G/B \xleftarrow{\text{Langlands}} T^*{}^L G/{}^L B.$$

It has its origin in supersymmetric gauge theories, see e.g. [20, 28] for a recent treatment, and is known in mathematics under the name of *symplectic duality*. Among other things, it should exchange equivariant and Kähler roots. For example, the Hilbert schemes of points in \mathbb{C}^2 are self-dual.

3.2. Braid groupoid.

3.2.1. In classical Lie theory, a central role is played by the Weyl group of a root system. It is a natural group of symmetries which is large enough to act transitively on the set of cones. Further, the alcoves of the *affine arrangement*

$$(29) \quad \langle \alpha, x \rangle \in \mathbb{Z}$$

are permuted transitively by the affine Weyl group.

There are natural finite groups of symmetries for both equivariant and Kähler roots. On the equivariant side, we have the Weyl group

$$\text{Weyl}(\mathbf{A}) = \text{Normalizer}(\mathbf{A})/\mathbf{A}$$

where the normalizer is taken inside $\text{Aut}(X, \omega)$. On the Kähler side, there is an analog of the Weyl group constructed by Y. Namikawa in [84], see also e.g. the discussion of the topic in Section 2.2 of [19]. In the affine case, one can take the semidirect product of this finite group with a suitable lattice.

However, roots of either kind are intrinsically just not symmetric enough, e.g. for $X = \text{Hilb}(\mathbb{C}^2, n)$, the alcoves are separated by the walls

$$\left\{ \frac{a}{b} \right\} \subset \mathbb{R}, \quad a \in \mathbb{Z}, b \in \{1, \dots, n\}$$

and they are very far from being transitive under $\{\pm\} \times \mathbb{Z}$.

It is best to embrace the idea that different chambers and alcoves are different entities that only become related in some deeper way. In this new paradigm, the place of a transitive symmetry group of a real hyperplane arrangement will be taken by the *fundamental groupoid* of the same arrangement. I learned this point of view from R. Bezrukavnikov.

3.2.2. A groupoid is a category in which every morphism is invertible. The fundamental groupoid $\pi_1(U)$ of a topological space U is formed by paths, taken up to homotopy that fixes endpoints. Let $U = \mathbb{C}^n \setminus \bigcup H_i$ be the complement of a complexification of a real hyperplane arrangement. Here

$$H_i = \{f_i(z) = 0\},$$

where f_i is a real affine-linear function. The groupoid $\pi_1(U)$ was studied by Deligne [35], Salvetti [96], and many others. As in the survey [108], define

$$\mathbb{C}^\times = \mathbb{C}_> \sqcup \mathbb{C}_< \sqcup \mathbb{C}_\uparrow \sqcup \mathbb{C}_\downarrow$$

where

$$\mathbb{C}_{\geq} = \{\Re x \geq 0\}, \quad \mathbb{C}_{\uparrow\downarrow} = \{\Re x = 0, \Im x \geq 0\}.$$

For any sequence $*_1, *_2, \dots$ of symbols from $\{>, <, \uparrow, \downarrow\}$, we set

$$(30) \quad U_{*_1, *_2, \dots} = \bigcap f_i^{-1}(\mathbb{C}_{*_i}).$$

Nonempty intersections of the form (30) decompose U . The Salvetti complex is, by definition, dual to this decomposition. It is a deformation retract of U .

We get 0-cells of the Salvetti complex when we choose $*_i \in \{\geq\}$ for all i ; these corresponds to the alcoves of $\mathbb{R}^n \setminus \bigcup H_i$ and give the objects of $\pi_1(U)$. The morphisms in $\pi_1(U)$ are generated by 1-cells of the Salvetti complex and those correspond to a wall between two alcoves plus a choice of over- or under-crossing, that is, a choice of \uparrow versus \downarrow for the corresponding equation f_i .

The relations correspond to 2-cells and those correspond an alcove ∇ meeting several others along a stratum S of codimension 2, as in Figure 1. We set

$$*_i = \begin{cases} \geq, & f_i(S) \geq 0, \\ \updownarrow, & f_i(S) = 0, f_i(\nabla) \geq 0. \end{cases}$$

The corresponding $U_{**\dots}$ fibers over S with fiber $i\text{Cone}_S\nabla$, where $\text{Cone}_S\nabla$ denotes the tangent cone to ∇ at S . We get a braid relations of the form shown in Figure 1, where the path stays in $\mathbb{R}^n + i\text{Cone}_S\nabla$.

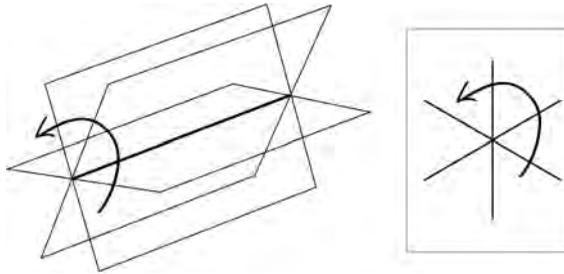


FIGURE 1. Product of morphisms around a stratum of codimension 2 is the identity in the dynamical groupoid. In the fundamental groupoid, the imaginary parts of the paths have to lie in one of the cones.

3.2.3. For future use, we note a generalization of $\pi_1(U)$ called the *dynamical* groupoid of the arrangement. It has the same objects and for every wall w between two alcoves a single invertible matrix $B_w(f_i(z))$ which *depends* on the variables $z \in \mathbb{C}^n$ through the equation f_i of hyperplane H_i containing w . The variables z are called the dynamical variables.

These matrices are required to satisfy one braid relation as in Figure 1 for any stratum of codimension 2. In other words, for the dynamical groupoid, there is a bijection

$$\begin{aligned} \text{objects} & \leftrightarrow \text{codim} = 0 \text{ strata} \\ \text{morphisms} & \leftrightarrow \text{codim} = 1 \text{ strata} \\ \text{relations} & \leftrightarrow \text{codim} = 2 \text{ strata} \end{aligned}$$

where strata refer to the stratification of \mathbb{R}^n . To go from the dynamical groupoid to the fundamental groupoid, we set

$$(31) \quad B_w^{\updownarrow} = \lim_{x \rightarrow \pm i\infty} B_w(x),$$

assuming this limit exists.

We will meet the dynamical groupoids in exponential form, that is, as an arrangement of codimension 1 subtori in an algebraic torus. The fundamental groupoid is then found from the values of $B_w(z)$ at the fixed points of a certain toric compactification.

The Yang-Baxter equation (8) is a very prominent example of a relation in a dynamical groupoid.

3.2.4. We now consider

$$U = H^2(X, \mathbb{C}) \setminus \text{Kähler arrangement}$$

where Kähler arrangement is the affine arrangement (29) corresponding to Kähler roots $\{\alpha\} \subset H_2(X, \mathbb{Z})$. It is locally finite and periodic under the action of

$$\text{Pic}(X) \cong H^2(X, \mathbb{Z}).$$

The fundamental groupoid $\pi_1(U)$ appears in two contexts, the relation between which is rather deep. These are:

- monodromy of the quantum differential equation (18),
- autoequivalences of $D^b \text{Coh}_T X$,

where $T \subset \text{Aut}(X)$ is a maximal torus.

In the early days of mirror symmetry, it was conjectured by M. Kontsevich that any X and X' that have a common Kähler moduli space \mathfrak{K} are derived equivalent and, moreover, there is a derived equivalence for any homotopy class of paths from X to X' in the regular locus of the quantum connection. This idea became more concrete with the advent of Bridgeland’s theory of stability conditions [25, 26]

More recently, it was conjectured by R. Bezrukavnikov (with perhaps some infinitesimal input from the author of these notes, which is how is often attributed in the literature, see [3, 37]) that, first, this works fully equivariantly¹¹ and, second, matches specific derived automorphisms of X obtained via quantization in characteristic $p \gg 0$.

3.2.5. It is clear from definitions that the groupoid $\pi_1(U)$ acts by monodromy, that is, analytic continuation of solutions of (18). Indeed, the natural isomorphism

$$(32) \quad H^2(X, \mathbb{C})/H^2(X, \mathbb{Z}) \cong \text{Spec } \mathbb{Z}H_2(X, \mathbb{Z}) \subset \mathfrak{K}$$

takes U to the regular locus of the quantum connection. Here $\mathbb{Z}H_2(X, \mathbb{Z})$ is the group algebra of the lattice $H_2(X, \mathbb{Z})$.

Let $z = 0_X$ be the origin in the chart $\text{Spec } \mathbb{Z}H_2(X, \mathbb{Z})_{\text{eff}}$. Via the monodromy of (18), the fundamental group of $U/\text{Pic}(X)$ acts on the fiber at 0. This fiber is $H_T^\bullet(X, \mathbb{C})$, but we identify it with $K_T(X) \otimes \mathbb{C}$ as in the work of Iritani [59]. Namely, one considers the map

$$(33) \quad K_T(X) \ni \mathcal{F} \rightarrow \Gamma \cup (2\pi i)^{\frac{\text{deg}(\cdot) - \dim_{\mathbb{C}} X}{2}} \text{ch } \mathcal{F} \in H_T^\bullet(X, \mathbb{C})_{\text{mero}}$$

where

$$\Gamma = \prod_{t_i = \text{Chern roots of } TX} \Gamma(1 + t_i),$$

$\text{deg}(\cdot)$ denotes the degree operator, and $H_T^\bullet(X, \mathbb{C})_{\text{mero}}$ denotes meromorphic functions on the spectrum of $H_T^\bullet(X, \mathbb{C})$. Because of the identity

$$\Gamma(1 + t) \Gamma(1 - t) = \frac{2\pi it}{e^{\pi it} - e^{-\pi it}},$$

the map (33) is an analog of Mukai’s map $\mathcal{F} \rightarrow \text{ch}(\mathcal{F})\sqrt{\text{Td}X}$ in that it is, up to a scalar multiple, an isometry for the natural *sesquilinear* inner products in the source and the target. It makes the monodromy act by unitary transformations of $K_T(X) \otimes \mathbb{C}$. Iritani’s motivation was to find an *integral* structure in the quantum connection, and this will match nicely with what follows.

¹¹Equivariant mirror symmetry is a notoriously convoluted subject, see for example [97].

We note that the q -analog of the Γ -function is the function

$$\begin{aligned} \Gamma_q(t) &= \prod_{n \geq 0} \frac{1}{1 - q^n t} \\ (34) \qquad &= \text{tr } \mathbb{C}[\text{Maps}(\mathbb{C}(q) \rightarrow \mathbb{C}(t^{-1}))] \end{aligned}$$

which is clearly relevant in enumerative K-theory of maps to X because of the second line in (34). Here the equivariant weight of the source $\mathbb{C}(q)$ and of the target $\mathbb{C}(t^{-1})$ are indicated in parentheses. See Section 8 of [90] for how these functions come up in the quantum difference equation for X .

3.2.6. Quantization of equivariant symplectic resolutions is a very fertile ground which is currently being explored by several teams of researchers, see for example [3, 6, 10–15, 19, 21, 63, 70–72, 78].

A quantization of X is, first, a sheaf $\mathcal{O}_{\widehat{X}}$ of noncommutative algebras deforming the sheaf $(\mathcal{O}_X, \{ \cdot, \cdot \})$ of Poisson algebras and, second, the algebra

$$\mathcal{X} = \Gamma(\mathcal{O}_{\widehat{X}})$$

of its global sections¹². In this way, very nontrivial algebras can be constructed as global section of sheaves of very standard algebras — a modern day replacement of the old dream of Gelfand and Kirillov [47]. Whenever \mathcal{X} is of finite homological dimension, one has a similar global description of $D^b \mathcal{X}\text{-mod}$.

Quantizations \mathcal{X}_λ come in families parametrized by the same data

$$\lambda \in \text{Pic}(X) \otimes \text{ground field}$$

as the commutative deformations. For Nakajima varieties, these can be described explicitly as so-called quantum Hamiltonian reductions [38].

3.2.7. While these notes are certainly not the place to survey the many successes of quantization of symplectic resolutions, certain important features of quantizations in characteristic $p \gg 0$, developed by Bezrukavnikov and Kaledin, are directly related to our narrative.

For an integral quantization parameter λ away from certain walls, that is for

$$\lambda \in \text{Pic}(X) \cap pU'$$

where

$$U' = \text{Pic}(X) \otimes \mathbb{R} \setminus \text{neighborhood of a periodic hyperplane arrangement,}$$

the theory produces an equivalence

$$(35) \qquad D^b \text{Coh}_{\mathbf{A}(1), \text{supp}} X^{(1)} \xleftarrow{\sim} D^b \mathcal{X}_\lambda\text{-mod}_{\mathbf{A}, \text{supp}}$$

where

$$\mathbf{A} \subset \text{Aut}(X, \omega) = \text{Aut}(\mathcal{O}_{\widehat{X}_\lambda})$$

is a maximal torus, $X^{(1)}$ denotes Frobenius twist, and one needs to impose a certain condition on the supports of sheaves and modules in (35). We will assume that the projection of the set-theoretic support to the affinization X_0 is contracted to $0 \in X_0$ by a certain 1-parameter subgroup in \mathbf{A} ; this means a version of the category \mathcal{O} on the quantization side.

On the commutative side, we have a larger torus $\mathbf{T} \supset \mathbf{A}$ of automorphisms, thus the quantization side gets an extra grading so that (35) is promoted to a

¹²The vanishing $H^i(\mathcal{O}_X) = 0$ for $i > 0$ and any symplectic resolution implies the same for $\mathcal{O}_{\widehat{X}}$, which is a very valuable property in the analysis of \mathcal{X} .

T-equivariant statement. The extra grading on $\mathcal{X}_\lambda\text{-mod}$ captures quite subtle representation-theoretic information. In fact, such graded lifts of representation categories are absolutely central to a lot of progress in modern representation theory, see e.g. [9, 102].

3.2.8. The map (32) send $U' \subset \text{Pic}(X) \otimes \mathbb{R}$ to the central torus $|z| = 1$ in \mathfrak{K} and one can associate the equivalence (35) to a straight path from the point $z = 0_X \in \mathfrak{K}$ to a component of U' .

One, meaning Bezrukavnikov, conjectures that:

- the hyperplane arrangement in the definition of U' is the affine Kähler arrangement¹³ as in Section 3.2.4,
- the equivalences (35) define a representation of $\pi_1(U)$, and
- their action on $K_T(X)$ equals the monodromy of the quantum differential equation,

where we use Iritani’s map as in Section 3.2.5 to lift monodromy to K-theory.

The second part of this conjecture means the following. For $\lambda \in pU'$, the algebras \mathcal{X}_λ are canonically identified for all flops of X and, for two quantization parameters λ, λ' separated by a wall w , we can consider the induced equivalence, that is, the horizontal row in the following diagram:

$$(36) \quad \begin{array}{ccc} & \text{D}^b \text{Coh } X^{(1)} & \\ & \swarrow \quad \searrow & \\ \text{D}^b \mathcal{X}_\lambda\text{-mod} & \xrightarrow{B_w^{\uparrow\downarrow}} & \text{D}^b \mathcal{X}_{\lambda'}\text{-mod}. \end{array}$$

The corresponding paths fall into two homotopy classes, according to the equation of the wall being effective or minus effective in X . Therefore, one expects all X in the same class to induces the same equivalence B_w^\uparrow , resp. B_w^\downarrow , in the diagram (36). For recent results in this direction, see [18].

Note that shifts λ by the lattice $p\text{Pic}(X)$ amount to twists of $\text{D}^b \text{Coh } X^{(1)}$ by a line bundle. This gives very interesting factorizations of twists by line bundles in the group $\text{Aut } \text{D}^b \text{Coh } X$.

3.2.9. For representation theory of \mathcal{X} , the monodromy group plays the role of Hecke algebra in the classical Kazhdan-Lusztig theory. It packages very valuable representation-theoretic information, which e.g. includes the classification of irreducible \mathcal{X} -modules according to their size. The latter problem inspired further conjectures by Etingof [37], which were proven in many important cases [15, 70, 100]. The general statement about monodromy is currently known [16] for Nakajima varieties X such that $\dim X^A = 0$.

As we will see, the monodromy of the quantum differential equation (18) lies somewhere between the differential equation itself and its q -difference analog. Possible categorical interpretations of the quantum q -difference equations will be discussed below.

¹³For uniformity, it is convenient to allow walls with trivial monodromy and derived equivalences, respectively. For example, the point $z = 1$ is not a singularity for the Hilbert scheme of points, as the pole there is cancelled by the scalar operator in (20).

4. Stable envelopes and quantum groups

4.1. Stable envelopes.

4.1.1. Let $X(\mathfrak{w})$ be a Nakajima variety as in (9). Our goal in this section is to produce interesting correspondences in $X(\mathfrak{w}) \times X(\mathfrak{w})$ which will result in actions of quantum groups and will be used to describe solutions to enumerative problems.

Note that $X(\mathfrak{w})$ is disconnected, and so a correspondence in it is really a collection of correspondences in $X(\mathfrak{w}, \mathfrak{v}) \times X(\mathfrak{w}, \mathfrak{v}')$. Even if one is interested in correspondences in $X \times X$ for a given symplectic resolution X , it proves beneficial to use correspondences $X \times X'$ as ingredients in the construction. In fact, perhaps the main obstacle in generalizing what is known for Nakajima varieties to general symplectic resolutions X is the shortage of natural relatives $\{X', X'', \dots\}$ with which X could meaningfully interact.

A rare general construction that one can use is the following. Let

$$\mathbf{A} \subset \text{Aut}(X, \omega)$$

be a torus, not necessarily maximal. Then a choice of a 1-parameter subgroup $\sigma : \mathbb{C}^\times \rightarrow \mathbf{A}$ determines a Lagrangian submanifold

$$(37) \quad \text{Attr} = \left\{ (x, y), \lim_{t \rightarrow 0} \sigma(t) \cdot x = y \right\} \subset X \times X^{\mathbf{A}}.$$

This is a very familiar concept, examples of which include e.g. conormals to Schubert cells in T^*G/P or the loci of extensions in the moduli of framed sheaves.

The choice of σ matters only up to the cone \mathfrak{C}

$$d\sigma \in \mathfrak{C} \subset \text{Lie } \mathbf{A}$$

cut out by the equivariant roots of X and containing $d\sigma$. We already saw these cones in Section 3.1.3. We will write $\text{Attr}_{\mathfrak{C}}$ when we need to stress this dependence.

4.1.2. The submanifold Attr cannot be used for our purposes for the simple reason of not being closed. Using its closure, and especially the structure sheaf of the closure, runs into all the usual problems with closure in algebraic geometry. These include $\overline{\text{Attr}}$ not being stable against perturbations, that is, not fitting into a family for \mathbf{A} -equivariant deformations X_λ of X .

For generic $\lambda \in H^2(X, \mathbb{C})$, the deformation X_λ is affine and (37) is closed for it. In cohomology, one gets good results by closing this cycle in the whole family, see Section 3.7 in [75]. In the central fiber X , this gives a Lagrangian cycle

$$\text{Stab}_{\mathfrak{C}} \subset X \times X^{\mathbf{A}},$$

called *stable envelope*, supported on the full attracting set Attr^f . By definition $(x, y) \in \text{Attr}^f$ if x can be joined to y by a chain of closures of attracting \mathbf{A} -orbits. With hindsight, one can recognize instances of this construction in such classical works as [48].

In practice, it is much more useful to have a characterization of $\text{Stab}_{\mathfrak{C}}$ in terms that refer to X alone instead of perturbations and closures. This becomes crucial in \mathbf{T} -equivariant K-theory, where $\mathbf{T} \supset \mathbf{A}$ is a torus which scales ω nontrivially. Since there are no \mathbf{T} -equivariant deformations of X , a perturbation argument does not yield a well-defined \mathbf{T} -equivariant K-class. In fact, as we will see, stable envelopes in K-theory crucially depend on certain data other than just a choice of a cone \mathfrak{C} .

4.1.3. The fixed locus has many connected components $X^A = \bigsqcup F_i$ and, by definition,

$$\left(\text{supp Attr}_{\mathfrak{e}}^f\right)^A = \bigsqcup_{i \geq j} F_i \times F_j,$$

where $i \geq j$ refers to the partial order on components of X^A determined by the relation of being attracted by σ . The main idea in definition of stable envelopes both in cohomology and K -theory is to require that

$$(38) \quad \text{Stab} \Big|_{F_i \times F_j} \text{ is smaller than } \text{Stab} \Big|_{F_i \times F_i},$$

for $i > j$ in a sense that will be made precise presently.

The torus A doesn't act on the fixed locus, so the restriction in (38) is a polynomial in equivariant variables with values in $H_{T/A}^\bullet(F_i \times F_j)$ and $K_{T/A}(F_i \times F_j)$, respectively. We denote deg_A the degree of this polynomial, defined as follows. In cohomology, we have a polynomial on $\text{Lie } A$, with the usual notion of degree. We require

$$(39) \quad \text{deg}_A \text{Stab} \Big|_{F_i \times F_j} < \text{deg}_A \text{Stab} \Big|_{F_i \times F_i} = \frac{1}{2} \text{codim } F_i,$$

and a simple argument, see Section 3 in [75], shows that an A -invariant cycle $\text{Stab}_{\mathfrak{e}}$ supported on Attr^f , equal to Attr near diagonal, and satisfying the degree bound (39)

- is always unique;
- exists for rather general algebraic symplectic varieties X , in particular, for all symplectic resolutions;
- for symplectic resolutions, may be obtained by a specialization argument from Section 4.1.2.

It can be constructed inductively by a version of Gram-Schmidt process, the left-hand side in (38) is then interpreted as the remainder of division by the right-hand side. Since we are dealing with multivariate polynomials, this is somewhat nontrivial. The key observation here is that for $A \subset Sp(2n)$, the class of any A -invariant Lagrangian $L \subset \mathbb{C}^{2n}$ in $H_A^{2n}(\mathbb{C}^{2n})$ is a \mathbb{Z} -multiple of the class of any fixed linear Lagrangian. A naive argument like this doesn't work in $K_A(\mathbb{C}^{2n})$ and things become more constrained there.

4.1.4. In K -theory, we deal with polynomials on A itself, for which the right notion of degree is given by the Newton polygon

$$\text{deg}_A \sum c_\mu a^\mu = \text{Convex hull}(\{\mu, c_\mu \neq 0\}) \subset A^\wedge \otimes_{\mathbb{Z}} \mathbb{R}$$

considered *up to translation*. The natural ordering on Newton polygons is that by inclusion and to allow for the up-to-translation ambiguity we require stable envelopes in K -theory to satisfy the condition

$$(40) \quad \text{deg}_A \text{Stab} \Big|_{F_i \times F_j} \subset \text{deg}_A \text{Stab} \Big|_{F_i \times F_i} + \text{shift}_{ij}$$

for a certain collection of shifts

$$\text{shift}_{ij} \in A^\wedge \otimes_{\mathbb{Z}} \mathbb{R}.$$

Condition (40) is known as the *window* condition. The uniqueness of K -classes satisfying it is, again, immediate. However, existence, in general, is not guaranteed. If the rank of A is one, one can argue inductively using the usual division with

remainder. However, for application we have in mind it is crucial to allow for tori of rank > 1 , and for them existence can only be shown for special shifts associated to fractional line bundles on X .

Let $\mathcal{L} \in \text{Pic}_A(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be a fractional A -linearized line bundle. Its restriction $\mathcal{L}|_{F_i}$ to a component of the fixed locus has a well-defined weight in $A^\wedge \otimes_{\mathbb{Z}} \mathbb{R}$ and we define

$$\text{shift}_{ij} = \text{weight } \mathcal{L}|_{F_i} - \text{weight } \mathcal{L}|_{F_j}.$$

Note that the choice of linearization cancels out and the shift depends on just a fractional line bundle $\mathcal{L} \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, called the *slope* of the stable envelope.

There is some fine print in the correct normalization of K -theoretic stable envelopes at the diagonal. It involves the notion of a *polarization*, see [90] for details. In fact, even in cohomology, it is best to require $\text{Stab} = \pm \text{Attr}$ near the diagonal, depending on a polarization, see [75]. With such normalization and for a certain class of X that includes all symplectic resolutions, we have

THEOREM 1 ([1, 55, 76]). *There exists unique stable envelope*

$$(41) \quad \text{Stab}_{\mathfrak{C}, \mathcal{L}} \in K_{\top}(X \times X^A)$$

for any cone \mathfrak{C} and any slope \mathcal{L} away from a certain locally finite $\text{Pic}(X)$ -periodic rational hyperplane arrangement in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proofs in the literature deduce this from more general statements. For Nakajima varieties, the existence of more general *elliptic* stable envelopes is shown in [1], see Section 3.8 there for specialization to K -theory. The paper [55] gives a Gram-Schmidt-style existence proof of *categorical* stable envelopes.

Note that for generic shifts, the inclusion in (40) is necessarily strict as the inclusion of an integral polytope into a very nonintegral one. Therefore, the \mathcal{L} -dependence of $\text{Stab}_{\mathfrak{C}, \mathcal{L}}$ is locally constant and can only jump if a lattice point gets on the boundary of the polytope on the right in (40).

There is a direct relation between stable envelopes and the monodromy of (18) proven in [1] in the more general q -difference case. It implies, in particular that

$$(42) \quad \left\{ \begin{array}{l} \text{hyperplanes in } \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \\ \text{where } \text{Stab}_{\mathfrak{C}, \mathcal{L}} \text{ jump} \end{array} \right\} \subset \left\{ \begin{array}{l} \text{Kähler arrangement} \\ \text{for } X \end{array} \right\},$$

and I think that we have an equality in (42) aside from trivialities like $X = X_1 \times X_2$ with A acting on X_2 only. Also note that the left-hand side in (42) does not depend on \mathfrak{C} .

4.1.5. Piecewise constant dependence on a fractional line bundle should certainly activate neurons in areas of the cerebral cortex responsible for multiplier ideal sheaves and related objects. Indeed, the similarity¹⁴ goes further in that

- window conditions are preserved by proper push-forwards;
- in a normal-crossing situation (e.g. for hypertoric varieties), stable envelope is a twist of a structure sheaf by a rounding of \mathcal{L} .

In particular, Nakajima varieties are related by certain Lagrangian *abelianization* correspondences to smooth hypertoric varieties, see [56], and this has been used to compute stable envelopes starting in [98, 101] and all the way to the elliptic level of generality in [1].

¹⁴It is easy to suspect an actual link here, and I would be grateful to experts for explaining what it is.

4.1.6. Favorable properties of stable envelopes are shared by their categorical lifts, which are defined as functors

$$(43) \quad \mathbf{Stab}_{\mathfrak{C}, \mathcal{L}} : D^b \text{Coh}_{\mathbb{T}} X^A \rightarrow D^b \text{Coh}_{\mathbb{T}} X$$

satisfying the same window conditions for the derived restriction to X^A . For $\text{rk } A = 1$, a detailed study of these, with many applications, can be found in the work of Halpern-Leistner [54], see also [5]. In this case, one doesn't need the shift to come from a line bundle. The case of tori of higher rank is considered in [55].

4.2. R-matrices.

4.2.1. If a construction in algebraic geometry requires additional choices, such as the choice of a cone \mathfrak{C} and a slope \mathcal{L} in the definition of stable envelopes, then this can be viewed as a liability or as an asset. The latter point of view is gaining popularity as people find more and more applications of wall-crossing phenomena of various kind. In the case at hand, it will be beneficial to study how stable envelopes change as the we cross the wall from \mathfrak{C} to another chamber $\mathfrak{C}' \subset \text{Lie } A$, or as we cross a wall of the Kähler arrangement from \mathcal{L} to \mathcal{L}' .

To formalize this, we introduce

$$(44) \quad R_{\mathfrak{C}' \leftarrow \mathfrak{C}}^{\mathcal{L}} = \text{Stab}_{\mathfrak{C}', \mathcal{L}}^{-1} \circ \text{Stab}_{\mathfrak{C}, \mathcal{L}}$$

and similarly

$$(45) \quad R_{\mathcal{L}' \leftarrow \mathcal{L}}^{\mathfrak{C}} = \text{Stab}_{\mathfrak{C}, \mathcal{L}'}^{-1} \circ \text{Stab}_{\mathfrak{C}, \mathcal{L}} .$$

Stable envelopes are isomorphisms¹⁵ after localization, so

$$\text{all } R\text{-matrices} \subset \text{End}(K_{\mathbb{T}}(X^A)) \otimes \mathbb{Q}(\mathbb{T}) .$$

They *depend* on equivariant variables $a \in A$ for the torus A which does not act on X^A , and this is, again, a bonus. We can expand R-matrices as $a \rightarrow 0$, where 0 is a fixed points of a toric compactification $\bar{A} \supset A$ that correspond to the fan $\{\mathfrak{C}_i\}$ of chambers. The coefficients of this expansion give countably many operators acting on *integral* \mathbb{T}/A -equivariant K-theory of X^A , that is, we can view R -matrices as interesting generating functions for correspondences in X^A .

4.2.2. Let \mathfrak{C} and \mathfrak{C}' be separated by a wall in $\text{Lie } A$. This wall is of the form $\text{Lie } A'$ for some codimension 1 subtorus $A' \subset A$. One shows, see Section 9.2 in [90] that

$$(46) \quad R_{\mathfrak{C}' \leftarrow \mathfrak{C}} \text{ for } A\text{-action on } X = R_{\mathfrak{C}' \leftarrow \mathfrak{C}} \text{ for } A/A'\text{-action on } X^{A'}$$

and, in particular, $R_{\mathfrak{C}' \leftarrow \mathfrak{C}}$ factors through the map $A \rightarrow A/A'$, that is, through the equation of the wall. This gives

PROPOSITION 4.1. *Matrices $R_{\mathfrak{C}' \leftarrow \mathfrak{C}}$ form a representation of the dynamical groupoid for the central arrangement defined by the equivariant roots of X .*

Here central is used as opposite of affine: we equate equivariant roots to 0 and not to an arbitrary integer.

¹⁵The inverse of a stable envelope is a transpose of another stable envelope, see Section 9 in [90].

4.2.3. In the situation of Section 2.2.5, the root hyperplane are $a_i/a_j = 1$ and the groupoid relation becomes the Yang-Baxter equation.

This geometric solution of the YB equation produces a geometric action of a quantum group $\mathcal{A} = \mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ for a certain Lie algebra \mathfrak{g} as indicated in Section 1.12. See [75, 92] for the details of the reconstruction.

The parallel procedure in cohomology constructs a certain degeneration $Y(\mathfrak{g})$ of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ known as the Yangian of \mathfrak{g} . This is a Hopf algebra deformation of $\mathcal{U}(g[t])$ and the universal enveloping of $\mathfrak{g} \subset \mathfrak{g}[t]$

$$(47) \quad \mathcal{U}(\mathfrak{g}) \subset Y(\mathfrak{g})$$

remains undeformed as a Hopf algebra.

4.2.4. The structure of \mathcal{A} may be analyzed using the following useful connection between R-matrices of the two kinds (44) and (45).

We can include \mathcal{L} into an infinite sequence of slopes

$$(48) \quad \dots, \mathcal{L}_{-2}, \mathcal{L}_{-1}, \mathcal{L}_0 = \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \dots$$

that go from a point $\mathcal{L}_{-\infty}$ on the minus ample infinity of $\text{Pic}(X)$ to a point \mathcal{L}_{∞} on the ample infinity of $\text{Pic}(X)$. A choice of such path is analogous to a choice of the factorization of the longest element in Weyl group in the usual theory.

Tautologically,

$$(49) \quad R_{(-\mathfrak{c}) \leftarrow \mathfrak{c}}^{\mathcal{L}} = \prod_{i < 0}^{\rightarrow} R_{\mathcal{L}_{i+1} \leftarrow \mathcal{L}_i}^{-\mathfrak{c}} R_{(-\mathfrak{c}, \mathcal{L}_{-\infty}) \leftarrow (\mathfrak{c}, \mathcal{L}_{\infty})} \prod_{i \geq 0}^{\leftarrow} R_{\mathcal{L}_{i+1} \leftarrow \mathcal{L}_i}^{\mathfrak{c}},$$

assuming the term in the middle makes sense, that is, infinite product converge in a suitable topology. This is indeed the case:

PROPOSITION 4.2 ([92]). *The operator $R_{(-\mathfrak{c}, \mathcal{L}_{-\infty}) \leftarrow (\mathfrak{c}, \mathcal{L}_{\infty})}$ is well-defined and acts by an operator of multiplication in $K_T(X^A)$.*

In fact, this operator acts by a very specific Schur functor of the normal bundle N_{X/X^A} to the fixed locus.

Formula (49) generalizes factorization of R-matrices constructed in the conventional theory of quantum groups, see for example [65–67].

Recall that $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ is spanned by coefficients in $u \in \mathbf{A} \cong \mathbb{C}^\times$ of the matrix coefficients of R-matrices (44). In this procedure, one can, in fact, pick out the matrix coefficients of each individual term in the factorization (49) and this gives

$$(50) \quad \mathcal{U}_\hbar(\mathfrak{g}_w) \hookrightarrow \mathcal{U}_\hbar(\widehat{\mathfrak{g}})$$

for each wall w crossed by the path (48). These deform the decomposition

$$(51) \quad \widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] = \widehat{\mathfrak{h}} \oplus \bigoplus_w \mathfrak{g}_w, \quad \mathfrak{g}_w = \bigoplus_{\alpha = -n \text{ on } w} \mathfrak{g}_\alpha t^n,$$

in which the $\widehat{\mathfrak{h}}$ part corresponds to the middle factor in (49). Note that the subalgebra (50) depends on

- the wall w between two alcoves and not only on the hyperplane containing it and, moreover, on
- the choice of the path (48).

These features are already familiar from the classical theory of quantum groups. Concretely, in textbooks, quantum groups are usually presented by generators and relations and, consequently, come equipped with root subalgebras corresponding to

simple roots. Constructing other root subalgebras requires choices just like we had to make in (48).

This argument also shows that the quantum group constructed using the R-matrix (49) is independent of the choice of \mathcal{L} .

4.3. Enumerative operators.

4.3.1. We go back to curve-counting in cohomology and reexamine the operator (17) for Nakajima varieties. Recall the inclusion (47). The root decomposition in (25) is according to the components of $X(\mathbf{w})$, that is,

$$(52) \quad \mathfrak{g}_\alpha : H^\bullet(X(\mathbf{v}, \mathbf{w})) \rightarrow H^\bullet(X(\mathbf{v} - \alpha, \mathbf{w})),$$

and all these root spaces are finite-dimensional. In parallel to Kac-Moody theory, one shows that all roots are either positive or negative, that is,

$$\text{roots of } \mathfrak{g} \subset \mathbb{N}^I \cup (-\mathbb{N})^I.$$

Duality (26) gives a canonical element

$$\text{Casimir}_\alpha \in : \mathfrak{g}_{-\alpha} \mathfrak{g}_\alpha : \subset \mathcal{U}(\mathfrak{g})$$

which preserves $H^\bullet X(\mathbf{v}, \mathbf{w})$ and annihilates it unless $|\alpha| \leq \mathbf{v}$ because of (52). Normal ordering here means that we act by lowering operators first.

The main result of [75] is the following.

THEOREM 2. *For Nakajima quiver varieties, formula (20) holds with*

$$L_\alpha = -\text{Casimir}_\alpha.$$

In particular, this gives a new computations of the quantum cohomology of Hilbert schemes of ADE surfaces, first obtained in [74, 91] by a more direct analysis.

4.3.2. The proof of Theorem 2 given in [75] is rather indirect. Recall that the group $\prod GL(W_i)$ in (15) acts on $X(\mathbf{v}, \mathbf{w})$, and this action is nontrivial if $\sum w_i > 1$. There is a flat difference connection in the corresponding equivariant variables which commutes with the quantum connection (18). It is constructed geometrically counting sections in certain twisted X -bundles over \mathbb{P}^1 . Operators of this kind, often called *shift operators*, find many other applications in enumerative geometry, see for example [60].

The main step of the proof identifies this commuting connection with what is known as the *quantum Knizhnik-Zamolodchikov* equations. These were introduced by Frenkel and Reshetikhin in [46] in the context of quantum loop algebras of finite-dimensional Lie algebras. They make sense for any solution of the Yang-Baxter equation with a spectral parameter and, in particular, for our geometrically constructed R-matrices. The support and the degree conditions satisfied by stable envelopes enter the argument at this step.

The K-theoretic computation in [90, 92], while much more involved on the technical level, follows the same general strategy: one identifies a commuting difference connection first, and uses that to constrain the connection of principal geometric interest.

4.3.3. One may phrase the answer in Theorem 2 as the identification of the quantum connection (18) with the trigonometric *Casimir connection* for the Yangian $Y(\mathfrak{g})$. For finite-dimensional Lie algebras, such connections were studied by Toledano Laredo in [107], see also the work [105, 106] by Tarasov and Varchenko.

A connection with Yangians was expected from the ideas of Nekrasov and Shatashvili. The precise identification with the Casimir connection was suggested

by Bezrukavnikov, Etingof, and their collaborators. Back then, it was further predicted by Etingof that the correct K-theoretic version of the quantum connection should be a generalization of the dynamical Weyl group studied by Tarasov, Varchenko, Etingof, and others for finite-dimensional Lie algebras \mathfrak{g} , see e.g. [41, 104]. Indeed, for such \mathfrak{g} , Balagovic showed [4] that the dynamical q -difference equations degenerate to the Casimir connection in an appropriate limit.

At that moment in time, there was neither geometric, nor a representation-theoretic construction of the required q -difference connection.

4.3.4. Referring the reader to [31] for all details, we will define a quasimap

$$f : C \dashrightarrow X = \tilde{X} // G$$

to a GIT quotient as a section of a bundle of prequotients $\mathcal{P} \times_G X$, up to isomorphism. Here \mathcal{P} is a principal G -bundle over C which is a part of the data and is allowed to vary. Stability conditions for quasimaps are derived from stability on \tilde{X} ; we will use the simplest one that requires the value of the quasimap be stable at the generic point of C .

To have an evaluation map at a specific point $p \in C$ one introduces quasimaps relative p , formed by diagrams of the form

$$(53) \quad \begin{array}{ccc} p' & \xrightarrow{\quad} & C' & \dashrightarrow & \tilde{X} // G \\ \downarrow & & \downarrow \pi & & \\ p & \xrightarrow{\quad} & C & & \end{array}$$

in which π is an isomorphism over $C \setminus p$ and contracts a chain of rational curves joining p' to $\pi^{-1}(C \setminus p)$. The evaluation map records the value at p' .

For quasimaps to $X(\mathbf{v}, \mathbf{w})$, we have $G = \prod GL(V_i)$ and so a principal G -bundle is the same as a collection of vector bundles of ranks \mathbf{v} . For a regular map f , these are the pullbacks $f^*\mathcal{V}_i$ of the tautological vector bundles \mathcal{V}_i on X . We denote them by $f^*\mathcal{V}_i$ for all quasimaps; they are a part of the data of a quasimap.

Consider the line bundle

$$\mathcal{L}_i = \det \mathcal{V}_i \in \text{Pic}(X).$$

For our difference connection, we need a correspondence in X that generalizes the operator $\otimes \mathcal{L}_i$ in $K_{\mathbb{T}}(X)$. We take

$$(54) \quad \mathbf{M}_{\mathcal{L}_i} = \sum_d z^d \text{ev}_* \left(\widehat{\mathcal{O}}_{\mathcal{M}_d} \otimes \det H^\bullet(f^*\mathcal{V}_i \otimes \pi^*\mathcal{O}_{p_1}) \right) \mathbf{G}^{-1},$$

where \mathcal{M}_d is the moduli space of quasimaps

$$(\mathbb{P}^1, 0, \infty) \cong (C, p_1, p_2) \dashrightarrow \tilde{X} // G$$

of degree d and \mathbf{G} is the glue matrix from (24). See [90] for detailed discussion of how one arrives at the definition (54) and what is the geometric significance of the solutions of the corresponding q -difference equations.

4.3.5. On the algebraic side, recall the subalgebras (50) associated to a wall of the Kähler arrangement. It is by itself a quantum group, in particular, it has its own R-matrix R_w , which is the corresponding factor in (49) with a suitable normalization. This is not a quantum loop algebra, in fact it is a quantum group of rank 1. Correspondingly, R_w does not have a spectral parameter and satisfies the

YB equation (8) for constant (in u) operators. Examples of these wall subalgebras are $\mathcal{U}_\hbar(\mathfrak{sl}(2))$ and the quantum Heisenberg algebra.

By definition, quantum Heisenberg algebra has two group-like invertible elements

$$\Delta H = H \otimes H, \quad \Delta K = K \otimes K,$$

of which K is central, while

$$HE = \hbar EH, \quad HF = \hbar^{-1}FH.$$

Here the generators E and F satisfy

$$[E, F] = \frac{K - K^{-1}}{\hbar - \hbar^{-1}}$$

and comultiply as follows:

$$\begin{aligned} \Delta E &= E \otimes 1 + K^{-1} \otimes E, \\ \Delta F &= F \otimes K + 1 \otimes F. \end{aligned}$$

This algebra acts on $\mathbb{Z}[x, \hbar^{\pm 1}]$ by

$$E = x, \quad F = -\frac{d}{dx}, \quad H = \hbar^x \frac{d}{dx}, \quad K = \hbar.$$

Its R-matrix is

$$R = \hbar^\Omega \exp(-(\hbar - \hbar^{-1})F \otimes E),$$

where

$$\Omega = -\log_\hbar H \otimes \log_\hbar H = -x \frac{d}{dx} \otimes x \frac{d}{dx}.$$

In its geometric origin, the \mathbb{Z} -valued operator Ω records the codimension of the fixed locus.

For quivers of affine type, every root is either real, in which case the wall subalgebra is isomorphic to $\mathcal{U}_\hbar(\mathfrak{sl}(2))$, or has zero norm. In the latter case, all multiples of the root enter \mathfrak{g}_w in (51) and $\mathcal{U}_\hbar(\mathfrak{g}_w)$ is a countable product of Heisenberg subalgebras, with their tori identified. For example, all root subalgebra of the algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}(1))$ associated with the quiver with one vertex and one loop are of this type. The structure of this algebra has been studied by many authors, see for example [29, 43, 44, 85].

4.3.6. With any R-matrix, we can associate

$$(55) \quad \text{qKZ operator} = (z^{\log_\hbar H} \otimes 1) R = (z^{\log_\hbar H} \otimes 1) \hbar^\Omega + \dots$$

acting in the tensor product of two $\mathcal{U}_\hbar(\mathfrak{g}_w)$ -modules. Here dots stand for an upper-triangular operator.

Since $\mathcal{U}_\hbar(\mathfrak{g}_w)$ is not a loop algebra, this is not a difference operator. Instead of a fundamental solution of a difference equation, we can ask for a unipotent matrix J that conjugates (55) to its eigenvalues

$$(56) \quad (z^{\log_\hbar H} \otimes 1) R J = J (z^{\log_\hbar H} \otimes 1) \hbar^\Omega.$$

Such operator exist universally and for the quantum Heisenberg algebra it equals

$$J = \exp\left(-(\hbar - \hbar^{-1}) \frac{z}{1-z} F \otimes E\right) \in \mathcal{U}_\hbar(\mathfrak{g}_w)^{\otimes 2}[[z]].$$

It is called the *fusion operator*. It gives a certain canonical way to promote the R-matrix to a rational function on the maximal torus.

We have a maximal torus

$$(57) \quad \text{Kähler torus} \times \mathbb{C}_q^\times \subset \text{Aut } \widehat{\mathfrak{g}}$$

where

$$\text{Kähler torus} = \text{Pic}(X) \otimes \mathbb{C}^\times = \text{maximal torus of } \mathfrak{g}/\text{center},$$

and \mathbb{C}_q^\times acts naturally by automorphisms of domains of quasimaps and on $\widehat{\mathfrak{g}}$ via the loop rotation automorphism (7). The adjoint action of (57) on $\mathcal{U}_\hbar(\mathfrak{g}_w)$ factors through a map to its maximal torus, so J can be promoted to a function on (57). Since the larger torus includes a shift by q , the equation (56) is trivially promoted to a q -difference equation.

4.3.7. From the fusion operator J , one can make the following dynamical operator

$$(58) \quad B_w = m \left((1 \otimes S_w) J_{21}^{-1} \right)$$

where

m = multiplication,

S_w = antipode of $\mathcal{U}_\hbar(\mathfrak{g}_w)$,

and $J_{21} = (12)J$.

Compare this with the operator

$$\text{Casimir}_\alpha = m \mathbf{r}_\alpha, \quad \mathbf{r}_\alpha \in \mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_\alpha,$$

in which the canonical tensor \mathbf{r}_α is the α -component of the *classical* R -matrix, see Section 4.8 in [75].

This is not how the dynamical Weyl group is defined in [41], but one can find a formula equivalent to (58) for $\mathfrak{g}_w = \mathfrak{sl}(2)$ there. The abstract formula (58) makes sense in complete generality, for any R -matrix.

The main result of [92] may be summarized as follows. Let $\nabla \subset H^2(X, \mathbb{R})$ be the Kähler alcove which coincides with the negative ample cone near zero.

THEOREM 3. *The operators B_w define a representation of the dynamical groupoid of the affine Kähler arrangement. We have*

$$(59) \quad \mathbf{M}_\mathcal{L} = \text{const } \mathcal{L} \dots B_{w_3} B_{w_2} B_{w_1}$$

where w_1, w_2, \dots is the ordered set of wall crossed on the way from ∇ to $\nabla - \mathcal{L}$, and

$$(60) \quad \mathbf{G} = \text{const } B_{w'_1} B_{w'_2} \dots$$

where w'_1, w'_2, \dots are the walls crossed on the way from ∇ to $-\nabla$.

In other words, the operators $\mathbf{M}_\mathcal{L}$ come from the lattice part of the dynamical groupoid, while the glue matrix \mathbf{G} is the dynamical analog of the longest element of the *finite* Weyl group.

The constant factor ambiguity in Theorem 3 is of the same nature as the diagonal ambiguity in (20).

4.3.8. The factorizations of the operators $\mathbf{M}_{\mathcal{L}}$ provided by Theorem 3 generalize the additive decomposition (20) of the operators of the quantum differential connection.

It is certainly natural to expect parallel factorizations and a dynamical groupoid action for an arbitrary symplectic resolution X . Note that if factorizations of the required kind exists, they are unique and determine the whole structure.

Some speculations of what this might look like will be offered below. One should bear in mind, however, that both the quasimap moduli spaces *and* the operators $\mathbf{M}_{\mathcal{L}}$, \mathbf{G} depend on the choice of the presentation (23). This can be seen in example (28).

4.3.9. The knowledge of the quantum difference equation opens the door to the reconstruction of the K-theoretic DT theory of ADE fibrations and toric varieties in the same way as it was done in cohomology [73]. For a technical introduction to the subject, see e.g. [90].

4.3.10. As in cohomology, one can define a flat q -difference connection on the space of equivariant parameters by counting twisted quasimaps to X . For twisted quasimaps, the spaces W_i, Q_{ij} from Section 2.2.1 and their duals $\hbar^{-1} \otimes W_i^\vee, \hbar^{-1} \otimes Q_{i,j}^\vee$ form fixed but possibly nontrivial bundles over the domain C of the quasimap. The \hbar^{-1} -weight here may also be twisted, that is, replaced by a line bundle on C .

On general grounds, the quantum q -difference connection commutes with these shift operators. A key step in the proof Theorem 3 it to find the quantum Knizhnik-Zamolodchikov connection among the shift operators. See Section 10 of [90] for details.

If we don't shift \hbar , that is, if we restrict ourselves to twisting by a maximal torus $\mathbf{A} \subset \text{Aut}(X, \omega)$, then the resulting q -difference connection is a connection with regular singularities on the toric variety $\bar{\mathbf{A}}$ defined by the fan of cones $\{\mathfrak{C}\} \subset \text{Lie } \mathbf{A}$. Dually, the quantum difference connection is a connection with regular singularities on \mathfrak{K} , which is a toric variety defined by the fan of ample cones. These two connections commute but are *not* regular jointly — this will be important below.

The two connections are conjectured to be exchanged by symplectic duality, up-to an explicit gauge transformation.

5. Further directions

5.1. Categorical dynamical groupoid.

5.1.1. The operators \mathbf{B}_w of the dynamical groupoid from Theorem 3 are normalized so that $\mathbf{B}_w|_{z=0} = 1$ where $0 = 0_X \in \mathfrak{K}$ is the origin in the space of Kähler variables for X . This is natural from the curve-counting point of view. One can choose a more symmetric normalization, in which the specialization to K-theoretic Bezrukavnikov's groupoid from Section 3.2.8 is the standard (31). In that normalization, one takes the factorization of \mathcal{L} provided by Bezrukavnikov's theory and distributes it over the factors in (59).

5.1.2. It is natural to ask for a categorical lift of the dynamical groupoid. The dependence on the Kähler variables may be treated either formally or not formally. In the formal treatment, we introduce an extra grading by the characters $H_2(X, \mathbb{Z})$ of the Kähler torus, otherwise nothing is changed.

The functors B_w from the triangle (36) are *perverse equivalences* as in the work of Rouquier and collaborators [34], see [3, 71, 72]. This means, in particular, that

the category

$$D^b \mathcal{X}_\lambda\text{-mod} = \mathcal{C} \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_k$$

has a finite filtration by thick subcategories such that the action of B_w on each successive quotient $\mathcal{C}_i/\mathcal{C}_{i+1}$ is a shift of homological degree and equivariant grading.

From (59) and (60) it is natural to expect the dynamical lift $B_w(z)$ to be the glue matrix $\mathbf{G}_{\mathcal{C}_i/\mathcal{C}_{i+1}}$ on each successive quotient, with a shift as before. This requires a geometric realization of the quotient category $\mathbf{G}_{\mathcal{C}_i/\mathcal{C}_{i+1}}$. In fact, if one wants to use quasimaps to define the glue matrix as in (24), one needs a presentation

$$\mathcal{C}_i/\mathcal{C}_{i+1} \cong D^b \text{Coh}_{\text{reductive group}} \left(\begin{array}{c} \text{lci affine} \\ \text{prequotient} \end{array} \right) / \text{unstable}.$$

Perhaps such presentation for each possible quotient may be obtained from a GIT presentation (23) for X itself. Perhaps there are ways to define the curve-counting operators which are more general than those based on quasimap spaces. In any case, it will require some additional data, as our running example (28) shows.

5.2. Elliptic theory.

5.2.1. The Yang-Baxter equation (8) has a generalization, in which the matrix $R(u)$ is allowed to depend on a variable z in the maximal torus for the corresponding group. This is known as the *dynamical* Yang-Baxter equations and it has very interesting solutions in elliptic functions of u and z . These have to do with the theory of *elliptic* quantum groups, which was started by Felder [45] and has since grown into a very rich subject.

As anticipated in [49] at the dawn of the subject, one gets elliptic quantum groups by lifting the constructions of the geometric representation theory to equivariant elliptic cohomology.

Elliptic stable envelopes were constructed in [1]. Their main conceptual difference with both K-theoretic and categorical stable envelopes is the following. While stable envelopes from Section 4 have a *piecewise constant* dependence on the slope $\mathcal{L} \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, elliptic stable envelopes depend *analytically* on their dynamical variable

$$z \in \text{Pic}(X) \otimes_{\mathbb{Z}} E, \quad E = \mathbb{C}^\times / q^{\mathbb{Z}}.$$

The piecewise constant dependence is recovered in the $q \rightarrow 0$ limit¹⁶ if

$$-\Re \frac{\ln z}{\ln q} \rightarrow \mathcal{L}.$$

It is, of course, well-known that elliptic functions have a piecewise analytic asymptotics as the elliptic curve degenerates.

The walls across which K-theoretic stable envelopes jump played an absolutely key role in the preceding discussion. In the elliptic theory, walls become poles and all alcoves melt into a single elliptic function. The only remaining discrete data is the choice of a cone $\mathfrak{C} \subset \text{Lie } A$, and the choice of the ample cone in $H^2(X, \mathbb{R})$ if one thinks about all flops of X at the same time. Note that this data is now symmetric on the two sides of the symplectic duality.

¹⁶Using modularity of elliptic stable envelopes, one can replace the $q \rightarrow 0$ limit by $q \rightarrow 1$. The important part is that the elliptic curve E degenerates to a nodal rational curve.

5.2.2. A categorical lift of elliptic stable envelopes along the lines of (43) should incorporate their dependence on the Kähler variables z together with the elliptic modulus q . A few paragraphs ago we discussed adding such extra variables formally. A real need to incorporate these variables arises in certain categories of boundary conditions, see [64] for a general introduction to the topic.

Recall that the Nakajima varieties may be interpreted as the Higgs branches of the moduli spaces of vacua in certain supersymmetric gauge theories. K-theoretic counts of quasimaps $C \dashrightarrow X$, where C is a Riemann surface, may then be interpreted as supersymmetric indices in 3-dimensional gauge theories with spacetime $C \times S^1$.

A boundary condition in such theory is a coupling of the bulk gauge theory to some $(0, 2)$ -supersymmetric theory on $\partial C \times S^1$. The Hilbert space of the boundary theory is graded by

$$\text{Kähler torus of } X = \text{Center of the gauge group}$$

together with the action of q by rotations of ∂C . These Hilbert spaces form a sheaf over X and taking their graded index one gets an element of $K_{\mathbb{T}}(X)(z)[[q]]$ which represents the q -expansion of an elliptic function. This is how elliptic stable envelopes should appear from a functor

$$(61) \quad \begin{array}{ccc} \text{boundary condition} & \longrightarrow & \text{boundary condition} \\ \text{in gauge theory for } X^A & & \text{in gauge theory for } X \end{array} ,$$

as will be discussed in [2].

Note that elliptic stable envelopes depend only on X itself and not on its presentation as a Nakajima variety.

5.2.3. While the monodromy of a differential equation gives a representation of the fundamental group of its regular locus, the monodromy of a q -difference equation in z is a collection of elliptic matrices in z that relate different fundamental solutions. In particular, q -connections with regular singularities on a toric variety produce elliptic connection matrices for any pair of fixed points. For us, those are labeled by a pair of birational symplectic resolutions X for the Kähler q -connection or by a pair of cones in $\text{Lie } A$ for the equivariant q -connection.

This difference in how monodromy packages the information is parallel to the differences highlighted in Section 5.2.1 and, in fact, elliptic stable envelopes give a very powerful tool for the analysis of the monodromy.

5.2.4. Recall from Section 4.3.10 that the Kähler and equivariant connections are regular separately but not jointly. This can never happen for differential equations by a deep result of Deligne [36], but can happen very easily for q -difference equations as illustrated by the following system:

$$f(qz, a) = af(z, a), \quad f(z, qa) = zf(z, a).$$

This proves to be a feature and not a bug, for the following reason.

Let $0_X \in \mathfrak{A}$ be the origin in the Kähler moduli space for X and choose a point $0_A \in \bar{A}$ on the infinity of the equivariant torus. Quasimap counting produces solutions of both Kähler and equivariant equations known as *vertex functions*, or *I-functions* in the mirror symmetry vernacular. These are born as power series in z , the terms in which record the contribution of quasimaps of different degrees. In particular, vertex functions are *holomorphic* in z in a punctured neighborhood of 0_X . However, they have *poles* in a accumulating to 0_A .

We can call these the z -solutions, and there are the dual a -solutions that are holomorphic in a punctured neighborhood of 0_A . The connection matrix between the z -solutions and the a -solutions is an elliptic matrix in both z and a . It may be called the *pole subtraction* matrix because it may be, in principle, computed by canceling the unwanted poles. It may be compared to a monodromy matrix, or to a Stokes matrix, except there is no analytic continuation involved, or sectors. One of the main results of [1] is the identification of pole subtraction matrices with stable envelopes. Note they depend on the same discrete data, namely a choice of $(0_X, 0_A) \in \mathfrak{K} \times \overline{A}$, and act in the direction

$$(62) \quad K_{\top}(X^A) \otimes \text{functions}(z, a, \hbar, q) \rightarrow K_{\top}(X) \otimes \text{same functions}.$$

K-theories in the source and the target of (62) are naturally identified with initial conditions for a -solutions and z -solutions, respectively.

5.2.5. A combination of two steps of the form

$$\begin{array}{ccc} \text{fundamental} & \xrightarrow{\text{elliptic stable envelope}} & \text{fundamental} \\ \text{solution at } 0_A & & \text{solution at } 0_X \end{array}$$

connect any two fixed points of \mathfrak{K} or \overline{A} . As a result, elliptic stable envelopes:

- compute the monodromy of the equivariant equations,
- intertwine the monodromy of the Kähler equations for X and X^A .

The second constraint is particularly powerful for Nakajima varieties, as it determines the behavior of the monodromy with respect to tensor product.

5.2.6. We recall that Kähler difference equations depend not only on X but also on a presentation of X as a Nakajima variety. In particular, even if X^A is a bunch of points, its Kähler difference equation may still be a nontrivial abelian difference equation. As a result, even through elliptic stable envelopes depend on X only, the monodromy depends on its quiver presentation.

5.2.7. One of the main results of [16] is the identification of the categorical stable envelopes (43) with the parabolic induction functors for the quantizations under the derived equivalences (35). In particular, in the case $\dim X^A = 0$, this gives a complete control on the action of the triangles (36) in K-theory and thus lets one prove the conjectures of Section 3.2.8 in the case of isolated fixed points.

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