The sixth moment of automorphic L-functions

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joint work with
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Clay Math workshop – Analytic Number Theory
Oxford, UK
October 2nd, 2013

Organization of the talk

- Some history of moments of the Riemann zeta function, Dirichlet L-functions, automorphic L-functions
- Main result
- A few words about the proof

The Riemann zeta function

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- It can be analytically continued to the whole complex plane except a simple pole at $s = 1$.
- Define
  $$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$
  Then it satisfies the functional equation
  $$\xi(s) = \xi(1-s).$$
- The Riemann hypothesis (RH): all non-trivial zeros of the Riemann zeta function lie on the critical line $\mathrm{Re}(s) = 1/2$.

The Lindelöf hypothesis

One consequence of the Riemann hypothesis is the **Lindelöf hypothesis** (LH), which states that

$$\zeta\left(\frac{1}{2}+it\right) = O(t^{\epsilon}).$$

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$$\zeta\left(\frac{1}{2}+it\right) = O(t^{\epsilon}).$$

An equivalent statement of LH is connected to the moments of the Riemann zeta function. In particular,

Theorem

LH is true if and only if for all $k \in \mathbb{N}$,

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = O(T^{\epsilon}).$$

Moreover, understanding higher moments of $\zeta(s)$ gives us progressively better bounds for $\zeta(1/2+it)$.

History of the moments of the Riemann zeta function

Let $I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \, dt$.

$k=1$

Hardy and Littlewood (1918) showed that

$$I_1(T) \sim T(\log T).$$

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$k \geq 3$

It remains unsolved! However, we have a good conjecture, lower bound and upper bound for it.

Why is the case $k \geq 3$ difficult?

When $\mathrm{Re}(s) > 1$, we can write

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s},$$

where $d_k(n) = \sum_{n_1 \ldots n_k = n} 1$ is the k^{th} divisor function. When $\mathrm{Re}(s) \leq 1$, the series does not converge, but we can approximate $\zeta^k(s)$ by using the functional equation of $\zeta(s)$. We can write

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \sim \sum_{\substack{m,n \\ mn \leq t^k}} \frac{d_k(n) d_k(m)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} + \textit{ small error},$$

Hence

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim \sum_{\substack{m,n \\ mn \leq T^k}} \frac{d_k(n) d_k(m)}{\sqrt{mn}} \frac{\left(\frac{m}{n}\right)^{iT} - 1}{\log(m/n)}.$$

- $\log(m/n)$ is big when $m \sim n$, so the main contribution comes from this case.
- When $k = 1$, the main term solely comes from the diagonal term $m = n$, and the off-diagonal term $(m \neq n)$ is easy to handle.

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- When $k = 2$, $mn \leq T^2$. The main term comes from $m \sim n \lesssim T^{1+\epsilon}$. We can still handle the off-diagonal terms, but it's not simple.
- When $k \geq 3$, we do not know how to deal with the off-diagonal terms. It is related to shifted convolution sums of the form $\sum_{n \leq x} d_k(n) d_k(n+f)$.

Conjecture for the moments of $\zeta(s)$

Based on heuristics for shifted divisor sums, Conrey and Ghosh (1998) derived conjecture when $k = 3$.

$$I_3(T) \sim 42 a_3 \frac{T(\log T)^9}{9!}.$$

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Conrey and Gonek (2001) has a conjecture when $k = 4$.

$$I_4(T) \sim 24024 a_4 \frac{T(\log T)^{16}}{16!}.$$

Conjecture (cont'd)

Keating and Snaith (2000): Through random matrix theory and relating distribution of the zeros $\zeta(s)$ with distribution of eigenvalues of matrices in circular unitary ensembles, they conjectured that for any positive integers k,

$$I_k(T) \sim \frac{g_k a_k}{k^2!} T(\log T)^{k^2},$$

where a_k is the coefficient of the leading term of $\sum_{n\leq T} \frac{d_k^2(n)}{n}$, and

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

Conjecture (cont'd)

Conrey, Farmer, Keating, Rubinstein, Snaith (2005) gave a more precise conjecture of $I_k(T)$ including an asymptotic expansion for lower order terms through the shifted moments. Their recipe also applies to moments of other families of L-functions.

Diaconu, Goldfeld and Hoffstein (2003) gave an alternative approach, based on multiple Dirichlet series, to give the same conjectures.

Bounds for the moments of $\zeta(s)$

- Unconditionally, Ramachandra (1978), Heath-Brown (1980), Conrey-Ghosh (1984) showed that

$$I_k(T) \gg_k T(\log T)^{k^2}$$

for $k > 0$ and $k \in \mathbb{Q}$.

- On RH, Soundararajan (2007), Harper (2013) showed that for any positive real k

$$I_k(T) \ll_k T(\log T)^{k^2}.$$

- Radziwiłł and Soundararajan (2014) showed that for any positive real $0 < k \leq 2$

$$I_k(T) \ll_k T(\log T)^{k^2}.$$

Dirichlet L-functions

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where χ is a primitive Dirichlet character modulo q.

- The family of all primitive Dirichlet L-functions of modulus q is analogous in some ways to the Riemann zeta function in t-aspects.
- In fact, this family is associated to unitary ensemble.
- The moments of this family should behave similarly to the moments of the $\zeta(1/2+it)$.

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Notation: $\sum^*$ is the sum over all primitive characters mod q. $\phi^*(q)$ is number of primitive characters mod q.

Moments of $L(s, \chi)$

Let $M_k(q) = \sum^*_{\chi \pmod q} |L(1/2, \chi)|^{2k}$.

$k = 1$: $M_1(q) \sim \phi^*(q) \log q$.

$k = 2$: Heath-Brown (1981), showed that

$$\sum^*_{\chi \pmod q} |L(1/2, \chi)|^4 \sim \frac{1}{2\pi^2} a_2 \; \phi^*(q)(\log q)^4$$

for most q. Later Soundararajan (2007) derived the asymptotic formula for the fourth moment for all q.

Young (2010) obtained the asymptotic formula for the fourth moment with a power saving for prime q. In particular

$$M_2(q) = \phi^*(q) \sum_{i=0}^{4} c_i (\log q)^i + O(q^{1-5/512+\epsilon}).$$

The sixth moment of Dirichlet L-functions

Like the moment of $\zeta(1/2+it)$, we do not know asymptotic formula for $M_k(q)$ when $k \geq 3$.

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\frac{1}{\phi^\star(q)} \sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^6 \sim 42 a_3 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \frac{(\log q)^9}{9!}.$$

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On GRH, by the work of Harper (2013),

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By using large seive inequality, Huxley (1970) showed that

$$\sum_{q \leq Q} \sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^6 \ll Q^2(\log Q)^9.$$

The average over q significantly increases the size of our family of L-functions from a family of about q L-functions of conductor q we move to a family of about Q^2 L-functions with conductor up to Q.

The sixth moment (cont'd)

By enlarging the family of Dirichlet L-functions, Conrey, Iwaniec and Soundararajan (2012) can prove the following

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \pmod q}\int_{-\infty}^{\infty}\left|L(\frac{1}{2}+it,\chi)\right|^6\left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt$$

$$\sim 42a_3\sum_{q\le Q}\prod_{p|q}\frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)}\phi^*(q)\frac{(\log q)^9}{9!}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt$$

$$\sim 42\ \widetilde{a}_3 Q^2\frac{(\log Q)^9}{9!}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt.$$

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$$\sim 42 a_3 \sum_{q\le Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt$$

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The sixth moments (cont'd)

- They also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term.
- The integration over t is fairly short due to the rapid decay of the Γ function along vertical lines.
- Deriving an analogous result without the average over t remains an open problem.

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- The integration over t is fairly short due to the rapid decay of the Γ function along vertical lines.
- Deriving an analogous result without the average over t remains an open problem.
- The contribution of the main term comes from both diagonal and off-diagonal terms.
- The limitation of the method should be the eighth moment.

The eighth moment of Dirichlet L-functions

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^8 \sim 24024 a_4 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \frac{(\log q)^{16}}{16!}.$$

Note that the constant 24024 appears in the leading term of the eighth moment of $\zeta(s)$.

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Note that the constant 24024 appears in the leading term of the eighth moment of $\zeta(s)$. By using large seive inequality, Huxley (1970) showed that

$$\sum_{q \le Q} \sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^8 \ll Q^2 (\log Q)^{16}.$$

Theorem (C., Li (2013))

On GRH, we have

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} \left| L(\frac{1}{2}+it,\chi) \right|^8 \left| \Gamma\left(\frac{1/2+it}{2} \right) \right|^8 dt$$

$$\sim 24024\ a_4 \sum_{q \le Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \phi^*(q) \frac{(\log q)^{16}}{16!}$$

$$\times \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2+it}{2} \right) \right|^8 dt$$

$$\sim 24024\ \widetilde{a}_4 Q^2 \frac{(\log Q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2+it}{2} \right) \right|^8 dt.$$

Automorphic L-functions

Let q be a prime number. Let $S_k(\Gamma_0(q), \chi)$ be the space of cuspidal holomorphic forms of weight k with respect to the congruence subgroup $\Gamma_0(q)$ and the character $\chi \mod q$, where

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod q \right\},$$

and we define $S_k(\Gamma_1(q))$ be the space of cuspidal holomorphic forms of weight k with respect to the congruence subgroup $\Gamma_1(q)$, where

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod q,\ a \equiv d \equiv 1 \mod q \right\}$$

Let $\mathcal{H}_k(q,\chi) \subset S_k(\Gamma_0(q),\chi)$ be the set of orthogonal basis of $S_k(\Gamma_0(q),\chi)$. Each normalized cusp form $f \in \mathcal{H}_k(q,\chi)$ has a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} e(nz),$$

where $\lambda_f(1) = 1$.

An L-function $L(f,s)$ associated to the normalized cusp form f is defined for $\mathrm{Re}(s) > 1$ as

$$L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}.$$

Note: We consider q to be prime to eliminate old forms.

The completed L-functions is

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = \left(\tfrac{q}{4\pi}\right)^{\frac{s}{2}} \Gamma\left(s+\tfrac{k}{2}\right) L\left(f, \tfrac{1}{2}+s\right).$$

It satisfies the following functional equations

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = i^k \overline{\eta}_f \Lambda\left(\bar{f}, \tfrac{1}{2}-s\right),$$

where $|\eta_f| = 1$.

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Harmonic average:

$$\sum_{f\in\mathcal{H}_k(q,\chi)}^{h} \alpha_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{H}_k(q,\chi)} \frac{\alpha_f}{\|f\|^2},$$

where $< f, g >$ is the Petersson inner product on $\Gamma_0(q)\backslash\mathbb{H}$.

Some results on moments of automorphic L-functions

The second moment: Iwaniec and Sarnak showed that

$$\sum_{f\in\mathcal{H}_2(q,\chi_0)}^{h} L(f,1/2)^2 \sim \log q.$$

The fourth moment: Kowalski, Michel and Vanderkam (2000) showed that

$$\sum_{f\in\mathcal{H}_2(q,\chi_0)}^{h} L(f,1/2)^4 = \frac{1}{60\pi^2}(\log q)^6 + \sum_{i=0}^{5} a_i(\log q)^5 + O(q^{-1/12+\epsilon}),$$

where χ_0 is a trivial character mod q.

The root numbers of this family are of the form $i^k\overline{\eta}_f = \pm 1$. This family has orthogonal symmetry. Therefore the leading term is of the form $\frac{(\log q)^6}{60\pi^2}$ instead of $\frac{(\log q)^4}{2\pi^2}$.

- No asymptotic formula is available for higher moments for any family of L-functions associated to holomorphic modular forms.
- To obtain asymptotic formulae for the sixth and the eighth moment of Dirichlet L-functions, we need to enlarge the size of the family we average on.
- In GL_2 case, we will also increase the size of family of L-functions to get an asymptotic formula for the sixth moment.

The spaces $S_k(\Gamma_0(q))$ vs $S_k(\Gamma_1(q))$

Dimension of the spaces

$$\dim S_k(\Gamma_0(q)) \sim \frac{k-1}{12} q \prod_{p|q} (1 + p^{-1}),$$

and

$$\dim S_k(\Gamma_1(q)) \sim \frac{k-1}{24} q^2 \prod_{p|q} (1 - p^{-2}).$$

They are connected by

$$S_k(\Gamma_1(q)) = \bigoplus_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} S_k(\Gamma_0(q), \chi).$$

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GL_1	GL_2
Dirichlet characters mod q (size q)	$\Gamma_0(q)$ modular forms
All Dirichlet characters mod $q \sim Q$ (size Q^2)	$\Gamma_1(q)$ modular forms

Djankovic's work

For fixed weight k and prime q, Djankovic (2011) considered the sixth moment of family of L-functions associated with modular forms in $S_k(\Gamma_1(q))$. In particular, he considered

$$M_6(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sum_{f \in \mathcal{H}_k(q,\chi)}^{h} |L(f, 1/2)|^6 .$$

However, this family admits the unitary symmetry instead of orthogonal symmetry (note that the root numbers $i^k \overline{\eta}_f$ are uniformly distributed around the unit circle).

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However, this family admits the unitary symmetry instead of orthogonal symmetry (note that the root numbers $i^k \overline{\eta}_f$ are uniformly distributed around the unit circle).

Theorem (Djankovic)

For $k \geq 3$,

$$M_6(q) \ll q^{\epsilon}.$$

- This bound is consistent with the Lindelöf hypothesis on average.
- A main tool of the proof is a large sieve for the family of $\Gamma_1(q)$ developed by Iwaniec and Xiaoqing Li (2007).

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Question: Is there an asymptotic formula for $M_6(q)$?

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- A main tool of the proof is a large sieve for the family of $\Gamma_1(q)$ developed by Iwaniec and Xiaoqing Li (2007).

Question: Is there an asymptotic formula for $M_6(q)$?

Answer: Almost! Instead, we can obtain an asymptotic formula if we include a short integration over t.

$$\mathcal{I}_6(q)$$
$$:= \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)} \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2}+it\right) \right|^6 \left| \Gamma\left(\frac{k}{2}+it\right) \right|^6 dt.$$

Main result

Theorem (C. and X.Li)

For $k \geq 4$, we have

$$\mathcal{I}_6(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sum_{f \in \mathcal{H}_k(q,\chi)}^h \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2} + it\right) \right|^6 \left| \Gamma\left(\frac{k}{2} + it\right) \right|^6 dt$$

$$\sim 42 b_3 \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{k}{2} + it\right) \right|^6 dt$$

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$$\sim 42 b_3 \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{k}{2} + it\right)\right|^6 dt$$

This is consistent with a conjecture of $M_6(q)$ from the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, which is

$$M_6(q) \sim 42 b_3 \frac{(\log q)^9}{9!}.$$

A few words about the proof

The main result follows from calculating the shifted moment of the following form:

$$\mathcal{I}_6(q,\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}_k(q,\chi)} \int_{-\infty}^{\infty} \left(\frac{q}{4\pi^2}\right)^{\delta(\alpha,\beta)}$$

$$\times \prod_{j=1}^{3} \Gamma\left(\frac{k}{2}+\alpha_j+it\right)\Gamma\left(\frac{k}{2}-\beta_j-it\right)$$

$$\times \prod_{j=1}^{3} L\left(f,\frac{1}{2}+\alpha_j+it\right)L\left(f,\frac{1}{2}-\beta_j-it\right)\, dt,$$

where $\delta(\alpha,\beta) = \frac{1}{2}\sum_{j=1}^{3}(\alpha_j-\beta_j)$, and $\alpha_j,\beta_j \ll \frac{1}{\log q}$.

Let

$$\mathcal{D}(s,\boldsymbol{\alpha},\boldsymbol{\beta}) = \Big(\frac{q}{4\pi^2}\Big)^{\delta(\alpha,\beta)} \prod_{j=1}^{3} \Gamma\left(\frac{k}{2}+\alpha_j\right)\Gamma\left(\frac{k}{2}-\beta_j\right)\mathcal{A}\mathcal{Z}(0,\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where $\mathcal{Z}(s,\boldsymbol{\alpha},\boldsymbol{\beta}) := \prod_{i=1}^{3}\prod_{j=1}^{3}\zeta(1+2s+\alpha_i-\beta_j)$, and $\mathcal{A}(s,\boldsymbol{\alpha},\boldsymbol{\beta})$ is absolutely convergent when Re $(s) > -1/4$.

Theorem (C., Li)

We have $\mathcal{I}_6(q,\boldsymbol{\alpha},\boldsymbol{\beta})$ is

$$\int_{-\infty}^{\infty} \sum_{\pi\in S_6/(S_3\times S_3)} \mathcal{D}(q,\pi(\boldsymbol{\alpha}+it),\pi(\boldsymbol{\beta}+it))\,dt + O(q^{-\frac{1}{4}+\varepsilon}),$$

where S_j is the permutation group of j variables. If $\pi\in S_{2k}$, we define $\pi(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\alpha_{\pi(1)},\ldots,\alpha_{\pi(2k)})$, and we take the first k coordinates to be $\pi(\boldsymbol{\alpha})$ and the rest to be $\pi(\boldsymbol{\beta})$.

Approximate functional equation

For simplicity in this talk, we will consider the moment without shifts.

Notation: $d_3(n) = \sum_{n_1 n_2 n_3 = n} 1$, and the Hecke relation is

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m.n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right).$$

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By Hecke relation and the functional equation of L-functions, We have

$$\left|\Lambda\left(\frac{1}{2}+it, f\right)\right|^6 = \left|L\left(f, \frac{1}{2}+it\right)\right|^6 \left|\Gamma\left(\frac{k}{2}+it\right)\right|^6$$

$$= \sum\sum\sum\sum_{a_1,b_1,a_2,b_2} \frac{\mu(a_1)\chi(a_1)}{a_1^{3/2}} \frac{\chi(b_1)d_3(b_1)}{b_1} \frac{\mu(a_2)\overline{\chi}(a_2)}{a_2^{3/2}} \frac{\overline{\chi}(b_2)d_3(b_2)}{b_2}$$

$$\cdot \sum\sum_{n,m} \frac{\lambda_f(a_1 n)d_3(n)}{n^{1/2}} \frac{\overline{\lambda_f}(a_2 m)d_3(m)}{m^{1/2}} W_t\left(\frac{a_1^3 b_1^2 a_2^3 b_2^2 nm}{q^3}\right),$$

where W_t has a rapid decay.

Model

Here we consider

$$\sum_{n,m}\sum \frac{\lambda_f(n)\sigma_3(n)}{n^{1/2}} \frac{\overline{\lambda_f}(m)\sigma_3(m)}{m^{1/2}} W_t\left(\frac{nm}{q^3}\right).$$

After integration over t, we have

$$\sum_{n,m}\sum \frac{\lambda_f(n)\sigma_3(n)}{n^{1/2}} \frac{\overline{\lambda_f}(m)\sigma_3(m)}{m^{1/2}} V(m,n),$$

where $V(m,n) \ll \exp\left(-\frac{c\max\{m,n\}}{q^{3/2}}\right)$.

Remark

- *The most important region is when* $mn \asymp q^{3+\varepsilon}$. *In a simple model, we will do analysis there.*
- *If we do not have the integration over* t, *then the main contribution comes from when* mn *go up to about size of* q^3, *and it is possible that one variable is very large* ($\sim q^3$), *and the other is small.*
- *It is difficult to bound off-diagonal terms for this range with the current method.*
- *The integrated approximate functional equation leads to the term where* m, n *are both at most* $q^{3/2+\epsilon}$.

We want to understand $M(q)$ which is defined by

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sum_{f \in \mathcal{H}_k(q,\chi)}^{h} \sum_{n,m}\sum \frac{\lambda_f(n) d_3(n)}{n^{1/2}} \frac{\overline{\lambda_f}(m) d_3(m)}{m^{1/2}} V(m,n).$$

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We then apply Petersson's formula

$$\sum_{f\in\mathcal{H}_k(q,\chi)}^{h} \lambda_f(m)\lambda_f(n) = \delta_{m=n} + \sigma_\chi(m,n),$$

where

$$\sigma_\chi(m,n) = 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{1}{cq} S_\chi(m,n;cq) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right).$$

S_χ is the Kloosterman sum defined by

$$S_\chi(m,n;cq) = \sum_{a \bmod cq}^{*} \chi(a) e\left(\frac{am+\overline{a}n}{cq}\right),$$

where $e(x) = \exp(2\pi i x)$, and J_{k-1} is the Bessel function.

We can write

$$M(q) = D(q) + G(q),$$

where

- $D(q)$ is the contribution from diagonal terms $m = n$. (This contributes to the main term.)
- $G(q)$ is the contribution from off-diagonal terms.

We consider $G(q)$, which is

$$\sideset{}{^*}\sum_{a \bmod cq} \sum_{n,m}\sum \frac{d_3(n)}{n^{1/2}} \frac{d_3(m)}{m^{1/2}} V(m,n) \sum_{c=1}^{\infty} \frac{1}{cq} J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right)$$

$$\cdot (2\pi i^{-k}) e\left(\frac{am + \overline{a}n}{cq}\right) \left\{ \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \chi(a) \right\}.$$

The Bessel function satisfies

$$J_{k-1}(x) \ll \min\{x^{-1/2}, x^{k-1}\}.$$

The most important region to consider is the transition region when $x \asymp 1$, in other words, when $c \asymp \frac{\sqrt{mn}}{q} \asymp q^{\frac{1}{2}+\varepsilon}$. (Recall that here we consider only the case when $mn \asymp q^{3+\varepsilon}$.)

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We then apply orthogonality relation for Dirichlet characters

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(m)\overline{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n \mod q \\ (-1)^k & \text{if } m \equiv -n \mod q \\ 0 & \text{otherwise.} \end{cases}$$

After applying the orthogonality relation and split the sum to dyadic sum, essentially, we need to understand the sum of the form

$$\sum_{n,m}\sum \frac{d_3(n)}{n^{1/2}}\frac{d_3(m)}{m^{1/2}} V(m,n) \sum_{c \asymp q^{\frac{1}{2}+\varepsilon}} \frac{1}{cq} J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right)$$

$$\cdot \sideset{}{^*}\sum_{\substack{a \mod cq \\ a\equiv 1 \mod q}} e\left(\frac{am+\bar{a}n}{cq}\right) f\left(\frac{m}{M}\right) f\left(\frac{n}{N}\right),$$

where f is a smooth partition of unity.

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where f is a smooth partition of unity.

We want to apply Voronoi summation to the sum over m and n.

Treatment of the exponential sum

By Chinese Remainder Theorem and reciprocity, we may factor our exponential sum as

$$\sideset{}{^*}\sum_{\substack{a \mod cq \\ a\equiv 1 \mod q}} e\left(\frac{am+\overline{a}n}{cq}\right)$$

$$= e\left(\frac{m+n}{cq}\right) \sideset{}{^*}\sum_{z \mod c} e\left(\frac{\overline{q}(z-1)m}{c}\right) e\left(\frac{\overline{q}(\bar{z}-1)n}{c}\right)$$

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- Originally, the conductor is cq. Before applying the Voronoi summation, the main contribution comes from when $m, n \ll q^{\frac{3}{2}+\varepsilon}$. After Voronoi summation, the main contribution of the dual sum comes from when $m' \ll \frac{(cq)^3}{M} \ll q^{3-\varepsilon}$ and $n' \ll \frac{(cq)^3}{M} \ll q^{3-\varepsilon}$.
- The important range of the dual sum is much longer after Voronoi summation, and it is more difficult to handle.

- So we reduce the conductor to c. After applying Voronoi summation, the main contribution of the dual sum is when $m' \ll \frac{c^3}{M} \asymp \frac{q^{\frac{3}{2}+\varepsilon}}{M}$ and $n' \ll \frac{c^3}{N} \asymp \frac{q^{\frac{3}{2}+\varepsilon}}{N}$.
- We consider the region when $MN \asymp q^{3+\varepsilon}$. Due to the decay rate of V, we have $M, N \ll q^{\frac{3}{2}+\varepsilon}$. Hence the important region is when M, N are of size around $q^{\frac{3}{2}+\varepsilon}$.
- Hence the main contribution of the dual sums is when $m', n' \ll q^{\varepsilon}$.

Applying Voronoi summation

We apply Voronoi summation formula to

$$H(q) := \sum_{c \asymp q^{\frac{1}{2}+\varepsilon}} \frac{1}{cq} \sideset{}{^*}\sum_{z \mod c} \sum_n \frac{d_3(n)}{n^{1/2}} F_1(n) e\left(\frac{\overline{q}(\bar{z}-1)n}{c}\right)$$
$$\times \sum_m \frac{d_3(m)}{m^{1/2}} F_2(m,n) e\left(\frac{q(z-1)m}{c}\right),$$

where F_1 and F_2 are smooth functions, defined as

$$F_1(y) := f\left(\frac{y}{N}\right) e\left(\frac{y}{cq}\right),$$

and

$$F_2(x,y) := V(x,y) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{xy}\right) f\left(\frac{x}{M}\right) e\left(\frac{x}{cq}\right).$$

Note: We can treat $e\left(\frac{x}{cq}\right)$ as smooth functions because its j-th derivative with respect to x is

$$\frac{1}{(cq)^j} \asymp \frac{1}{(q^{3/2+\varepsilon})^j} \ll \min\left\{\frac{1}{M^j}, \frac{1}{N^j}\right\}.$$

(N, M goes up to $q^{\frac{3}{2}+\varepsilon}$ due to the factor V). Hence

$$\frac{\partial^j F_1}{\partial y^j} \ll \frac{1}{N^j}$$

and

$$\frac{\partial^j F_2}{\partial x^j} \ll \frac{1}{M^j}, \quad \text{and} \quad \frac{\partial^j F_2}{\partial y^j} \ll \frac{1}{N^j}.$$

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If we do not have the integration over t in the moments, N could possibly go up to q^3, and $\frac{1}{(cq)^j} \gg \frac{1}{N^j}$. So the derivative of $e\left(\frac{x}{cq}\right)$ is too large.

Applying Voronoi summation for the sum over m (fixed n), we roughly have

$$\text{main terms} + \sum_{m'=1}^{\infty} a_{m'} \int_0^{\infty} F_2(x,n) U\left(\frac{xm'}{c^3}\right) dx.$$

Big m': When $m' \gg \frac{c^3 q^{\varepsilon}}{M}$, we can use integration by parts several times to show that the contribution from these terms is small. So we want the j-th derivative over F_2 to be small (i.e. $\ll \frac{1}{M^j}$.). The size of the derivative of $e\left(\frac{x}{cq}\right)$ is important here.

Small m': When $m' \ll \frac{c^3 q^{\varepsilon}}{M} \asymp q^{\varepsilon}$, we can bound it trivially because the sum is short.

Applying Voronoi summation, we have

$$H(q) = \text{main terms} + \text{short sums over m', n'} + \text{tail of the sums.}$$

- The contribution from the short sums and the tail is $O(q^{-1/4+\varepsilon})$.
- We combine the main terms with the diagonal terms ($m = n$).

$$\mathcal{I}_6(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sum_{f \in \mathcal{H}_k(q,\chi)}^{h} \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2} + it\right)\right|^6 \left|\Gamma\left(\frac{k}{2} + it\right)\right|^6 dt$$

$$\sim 42 b_3 \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{k}{2} + it\right)\right|^6 dt$$

In the case of shifted moments, we compare the main terms with the conjecture of CFKRS through the Euler products.

THANK YOU VERY MUCH!