Quaternionic and Hilbert modular forms and their Galois representations (second version)

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1 Notations ans some algebra

Let F be a totally real field of degree $d = [F : \mathbb{Q}]$, $\mathcal{O}_F$ its ring of integers. Let D be its discriminant. The set of prime factors dividing D is denoted by $\mathrm{Ram}(F)$.

Let $J_F = \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F, \overline{\mathbb{Q}})$ be the set of embeddings of F; we fix an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ by which we view also J_F as the set of archimedean places of F.

We fix a partition $J_F = S \sqcup S'$, and put $s = \mathrm{Card}(S)$, $s' = \mathrm{Card}(S')$, so that $s + s' = d$.

We refer to Vignéras' book [20] for details on this section.

Let B be a simple central F-algebra of degree 4 ramified at all places of S' and split at all places of S. When $S = J_F$, the case $B = M_2(F)$ is possible; if $B \neq M_2(F)$, B is called a quaternion division algebra over F. We denote by Δ the set of prime ideals of F at which B ramifies. The ramification set of B is $\mathrm{Ram}(B) = \Delta \cup S'$. Its cardinality is even.

We will be mostly interested in the cases $B = M_2(F)$ (Hilbert modular case), $s = 1$ (Shimura curve case) and $s = 0$ (zero-dimensional Shimura variety, called Hida set by K. Fujiwara).

A possible construction of B is to give an F-basis $(1, i, j, k)$ together with its multiplication table: choose $a, a' \in F^\times$, then define $B = [a, a']$ as the F-algebra with basis $1, i, j, k$ such that $i^2 = a$, $j^2 = a'$ and $k = ij = -ji$.

For $[a, a']$, the set S is the subset of embeddings for which either a or a' is positive while the set Δ consists of the primes $\mathfrak{p}$ such that the form $x^2 - ay^2 - a'z^2 + aa't^2$ does not represent zero over $F_\mathfrak{p}$.

We denote by $b \mapsto b'$ the main involution of B: $x + yi + zj + tk \mapsto x - yi - zj - tk$, $tr : B \to F, b \mapsto b + b' = 2x$ the reduced trace and $\nu : B^\times \to F^\times, b \mapsto bb' = x^2 - ay^2 - a'z^2 + aa't^2$ the reduced norm. We denote its kernel by B^1. For $B = M_2(F)$, tr is the usual trace, $\nu = \det$, $B^\times = GL_2(F)$ and $B^1 = SL_2(F)$.

We introduce various reductive algebraic groups defined over $\mathbb{Q}$: $G = \mathrm{Res}^F_{\mathbb{Q}} B^\times$, $G^* = \{g \in G; \nu(g) \in \mathbb{Q}^\times\}$ and $G^1 = \mathrm{Ker}\,\nu = \mathrm{Res}^F_{\mathbb{Q}} B^1$ (this last group is simple and is the derived group of G and G^*).

Any maximal order $R \subset B$ provides integral models G_R, G^*_R, resp. G^1_R of the reductive groups G, G^* and G^1 by applying the Weil restriction from R to $\mathbb{Z}$. These group schemes have good reduction outside $\mathrm{Ram}(B) \cup \mathrm{Ram}(F)$.

Note that for any quadratic extension K/F such that for any $v \in \mathrm{Ram}(B)$, the place v does not split in K, we have $B \otimes_F K \cong M_2(K)$ (by the theory of the Brauer group of a number field); note K is complex at each place in S'. Hence, for any extension L containing the Galois closure of K, there exists an isomorphism $B \otimes_{\mathbb{Q}} L \cong M_2(L)^{J_F}$. Actually, given a maximal order $R \subset B$ and such an extension L/F, one can find adjust the isomorhsim so that it sends $R \otimes \mathcal{O}_L$ into $M_2(\mathcal{O}_L)^{J_F}$ (Exercise! hint: fix any isomorphism over L and use a conjugation by an element $(g_\sigma) \in SL_2(L)^{J_F}$ with $g_\sigma \in SL_2(L)$ close to an adelic element $\widehat{g}_\sigma \in SL_2(L_f)$ realizing the desired conjugation into $M_2(\widehat{\mathcal{O}}_L)^{J_F}$).

Let $\mathbb{A} = \mathbb{Q}_f \times \mathbb{Q}_\infty$ be the ring of rational adeles, decomposed into the product of its finite part $\mathbb{Q}_f = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$ and its archimedean part $\mathbb{Q}_\infty = \mathbb{R}$. We write $F_{\mathbb{A}} = F \otimes \mathbb{A}$ (resp.$F_f = F \otimes \mathbb{Q}_f$, ...) and we consider the $F_{\mathbb{A}}$-algebra $B_{\mathbb{A}} = B \otimes \mathbb{A}$, $B^\times_{\mathbb{A}}$ its group of invertible elements, and $B^1_{\mathbb{A}} = B^1_f \times B^1_\infty$ its subgroup of norm 1 elements. Recall that if the Eichler condition $S \neq \emptyset$ ($s > 0$) holds, we have the strong approximation theorem: $B^1_{\mathbb{A}} = B^1 \cdot (V \times B^1_\infty)$ for any compact open subgroup V of B^1_f. It implies that for any compact open subgroup $U \subset B^\times_f$, **the reduced norm ν induces a bijection**

$$B^\times \backslash B^\times_{\mathbb{A}} / (U \times B^+_\infty) \to F^\times \backslash F^\times_{\mathbb{A}} / \nu(U) F^+_\infty$$

where B^+_∞ denotes the subgroup of elements whose reduced norm is totally positive (or equivalently S-positive). Note that the right-hand side Cl^+_U is a (finite) strict ray-class group. By choosing representatives $t_1, \ldots, t_h \in F^\times_f$ of Cl_U and then choosing $b_i \in B^\times_{\mathbb{A}}$ such that $\nu(b_i) = t_i$, one obtains:

$$(1) \quad B^\times_{\mathbb{A}} = \bigsqcup_{i=1}^{h} B^\times \cdot b_i \cdot (U \times B^+_\infty).$$

[Proof of the bijection:

surjectivity: write Cl_U as $F^{\times,S'+}\backslash(F_f^\times \times F_\infty^{\times,S'+})/\nu(U)F_\infty^+$, where $F^{\times,S'+}$ is the subgroup of S'-positive elements of $F^\times$;

injectivity: if $x,y \in B_f^\times$ and $\nu(y) \in F^{\times,S'+} \cdot \nu(x) \cdot \nu(U)F_\infty^+$, one can find $b \in B^\times$, $u \in U$, $b_\infty^+ \in B_\infty^+$ such that $b^{-1}y$ and xub_∞^+ have same ν. Therefore, $z = b^{-1}y(xub_\infty^+)^{-1} \in B_{\mathbb{A}}^1$. We apply $B_{\mathbb{A}}^1 = B^1 V B_\infty^1$ to $V = G_f^1 \cap xUx^{-1}$. We thus obtain $z = \beta v \beta_\infty$, or $y = b\beta v x u \beta_\infty b_\infty^+$; since $vxu = x(x^{-1}vx)u \in xU$, we see that $y \in B^\times xUB_\infty^+$ as desired.]

If the Eichler condition does not hold ($s = 0$), still, there is a finite decomposition (depending on U, not only on $\nu(U)$):

$B_{\mathbb{A}}^\times = \bigsqcup B^\times \cdot b_i \cdot (U \times B_\infty^\times)$. Note that in that case, $B_\infty^\times = B_\infty^+$. Actually, consider the modulus character $B_{\mathbb{A}}^\times \to \mathbb{R}^+$ given by $|b| = |\nu(b)|_{\mathbb{A}}$ and the topological space (not group!) $B^\times\backslash B_{\mathbb{A}}^{|\,|=1}$. In all cases ($s = 0$ or not), if $B \neq M_2(F)$, this quotient space is compact (it is called Fujisaki's theorem in Vignéras' book); this implies that the quotient sets $B^\times\backslash B_f^\times/U$ and $X_U = B^\times\backslash B_f^\times/UF_f^\times$ are finite. Indeed, $B^\times\backslash B_f^\times/U = B^\times\backslash B_{\mathbb{A}}^\times/UB_\infty^\times$ is finite (by Fujisaki), and X_U is the quotient of this set by the action of its finite subgroup $F^\times\backslash F_f^\times/(U \cap F_f^\times)$.

It also follows from Fujisaki's theorem that $B^1\backslash B_f^1$ is compact (because $B^1\backslash B_{\mathbb{A}}^1$ is).

2 Shimura varieties and sheaves

Let B as above. For $? = \emptyset, *, 1$, we put $G_f^? = G^?(\mathbb{Q}_f)$, $G_\infty^? = G^?(\mathbb{Q}_\infty)$, so that one has a decomposition of topological groups $G^?(\mathbb{A}) = G_f^? \times G_\infty^?$. Let H be the $\mathbb{R}$-algebra of Hamilton quaternions and H^1 its group of norm one elements; we write $K_\infty = SO_2(\mathbb{R})^S \times (H^1)^{S'}$ for the standard maximal compact subgroup of G_∞^1 and we put $C_\infty = F_\infty^\times \cdot K_\infty$.

2.1 Complex quaternionic varieties

For each compact open subgroup $U \subset G_f$, we introduce the set (called the Shimura variety for G of level U):

$$Sh_U = G(\mathbb{Q})\backslash G(\mathbb{A})/UC_\infty.$$

Let G_∞^+ be the subgroup of positive norm elements in G_∞. Note that $G_\infty^+ = GL_2^+(\mathbb{R})^S \times (H^\times)^{S'}$ and $G_\infty^+/C_\infty = (GL_2^+(\mathbb{R})/\mathbb{R}^\times SO_2(\mathbb{R}))^S \times (H^\times/\mathbb{R}^\times H^1)^{S'}$. This is diffeomorphic to the product $\mathfrak{h}_S$ of s copies of the upper half plane

$\mathfrak{h}$ by $g \mapsto g(i)$ (where $i = \sqrt{-1} \in \mathfrak{h}$). For any U, it is an analytic space of dimension s; moreover, Sh_U is compact for all U's, unless B is $M_2(F)$ ("there are no cusps for quaternionic Shimura varieties, except in the Hilbert case"). This follows immediately from Fujisaki's theorem. In particular, if $s = 0$, we have $C_\infty = B_\infty^\times$ so that the variety $Sh_U = B^\times \backslash B_f^\times / U$ is finite, consisting in the set $\{[G(\mathbb{Q})b_i U C_\infty]\}_i$; it is sometimes called the Hida set for G with level U.

In general, for any s, by weak approximation at archimedean places, one has $Sh_U = G(\mathbb{Q})^+ \backslash ((G_f/U) \times \mathfrak{h}_S)$ where $\mathfrak{h}_S$ denotes the product of copies of $\mathfrak{h}$ indexed by S, and one deduces from (1) the decomposition of Sh_U into connected components

$$Sh_U \cong \bigsqcup_i \Gamma_i \backslash \mathfrak{h}_S,$$

where $\Gamma_i = b_i U G_\infty^+ b_i^{-1} \cap G(\mathbb{Q})$ $(i = 1, \ldots, h)$ is a discrete subgroup of $G_\mathbb{Q}^+$ which acts properly modulo center on $\mathfrak{h}_S$. [Indeed, $G_\mathbb{A} = \bigsqcup G_\mathbb{Q} b_i U G_\infty^+$ and for any $b \in G_f$, the stabilizer of bUG_∞^+ in $G_\mathbb{Q}$ is $\Gamma_b = G_\mathbb{Q} \cap bUG_\infty^+ b^{-1}$.]

If U is sufficiently small, all the groups Γ_i and their quotients by $F^\times \cap \Gamma_i$ are torsion-free (we'll see this more precisely later). Thus, $\Gamma_i/(F^\times \cap \Gamma_i)$ acts properly and freely on $\mathfrak{h}_S$. This implies

Fact: For any s, if U is sufficiently small, Sh_U is a smooth complex variety.

Indeed, all its connected components are complex varieties.

Comments:

1) Each connected component $\mathcal{C}_i$ of Sh_U is a quotient of the connected Shimura variety $Sh_{V_i}^1 = \Gamma_i^1 \backslash \mathfrak{h}_S$ where $V_i = G_f^1 \cap b_i U b_i^{-1}$ is compact open in G_f^1 and $\Gamma_i^1 = V_i \cap G^1(\mathbb{Q})$ is a discrete subgroup of G_∞^1. More precisely, there is a finite étale covering $Sh_{V_i} \to \mathcal{C}_i$ with Galois group $E_i = \det(\Gamma_i)/(F^\times \cap \Gamma_i)^2$, a finite subquotient of $\mathcal{O}^{\times,+}$.

2) When $B \neq M_2(F)$, the variety Sh_U is called a quaternionic Shimura variety, resp. a Shimura curve when $s = 1$.

3) In the case $B = M_2(F)$, Sh_U is called the adelic Hilbert modular variety of level U. In this case, for $U^* = U \cap G_f^*$, and $C_\infty^* = \mathbb{Q}_\infty^\times \cdot K_\infty$, we also introduce $Sh_U^* = G^*(\mathbb{Q}) \backslash G^*(\mathbb{A})/U^* C_\infty^*$ (Shimura variety for G^* of level U^*). We have

$$Sh_{U^*}^* \cong \bigsqcup_j \Gamma_j^* \backslash \mathfrak{h}_S$$

where $\Gamma_j^* = c_j U c_j^{-1} \cap G^*(\mathbb{Q})$ for $j = 1, \ldots, h^*$. The reduced norm induces an isomorphism from the group $\pi_0(Sh_{U^*}^*)$ of connected components to

$$\mathbb{Q}^\times \backslash \mathbb{A}^\times / (\nu(U) \cap \mathbb{Q}_f^\times) \mathbb{Q}_\infty^+,$$

and each connected component coincide with $Sh^1_{V_j} = \Gamma^1_j \backslash \mathfrak{h}_S$ where $V_j = G^1_f \cap c_j U c_j^{-1}$. Thus, the inclusions $G^1 \subset G^* \subset G$ induce morphisms $Sh^1_{U^1} \to Sh^*_{U^*} \to Sh_U$, which are open, quasi-finite, the first one being an open immersion. The introduction of G^* and of its Shimura varieties is very useful because, even if U is neat, the canonical model of Sh_U will be only a coarse moduli space while the one of Sh^*_U will be a fine moduli space.

2.2 Coefficient systems

We fix a maximal order $R \subset B$, and we choose a finite extension K/F which splits B and contains a Galois closure of F, together with an isomorphism $B \otimes_{\mathbb{Q}} K \cong M_2(K)^{J_F}$ sending $R \otimes \mathcal{O}_K$ into $M_2(\mathcal{O}_K)^{J_F}$.

For any $n_\tau, v_\tau \in \mathbb{N}$, we introduce the $\mathcal{O}_K$-modules $\Lambda(n_\tau, v_\tau; \mathcal{O}_K) = \mathrm{Sym}^{n_\tau} \mathcal{O}_K^2 \otimes \det^{v_\tau} \mathcal{O}_K^2$ and its $\mathcal{O}_K$-dual $L(n_\tau, v_\tau; \mathcal{O}_K) = \Lambda(n_\tau, v_\tau; \mathcal{O}_K)^*$; one has $L(n_\tau, v_\tau; \mathcal{O}_K) = \mathrm{Sym}^{n_\tau} \mathcal{O}_K^2 \otimes \det^{-n_\tau - v_\tau} \mathcal{O}_K^2$. Since there is an ambiguity about the definition of symmetric powers over the integers, let us be clear that in the definition of $\Lambda(n_\tau, v_\tau; \mathcal{O}_K)$, resp. $L(n, v; \mathcal{O}_K)$, we view the symmetric algebra as the graded subalgebra of the tensor algebra, resp. as the quotient algebra by the graded two-sided ideal generated by $u \otimes v - v \otimes u$.

There is an $\mathcal{O}_K$-schematic action, "the τ-action", of the $\mathcal{O}_F$-group scheme $R^\times$ on $\Lambda(n_\tau, v_\tau; \mathcal{O}_K)$ via the τ-embedding $R \subset M_2(\mathcal{O}_K)$ fixed above. We endow $L(n_\tau, v_\tau; \mathcal{O}_K)$ with the contragredient of this representation of $R^\times$.

For $n = \sum_{\tau \in J_F} n_\tau \tau \in \mathbb{N}[J_F]$ and $v = \sum_{\tau \in J_F} v_\tau \tau \in \mathbb{Z}[J_F]$, we put $L(n, v; \mathcal{O}_K) = \bigotimes_{\tau \in J_F} L(n_\tau, v_\tau, \mathcal{O}_K)$. this finite free $\mathcal{O}_K$-module comes with a schematic action of G_R defined over $\mathcal{O}_K$. It is the tensor product of the τ-actions.

In order to define a sheaf over Sh_U, we'll need that scalar matrices $\epsilon \in \mathcal{O}_F^\times \cap U \subset G_R(\mathbb{Z})$ act trivially by this action. This is why we assume from now on that $n + 2v$ is diagonal: if $t = \sum_\tau \tau$, **we assume that there exists** $m \in \mathbb{Z}$ **such that** $n + 2v = mt$.

For any $\mathcal{O}_K$-module, we also put $L(n, v; M) = L(n, v; \mathcal{O}_K) \otimes M$; it is still a schematic module over $G_R \otimes \mathcal{O}_K$. If M is a K-vector space, we define a left action of $G(\mathbb{Q})$ on $L(n, v; M)$ by the left representation of $G \otimes K$ defined above, and take the trivial right action of UC_∞. Then consider the sheaf $\mathcal{L}(n, v; M)$ over Sh_U of continuous sections of the K-vector bundle

$$B_1 = G(\mathbb{Q}) \backslash G(\mathbb{A}) \times L(n, v; M) / UC_\infty \to Sh_U$$

induced by $pr_1 : G(\mathbb{A}) \times L(n, v; M) \to G(\mathbb{A})$ for the actions $\gamma \cdot (g, \ell) = (\gamma g, \gamma \cdot \ell)$ and $(g, \ell) u c_\infty = (g u c_\infty, \ell)$.

We will be interested in the Betti cohomology groups $H^\bullet(Sh_U, \mathcal{L}(n, v; M))$, $H^\bullet_c(Sh_U, \mathcal{L}(n, v; M))$, especially the sth cohomology group (middle degree).

Let p be a prime number. If M is p-adic, one can define a p-adic integral structure on these groups. More precisely, let v be a place of $\mathcal{O}_K$ above p and let $\mathcal{O}$ be a discrete valuation ring which is finite flat over the completion $\mathcal{O}_{K_v}$ of $\mathcal{O}_K$ at v. Let us assume that M_0 is an $\mathcal{O}$-module. Consider the trivial left action of $G(\mathbb{Q})$ on $L(n,v;M_0)$ and the right action of UC_∞ given by $v \cdot uc_\infty = u_p^{-1}v$. This defines another bundle

$$B_2 = G(\mathbb{Q})\backslash G(\mathbb{A}) \times L(n,v;M_0)/UC_\infty \to Sh_U$$

In case M is both a K-vector space and an $\mathcal{O}$-module, these two bundles are isomorphic by the Sh_U-morphism $B_1 \to B_2$ given by $(g,v) \mapsto (g, g_p^{-1}v)$. Let us define, for any $\mathcal{O}$-module M_0, the sheaf $\mathcal{L}(n,v;M_0)$ of continuous sections of $B_2 \to Sh_U$.

With this definition, for any $\mathcal{O}$-module M_0 and for $M = M_0[1/p]$, we have a natural integral structure on $H^\bullet_*(M) = H^\bullet_*(Sh_U, \mathcal{L}(n,v;M))$ $(* = \emptyset, c)$, given by the image of $H^\bullet_*(M_0) = H^\bullet_*(Sh_U, \mathcal{L}(n,v;M_0))$.

2.3 Hecke correspondences

The correspondence $[U\xi U]$ (for $\xi \in G_f$): Let $U(\xi) = \xi U \xi^{-1} \cap U$. We consider the two projections $\pi_i : Sh_{U(\xi)} \to Sh_U$ where π_1 is the transition map induced by the inclusion $U(\xi) \subset U$, and π_2 is induced by right multiplication by ξ on G_f. Moreover, by noticing that $\pi_1^*\mathcal{L}$ coincides with the sheaf $\mathcal{L}_\xi$ associated above to L on $Sh_{U(\xi)}$, we see that there is a natural K-linear morphism of sheaves $[\xi] : \pi_2^*\mathcal{L}(n,v;M) \to \pi_1^*\mathcal{L}(n,v;M)$: if $s(g) = s(gu)$ for $u \in U$, $\pi_2^*s : g \mapsto s(g\xi)$ is right invariant by $U(\xi)$ hence defines a section of $\pi_1^*\mathcal{L} = \mathcal{L}_\xi$. Since π_i are finite (étale if U is neat), we can define the trace π_{1*}; define $T_\mathfrak{q}$ as $\pi_{1*} \circ [\xi] \circ \pi_2^*$. This correspondence acts on $H^\bullet_*(M)$. Moreover, if $M = M_0[1/p]$ for a finite free $\mathcal{O}$-module M_0, we see that if $\xi_p = 1$, the action of $[U\xi U]$ preserves the p-adic integral structure given by the image of $H^\bullet_*(M_0)$ in $H^\bullet_*(M)$.

Remark: If one wants to include the case where ξ_p is non trivial, one needs to assume that $\xi_p \in R_p \cap B_p^\times$ and we need to modify the morphism $[\xi]$ using the description via the bundle B_2: $[\xi](s) = \xi'_p \cdot s(g\xi)$ where the prime denotes the main involution on B.

For a compact open subgroup $U \subset G_f$, we denote by $Ram(U)$ the set of places of F such that $U_v \neq R_v^\times$. For a prime $\mathfrak{q} \notin \mathrm{Ram}(U) \cup \mathrm{Ram}(B)$, let $\varpi_\mathfrak{q}$ be a uniformizing parameter in $F_\mathfrak{q}$ and $\xi \in B_f^\times$. We define the Hecke correspondence $T_\mathfrak{q}$, resp. $S_\mathfrak{q}$, by choosing ξ whose components are 1, except at $\mathfrak{q}$ where it is $\begin{pmatrix} \varpi_\mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix}$, resp. $\begin{pmatrix} \varpi_\mathfrak{q} & 0 \\ 0 & \varpi_\mathfrak{q} \end{pmatrix}$. As observed above, for $\mathfrak{q}$ prime to p, these operators preserve the p-adic integral structure on the cohomology.

3 Hilbert and Quaternionic forms

3.1 Automorphic vector bundles on quaternionic Shimura varieties

In the decomposition $G_\infty = G(\mathbb{R}) = GL_2(\mathbb{R})^S \times (H^\times)^{S'}$, any element $g \in G_\infty$ is also written as $g = (g_\tau)_\tau$; we write sometimes G_S for the first factor, and $G_{S'}$ or G_∞^S, for the second. Consider the subgroup G_∞^+ of G_∞ of elements with totally positive reduced norm (or, equivalently, S-positive reduced norm).

The product $\mathfrak{h}_S$ of copies of $\mathfrak{h}$ indexed by S carries a left action of G_∞^+ defined, for $z = (z_\tau)_{\tau\in S}$, by $g(z) = (g_\tau(z_\tau))_{\tau\in S}$; here, for each $\tau \in S$, $g_\tau(z_\tau)$ denotes the action of $g_\tau \in GL_2(\mathbb{R})^+$ by linear fractional transformation on $\mathfrak{h}$. This action factors through G_S^+ and even through G_S^+/Z_S where Z_S denotes the center of G_S. Consider the $\mathbb{Q}$-torus $T = \mathrm{Res}_\mathbb{Q}^F F^\times$; its group of characters $X^*(T)$ is identified to $\mathbb{Z}[J_F]$ by $t^{\sum_\tau m_\tau \tau} = \prod_\tau (t^\tau)^{m_\tau}$. Consider the group decomposition $T(\mathbb{C}) = T_S \times T_{S'}, t \mapsto (t_S, t_{S'})$. Define the S-automorphic factor

$$G_\infty^+ \times \mathfrak{h}_S \to T_S, \quad (g,z) \to j_S(g,z) = (c_\tau z_\tau + d_\tau)_{\tau\in S},$$

for $g_\tau = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}$. It satisfies $j_S(gg', z) = j_S(g, g'(z)) \cdot j_S(g', z)$.

For any $m = \sum_\tau m_\tau \tau \in \mathbb{Z}[J_F]$, we put $m_S = \sum_{\tau\in S} m_\tau \tau \in \mathbb{Z}[S]$; it defines an algebraic character of the $\mathbb{R}$-torus T_S by $t_S \mapsto t_S^{m_S}$. Let $k_S \in \mathbb{Z}[S]$ and define the automorphic factor of weight k_S: $G_\infty^+ \times \mathfrak{h}^S \to \mathbb{C}^\times$, $(g,z) \mapsto j_S(g,z)^{k_S} = \prod_{\tau\in S}(c_\tau z_\tau + d_\tau)^{k_\tau}$.

Let us assume that for any $\tau \in S'$, we have $k_\tau \geq 2$; so, for these τ's, we have $n_\tau = k_\tau - 2 \geq 0$. For completeness, we also introduce $n_\tau = k_\tau - 2$ even for $\tau \in S$, but, *a priori* we don't assume $n_\tau \geq 0$ for those τ's.

Choose $v^S = (v_\tau)_{\tau\in S'}$. Let $L(n^S, v^S; \mathbb{C}) = \bigotimes_{\tau\in S'} L(n_\tau, v_\tau; \mathbb{C})$. For any $w_S \in \mathbb{Z}[S]$, we denote by ν^{w_S} the composition $G_S \overset{\nu}{\to} T_S \overset{w_S}{\to} \mathbb{C}^\times$. Let us consider

- the representation $\nu^{-w_S} \otimes L(n^S, v^S; \mathbb{C})$ of $G_\infty = G_S \times G_\infty^S$,
- the one-dimensional representation $\mathbb{C}(k_S)$ of T_S given by $t_S \mapsto t_S^{k_S}$.

We have seen that $Sh_U = G(\mathbb{Q})^+ \backslash (G_f \times \mathfrak{h}_S)/U$; we define a holomorphic vector bundle over Sh_U by defining a left action of $G(\mathbb{Q})^+$ on $G_f \times \mathfrak{h}_S \times \mathbb{C}(k_S) \otimes$

$\nu^{-w_S} L(n^S, v^S; \mathbb{C})$ by $\gamma \cdot (g_f, z, \ell) = (\gamma g_f, \gamma(z), \nu(\gamma)^{-w_S} j_S(\gamma, z)^{k_S} \cdot \gamma^S \cdot \ell)$, and trivial right action of U.

A global section of this bundle is a function $\ell : G_f/U \times \mathfrak{h}_S \to \mathbb{C}(k_S) \otimes \nu^{w_S} \otimes L(n^S, v^S; \mathbb{C})$ such that for any $\gamma \in G(\mathbb{Q})^+$, $\ell(\gamma g_f, \gamma(z)) = \nu(\gamma)^{-w_S} j_S(\gamma, z)^{k_S} \cdot \gamma^S \cdot \ell(g_f, z)$.

Therefore, in order for this vector bundle to admit non zero global sections, it is necessary that any $\epsilon \in F^\times \cap U$ has trivial action. **This is why we add to our assumption $n + 2v = mt$ for some integer $m \geq 0$ the conditions**

1) $U \cap F^\times \subset \mathcal{O}_F^+$ **or** m **even, and**

2) $w = v + k - t$.

Indeed, under these assumptions, ϵ acts on the left by $\epsilon^{-2w_S + k_S - (n^S + 2v^S)}$. The exponent is equal to $-2(v_S + k_S - t_S) + k_S - (k^S - 2t^S + 2v^S) = -2v - k + 2t$; it is therefore an integral multiple of t, so that ϵ acts through a power of its norm, which is 1. We obtain a holomorphic vector bundle on Sh_U denoted by $\omega_S^{k_S} \otimes \nu^{-w_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C})$ (the tensor product is only a notation...)

Comment on the notation: For any i, the pull-back to $Sh^1_{V_i}$ of the restriction to the connected component $\mathcal{C}_i$ of the sheaf $\omega_S^{k_S} \otimes \nu^{-w_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C})$ can be decomposed as the tensor product of an invertible sheaf ω^{k_S} (defined by the same automorphic factor as before) and a locally constant sheaf $\mathcal{L}(n^S, v^S; \mathbb{C})$. In a similar way, note that the locally constant sheaf $\mathcal{L}(n, v; \mathbb{C})$ can also be decomposed as the tensor product of locally constant sheaves $\mathcal{L}(n_S, v_S; \mathbb{C}) \otimes \mathcal{L}(n^S, v^S; \mathbb{C})$. Moreover, if $k_S \geq 2t_S$, Hodge theory provides a canonical injective homomorphism

$$H^0(Sh^1_{V_i}, \omega^{k_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C})) \hookrightarrow H^s(Sh^1_{V_i}, \mathcal{L}(n_S, v_S; \mathbb{C}) \otimes \mathcal{L}(n^S, v^S; \mathbb{C}))$$

which identifies the left hand side with the last non zero term of the Hodge filtration.

Note however that this tensor product decomposition does not descend to $\mathcal{C}_i$ because of the presence of units $\epsilon \in F^\times \cap U$, by the calculation above.

3.2 Space of quaternionic cusp forms

Assume first $s > 0$:

If $d > 1$ or if $d = 1$ and $B \neq M_2(\mathbb{Q})$, we define the vector space of quaternionic forms of weight k, w and level U by $M_{k,w}(U) = H^0(Sh_U, \omega_S^{k_S} \otimes \nu^{-w_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C}))$. Again, $n = k - 2t$ with $n^S \geq 0$, $n + 2v = mt$ and $w = k - t + v$.

If $d = 1$, and $B = M_2(\mathbb{Q})$ the holomorphicity at cusps must be also imposed.

Apart from the Hilbert case $B = M_2(F)$ the space $M_{k,w}(U)$ consists only of cusp forms (in the sense of the Jacquet-Langlands correspondence); we thus write $S_{k,w}(U) = M_{k,w}(U)$ ("S" is the initial of Spitz=cusp):

$$S_{k,w}(U) = H^0(Sh_U, \omega_S^{k_S} \otimes \nu^{-w_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C})).$$

In the Hilbert case, the space $S_{k,w}(U)$ is defined as the subspace of $M_{k,w}(U)$ of sections vanishing at cusps.

Since $s > 0$, a global section of the sheaf $\omega_S^{k_S} \otimes \nu^{-w_S} \otimes \mathcal{L}(n^S, v^S; \mathbb{C})$ over Sh_U is a function $f : G_f \times \mathfrak{h}_S \to \nu^{-w_S} \otimes L(n^S, v^S; \mathbb{C})$, such that $f(g_f, z)$ is holomorphic in $z \in \mathfrak{h}_S$ and for any $\gamma = (\gamma_f, \gamma_\infty) \in G(\mathbb{Q})^+$ and any $u \in U$, one has $f(\gamma_f g_f u, \gamma_\infty(z)) = \nu(\gamma)^{-w_S} j_S(\gamma, z)^{k_S} \gamma^S \cdot f(g_f, z)$.

There is another description of the pull-back of the space $S_{k,w}(U)$. Namely, one maps a function $f : G_f/U \times \mathfrak{h}_S \to L(n^S, v^S; \mathbb{C})$ which is C_∞-invariant and $G(\mathbb{Q})$-equivariant, to the function $\phi : G(\mathbb{Q})^+ \backslash ((G_f/U) \times G_\infty^+) \to L(n^S, v^S; \mathbb{C})$ given by $\phi(g_f, g_\infty) = \nu(g_\infty)^{w_S} j(g_\infty, i)^{-k_S} \cdot g_\infty^{-1} \cdot f(g_f, g_\infty(i))$. This function is $G(\mathbb{Q})$-invariant and C_∞-equivariant: It is easy to see that the functional equation $f(\gamma_f g_f u, \gamma(z)) = \nu(\gamma)^{-w_S} j_S(\gamma, z)^{k_S} \gamma^S \cdot f(g_f, z)$ becomes $\phi(guc_\infty) = \nu(c_\infty)^{w_S} j_S(c_\infty, i)^{-k_S} c_\infty^{-1} \cdot \phi(g)$. Note that the domain of definition of ϕ is $G(\mathbb{Q})^+ \backslash ((G_f/U) \times G_\infty^+) = G(\mathbb{Q}) \backslash G(\mathbb{A})/U$.

One can use this trick to define p-adic integral structures on the space $S_{k,w}(U)$. The model for this is the theory of (A_0)-type Hecke characters of A. Weil ([1955c], Collected Works vol.II), Let us recall its gist.

Let $k \in \mathbb{Z}[J_F]$ and let $f : F_\mathbb{A}^\times / UF_\infty^+ \to \mathbb{C}^\times$ be a continuous homomorphism such that $f(\alpha x) = \alpha^k f(x)$ for all$\alpha \in F^\times$. Because of the following exercise, we call f a **classical avatar** of an (A_0)-type Hecke character.

Exercise: Such an f induces a classical Hecke character of weight k. That is, there exists a number field E such that f factors into a homomorphism from the group of non-zero fractional ideals prime to $Ram(U)$ to $E^\times$ such that if $\mathfrak{a} = (\alpha)$ with $\alpha \in F^\times \cap UF_\infty^\times$, one has $f(\mathfrak{a}) = \alpha^k$.

We define now the **complex avatar** ϕ_∞ of f is defined by $\phi_\infty(x) = x_\infty^{-k} f(x)$. It is a continuous function $\phi_\infty : F^\times \backslash F_\mathbb{A}^\times / U \to \mathbb{C}$ such that $\phi_\infty(xy_\infty) = y_\infty^{-k} \phi_\infty(x)$ for all $y_\infty \in F_\infty^+$.

Finally, let $U = U_p \times U^p$; the **p-adic avatar** ϕ_p of f is defined by $\phi_p(x) = x_p^{-k} f(x) = x_p^k x_\infty^k \phi_\infty(x)$. It is a continuous function $\phi_p : F^\times \backslash F_\mathbb{A}^\times / U^p F_\infty^+ \to \overline{\mathbb{Q}}_p$ such that $\phi_p(xu_p) = u_p^{-k} \phi_p(x)$ for all $u_p \in U_p$.

Exercise: Show that indeed ϕ_p takes values in the p-adic field completion of E in $\overline{\mathbb{Q}}_p$.

It is automatically integral valued because by continuity it factors through the separated quotient of $F^\times \backslash F_\mathbb{A}^\times / U^p F_\infty^+$ which is a compact group (it is isomorphic to $\mathrm{Gal}(F_\infty/F)$, where F_∞ is the maximal abelian extension of F unramified outside $p\infty$); thus its image is contained in $\overline{\mathbb{Z}}_p^\times$.

The analogy for G instead of GL_1 is elementary if $S = \emptyset$ (the Hida set case). If $S \neq \emptyset$ and $B \neq M_2(F)$, this construction involves Deligne-Carayol theory of (integral) strange models which are moduli schemes of PEL type. For their definition in the Shimura curve case, see Carayol [3] and Kassaei [9] for instance. Since it is enough in the sequel, let us focus on the easy case $S = \emptyset$.

Explicit description when $s = 0$: Good references for this situation are [7] Sect.1 and [6]. Let K be an extension of F which splits B and contains a Galois closure of F. For any K-algebra L, let us consider

$$\widetilde{S}_{k,w}(U; L) = \{f : G_f/U \to L(n, v; L); f(\gamma g_f) = \gamma \cdot f(g_f)\}$$

or equivalently, if L is a $\mathbb{C}$-algebra:

$$\widetilde{S}_{k,w}(U; L) = \{\phi_\infty : G(\mathbb{Q})\backslash G_{\mathbb{A}}/U \to L(n, v; L); \phi_\infty(gc_\infty) = c_\infty^{-1} \cdot \phi_\infty(g)\}$$

The bijection $f \mapsto \phi_\infty$ is given by $\phi_\infty(g_f, g_\infty) = g_\infty^{-1} \cdot f(g)$.

or else, if $\mathfrak{p}$ is a place of K above p and L is a $K_{\mathfrak{p}}$-algebra:

$$\widetilde{S}_{k,w}(U; L) = \{\phi_p : G(\mathbb{Q})\backslash G_f/U^p \to L(n, v; L); \phi_p(gu_p) = u_p^{-1} \cdot \phi_p(g)\}$$

The bijection $f \mapsto \phi_p$ is given by $\phi_p(g^p, g_p) = g_p^{-1} \cdot f(g)$.

If $k \neq 2t$ (so $w \neq t$), one defines the space of cusp forms as $S_{k,w}(U; L) = \widetilde{S}_{k,w}(U; L)$.

If $k = 2t$ and $w = t$, there is an element in $\widetilde{S}_{2t,t}(U)$ which contributes to the discrete series but not to the cuspidal spectrum, namely $\nu^t = N_{F/\mathbb{Q}} \circ \nu$; this is why one defines the space of cusp forms as $S_{2t,t}(U; L) = \widetilde{S}_{2t,t}(U; L)/ < \nu^t >$.

If L is a p-adically complete K_v-algebra with valuation ring $\mathcal{O}_L$ it is now easy to define an integral structure on $S_{k,w}(U; L)$; one simply puts

$$S_{k,w}(U; \mathcal{O}_L) = \{\phi_p : G(\mathbb{Q})\backslash G_f/U^p \to L(n, v; \mathcal{O}_L); \phi(gu_p) = u_p^{-1} \cdot \phi_p(g)\},$$

with an obvious modification for $(k, w) = (2t, t)$.

Remark: Note that contrary to the GL_1 case, despite the fact that the domain of the functions ϕ_p is compact, the integrality of their image is not automatic because these functions are not homomorphisms!

The description of the Hecke operators $T_{\mathfrak{q}}$ ($\mathfrak{q} \notin \mathrm{Ram}(B)$ and prime to $lev(U)$) acting on the space of cusp forms can be given in the two avatars of

$S_{k,w}(U;L)$ as follows. Let (x_i) be a finite set of representatives of $G_{\mathbb{Q}}\backslash G_f/U$, in the first description, we put $(f|T_q)(g) = \sum_i f(gx_i)$, and in the second, we put $(\phi_p|T_{\mathfrak{q}})(g) = \sum_i x'_{i,p} \cdot \phi_p(gx_i)$, where for $x \in G_f$, $x_p \in G_p = G(\mathbb{Q}_p)$ denotes the p-component of x. We choose the main involution instead of the inverse because thus, the integral form $S_{k,w}(U;\mathcal{O}_L)$ is preserved by all Hecke operators, including $T_{\mathfrak{p}}$ for $\mathfrak{p}$ dividing p, if B does not ramify above p and $p \notin Ram(U)$. So the two definitions differ slightly, only for the operators $T_{\mathfrak{p}}$ for $\mathfrak{p}$ dividing p; if we need those operators, for our purpose, the correct definition is the second, which amounts to a twist of the first by $(\det g_p)^{n+2v}$.

The group $F_f^\times$ of finite ideles acts continuously on $S_{k,w}(U;\mathcal{O}_L)$ by $(z \cdot \phi_p)(g) = \phi_p(gz)$. This action factors through the p-adic group $Cl_{U^p} = F_f^\times/\overline{F^\times(U \cap F_f^\times)^p}$. Moreover, this action is locally algebraic: on the image of $U_p \cap F_p^\times$, it is given by $z_p \mapsto z_p^{mt}$. The set of characters $\psi : Cl_{U^p} \to \mathcal{O}_L^\times$ of this type is a torsor under the group $\mathrm{Hom}(Cl_U, \mathcal{O}_L^\times)$ of characters of the finite class-group $Cl_U = F_f^\times/F^\times(U \cap F_f^\times)$. Since the image of $U_p \cap F_p^\times$ has finite index in Cl_{U^p}, this torsor is trivialized by a finite extension of L; for such an extension L, if we fix a point ψ_0 of this torsor, any character ψ can be written $\psi = \psi_\theta = \psi_0\theta$ where θ runs over the set of characters of Cl_U. For any prime ideal $\mathfrak{q}$ of F, $\mathfrak{q} \notin Ram(B) \cup Ram(U)$, we define $S_{\mathfrak{q}}$ by The operator $S_{\mathfrak{q}}$ acts as $f|S_{\mathfrak{q}}(g) = N\mathfrak{q}^m \cdot f(gz_{\mathfrak{q}})$ where $z_{\mathfrak{q}} \in Cl_{U^p}$ is the idele whose components are one at all places except at $\mathfrak{q}$ where it is $\varpi_{\mathfrak{q}}$.

Definition 3.1 *An eigenform $f \in S_{k,w}(U;\mathcal{O}_L)$ is a form which is eigenvector for all the $T_{\mathfrak{q}}$ and the $S_{\mathfrak{q}}$. We write $f|S_{\mathfrak{q}} = \psi_\theta(\mathfrak{q})f$, where θ is a finite order character of the ideal-class group Cl_U.*

3.3 The situation in [12]:

Let p be an odd prime. Let F, B, R as before. We assume that the quaternion algebra B splits at p and we fix $R_p = R \otimes \mathbb{Z}_p \cong M_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$. Let $L \subset \overline{\mathbb{Q}}_p$ be a p-adic field containing a Galois closure of F. We fix a level group $U = U^p \times U_p$ such that $U_p = R_p^\times$ (U is unramified at p) and $U_v = R_v^\times$ for any finite $v \in \mathrm{Ram}(B)$. Fix (k,w) such that $n_\tau := k_\tau - 2 \geq 0$ for all τ's, and $v := w - k + t \geq 0$ is such that $v_\tau = 0$ for one τ and $n + 2v = mt$. For any $\mathfrak{p}$ dividing p in F, let $W_{\mathfrak{p}} = \bigotimes_{\tau:F\to L\ \mathfrak{p}-adic} L(n_\tau, v_\tau; \mathcal{O}_L)$. We denote by $\sigma_{\mathfrak{p}}$ the representation of the $\mathfrak{p}$-component $U_{\mathfrak{p}}$ of U_p on $W_{\mathfrak{p}}$. The action of $U_p = \prod_{\mathfrak{p}} U_{\mathfrak{p}}$ on $W = \bigotimes_{\mathfrak{p}} W_{\mathfrak{p}} = L(n, v; \mathcal{O}_L)$ is denoted by $\sigma = \bigotimes_{\mathfrak{p}} \sigma_{\mathfrak{p}}$. We extend this action to U through the projection $U \to U_p$. Let $Cl_U = F^\times\backslash F_f^\times/(U \cap F_f^\times)$ be the finite class group "of level U". The set of continous homomorphisms

$\psi : F_f^\times/F^\times \to \mathcal{O}_L^\times$ such that $\psi(z) = z_p^{mt}$ for $z \in U \cap F_f^\times$ is a torsor under the finite group $\mathrm{Hom}(Cl_U, \mathcal{O}_L^\times)$. As explained above, we can write $\psi = \psi_\theta$ for $\theta : Cl_U \to \mathcal{O}_L^\times$. We can extend σ to $U \cdot F_f^\times$ by $\sigma(uz) = \sigma(u_p)\psi^{-1}(z)$. This is well defined because if $z \in U$, its p-component $z_p \in U_p$ acts on W by $\sigma(z_p) = z_p^{-(n+2v)} = z_p^{-mt} = \psi^{-1}(z)$. Of course, σ factors through $(U \cdot F_f^\times)/F^\times$. Let

$$S_{\sigma,\psi}(U;\mathcal{O}_L) = \{\phi_p : B^\times\backslash B_f^\times \to \mathcal{O}_L; \phi_p(guz) = \psi(z)\sigma^{-1}(u)\phi_p(g)\}.$$

In the description in terms of $f(g)$ instead of $\phi_p(g)$, it corresponds to forms f with central character θ: $f(gz) = \theta(z)f(g)$. Indeed, we have

$$S_{k,w}(U;\mathcal{O}_L) = \bigoplus_{\theta: Cl_U \to \mathcal{O}_L^\times} S_{\sigma,\psi_\theta}(U;\mathcal{O}_L)$$

Recall that $X_U = B^\times\backslash B_f^\times/(U \cdot F_f^\times)$ is finite by Fujisaki's theorem. Let $(b_i)_{i\in I}$ with $b_i \in B_f^\times$, be a set of representatives of this set: $X_U = \{\overline{b}_i; i \in I\}$. The stabilizer in $U \cdot F_f^\times$ of $B^\times b_i \in B^\times\backslash B_f^\times$ is $UF_f^\times \cap b_i^{-1}B^\times b_i$. For the action of $UF_f^\times$ on W by $uz_f \cdot w = \psi(z)^{-1} \cdot \sigma(u_p)w$, the choice of representatives above provides another description:

$$S_{\sigma,\psi}(U;\mathcal{O}_L) \cong \bigoplus_{i\in I} W_\sigma^{(U\cdot F_f^\times \cap b_i^{-1}B^\times b_i)/F^\times}, \quad \phi_p \mapsto (\phi_p(b_i))_i.$$

Indeed, for any i, and for any $x \in UF_f^\times \cap b_i^{-1}B^\times b_i$, we have $x \cdot \phi_p(b_i) = \phi_p(b_i)$ because $x = uz_f = b_i^{-1}\beta b_i$, so $x \cdot \phi_p(b_i) = \psi(z_f)^{-1}\sigma(u_p)\phi_p(b_i) = \phi_p(b_i u^{-1} z_f^{-1}) = phi_p(b_i b_i^{-1}\beta^{-1}b_i) = \phi_p(b_i)$. The map is clearly injective. To show the surjectivity, given a system of vectors (v_i), one defines $\phi_p(\beta b_i uz)$ by $\sigma^{-1}(uz)(v_i)$. One checks it is well-defined because v_i is invariant by $U \cdot F_f^\times \cap b_i^{-1}B^\times b_i$.

Definition 3.2 *We say that U is ad-neat if for any $b \in G_f$ we have $b^{-1}B^\times b \cap U \cdot F_f^\times = F^\times$.*

Definition 3.3 *The "strict Iwahori subgroup" U_v of $GL_2(\mathcal{O}_{F_v})$ consists of the matrices congruent to $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ modulo $\mathfrak{p}_v$.*

Lemma 3.4 *The principal level group $U(N)$ (with $N \geq 3$ prime to Σ) is ad-neat. If v is unramified in $F/\mathbb{Q}$ and the residual characteristic p_v is > 4, if U_v is the strict Iwahori at v, the group $U = U^v \times U_v$ is also ad-neat.*

Proof: Recall that G/Z is discrete in G_f/Z_f because B is totally definite, hence G_∞/Z_∞ is compact. Consider the adjoint representation $Ad : G \to SL_3$, of kernel Z. Let L be a finite extension of F which splits B and in which v remains unramified (it does exist since p splits B). For any U, $Ad(UF_f^\times b^{-1}B^\times b)/F^\times)$ is a finite subgroup of $\mathcal{U} \cap h^{-1}SL_3(L)h$, where $\mathcal{U} = \mathcal{U}^v \times \mathcal{U}_v$ is compact open in $SL_3(L_f)$, either of principal congruence type (still level N), or of strict Iwahori type:

$\mathcal{U}_v = \{u \in SL_3(\mathcal{O}_{L_v}); u \equiv \begin{pmatrix} * & * & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix} \text{mod.}\mathfrak{p}_v\}$. The congruence subgroup $h\mathcal{U}h^{-1} \cap SL_3(L)$ of $SL_3(L)$ can be checked to be torsion-free for v unramified such that $p_v > 4$. Hence, $Ad(UF_f^\times b^{-1}B^\times b)/F^\times)$ is trivial, and so is $UF_f^\times b^{-1}B^\times b)/F^\times$.

Corollary 3.5 *For U ad-neat as above, $S_{\sigma,\psi}(U;\mathcal{O}_L)$ is a finite projective $\mathcal{O}_L$-module and $W_\sigma \mapsto S_{\sigma,\psi}(U;\mathcal{O}_L)$ is an exact functor.*

We have $S_{\sigma,\psi}(U;\mathcal{O}_L) = \bigoplus_{x\in X_U} W_\sigma$, hence the statement is obvious.

3.4 Hecke algebras

Let B be a quaternion algebra; from now on, we denote by $Ram(B) = Ram_f(B) \sqcup Ram_\infty(B)$ the decomposition previously written as $Ram(B) = \Delta \sqcup S$, because we reserve S for a finite set of finite places of F. Let us fix such a finite set S of finite places containing $\mathrm{Ram}_f(B)$. Let L be a number field containing the Galois closure of F and splitting B. The universal Hecke algebra outside S is defined as the polynomial algebra $\mathbb{T}^{S,univ} = \mathcal{O}_L[T_\mathfrak{q}, S_\mathfrak{q}]_{\mathfrak{q}\notin S}$. Its representation on $S_{k,w}(U,L)$ defines a finite flat $\mathcal{O}_L$-algebra which we denote $\mathbb{T}_{k,w}^{S,B}(U;\mathcal{O}_L)$. The finiteness and flatness is obvious if B is totally definite. In the Hilbert case, it follows from the arithmetic geometric arguments sketched above. In general, it follows for instance from the Eichler-Shimura-Harder isomorphism (which brings back the question to that of the finite rank of the integral Betti cohomology of Sh_U, which is true by finite triangulation of the Borel-Serre compactification).

Another important tool is the Jacquet-Langlands correspondence, which allows to draw conclusions on Hilbert modular eigenforms, from a study of quaternionic eigenforms for a totally definite quaternion algebra.

Let $k \in \mathbb{Z}[J_F]$ a weight with $k \geq 2t$, $k_\sigma \equiv k_\tau \pmod 2$ for all σ, τ's in J_F. Let $v \in \mathbb{N}[J_F]$ such that $k + 2v$ is diagonal, and $w = k + v - t$.

Theorem 1 *(Jacquet-Langlands correspondence) Let B and B' two quaternion algebras such that $\mathrm{Ram}_f(B) = \mathrm{Ram}(B')_f \sqcup T$, T consisting of finite places. Let R be a maximal order in B. Let $U \subset B_f^\times$ be a level group such that $U_v = R_v^\times$ for every $v \in T$ and let $U' = U^T \times \prod_{v\in T} I_v$, where $I_v \subset GL_2(F_v)$ is the Iwahori subgroup. Let S be a finite set of places of F containing $\mathrm{Ram}(B)$, and $\mathbb{T}^{S,B}_{k,w}(U;\mathcal{O}_L)$, resp. $\mathbb{T}^{S,B'}_{k,w}(U;\mathcal{O}_L)$ be the Hecke algebra outside S acting on $S_{k,w}(B,U;L)$, resp. $S_{k,w}(B',U';L)$. There is a surjective $\mathcal{O}_L$-algebra homomorphism $\mathbb{T}^{S,B'}(U';\mathcal{O}_L) \to \mathbb{T}^{S,B}(U;\mathcal{O}_L)$ which sends the operators $T_\mathfrak{q}, S_\mathfrak{q}$ on $T_\mathfrak{q}, S_\mathfrak{q}$ and which induces an isomorphism $\mathbb{T}^{S,B'}(U';\mathcal{O}_L)^{T-new} \cong \mathbb{T}^{S,B}(U;\mathcal{O}_L)$.*

Comments: 1) Actually, if U_v is not maximal at $v \in T$, there is still a $U'_v \subset I_v$ with a similar statement: $\mathbb{T}^{S,B'}(U';\mathcal{O}_L)^{U'_T-sd} \cong \mathbb{T}^{S,B}(U;\mathcal{O}_L)$, where $U'_T - sd$ refers to the largest quotient associated to automorphic representations whose components π_v at $v \in T$ are in the discrete series and have U'_v-fixed vectors.

2) Note also that no condition is imposed for the archimedean places at which B or B' ramify. The condition on the weights make automatic the existence of a correspondence from the split to the ramified case for those places.

The situation in [12]: Let F be totally real in which p does not ramifiy. Actually, it will be necessary later to assume that p splits completely in F (for computations of local deformation rings). Let B be totally definite and Σ the set of finite places in $\mathrm{Ram}(B)$. Let $\mathcal{O} = \mathcal{O}_{L\mathfrak{p}}$ be the ring of integers in a p-adic field. Let $S = \Sigma \sqcup S' \sqcup \{v|p\}$ be a finite set of finite places in F. We assume that for any $v \in S'$, $Nv \not\equiv 1 \pmod p$; later we'll fix an eigenform f of level 1 on B and assume that for all $v \in S'$, the roots x_v, y_v of the Hecke polynomial $P_{v,f}(X)$ are such that $Nv \cdot (x_v + y_v)^2 - x_v y_v (1 + Nv)^2 \not\equiv 0 \pmod \varpi$ (such a set S' is called a set of auxiliary primes). Let $U \subset B_f^\times$ with $U_v = R_v^\times$ for $v \in \Sigma$, $U_v = GL_2(\mathcal{O}_{F_v})$ for $v|p$, and U_v is the strict Iwahori subgroup at $v \in S'$ Note that if S' contains a v with $p_v > 4$, U is ad-neat. Thus $\prod_{v\in S'} \mathfrak{p}_v$ is exactly the level of U; let Σ be the set of finite places of $\mathrm{Ram}(B)$. We define $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ as the quotient of the Hecke algebra $\mathbb{T}^{S,B}_{k,w}(U,\mathcal{O})$ generated by the operators outside S acting on $S_{\sigma,\psi} \subset S_{k,w}(U)$ and we denote by $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O}) = \mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})[X_v; v \in S']$ the subalgebra of $\mathrm{End}(S_{\sigma,\psi}(U;\mathcal{O}))$ generated over $T^S_{\sigma,\psi}(U,\mathcal{O})$ by the Hecke operators $X_{\pi_v} = U\xi_{\pi_v}U$, for all $v \in S'$, where ξ_{π_v} has all its components equal to 1 except the one at v, which is equal to $\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$. The algebras $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ and $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ are commutative, finite flat over $\mathcal{O}$, hence semilocal; moreover, $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ is reduced. The reducedness of $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ is not known, but we will consider only localizations of this ring at certain ideals which reduce the question to $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$.

Recall that since U is ad-neat, we have: $S_{\sigma,\psi}(U,\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F}=S_{\sigma,\psi}(U,\mathbb{F})$.

We have also the following corollary to Jacquet-Langlands and Shimizu correspondence:

Corollary 3.6 *The module $S_{\sigma,\psi}(U,\mathcal{O})$ is generically free of rank one over $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ (ie, it is free after inverting p).*

By the Jacquet-Langlands correspondence, it is enough to show the freeness of the space of Hilbert forms $S^{\Sigma-new}_{k,w}(U,\mathbb{C})$ over $\mathbb{T}^{\Sigma-new}_{k,w}(U,\mathbb{C})$, which follows from the action of Hecke operators on q-expansions, from multiplicity one theorem for Σ-newforms, and by Peterson duality on $S^{\Sigma-new}_{k,w}(U,\mathbb{C})$.

We also introduce two definitions

Definition 3.7 *An Eisenstein ideal $\mathfrak{m}$ is an ideal in $\mathbb{T}^{S,univ}\otimes_{\mathcal{O}_L}\mathcal{O}$, in $\mathbb{T}^{S,B}_{\sigma,\psi}(U,\mathcal{O})$, or in $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$, containing T_v-2 for almost all places v splitting completely in a fixed abelian extension of F. We say that $\mathfrak{m}$ is in the support of (σ,ψ) if $S_{\sigma,\psi}(U;\mathcal{O})_{\mathfrak{m}}\neq 0$.*

Example: For $F=\mathbb{Q}$, $S=\emptyset$ and k an even integer ≥ 4; let $\mathfrak{m}=(p,(T_q-(1+q^{k-1})_q)$ be the maximal ideal of $\mathbb{T}^{\emptyset}=\mathbb{Z}_p[T_q,S_q]_q$. By considering the abelian extension $\mathbb{Q}(\zeta_p)$ of $\mathbb{Q}$, one sees that $\mathfrak{m}$ is Eisenstein. Moreover, $\mathfrak{m}$ defines a maximal ideal of $\mathbb{T}_k^{\emptyset,M_2(\mathbb{Q})}$, or, equivalently, $\mathfrak{m}$ is in the support of $S_k(GL_2(\widehat{\mathbb{Z}}))$, if and only if p divides the Bernoulli number $B_k\in\mathbb{Z}_p$

3.5 Taylor-Wiles primes and Ihara's lemma

Let $S=\Sigma\sqcup S'\sqcup\{v|p\}$ and $U\subset B_f^\times$ be a level group which is strict Iwahori at S' and unramified elsewhere, in such a way that it is ad-neat. Let k,w as before, let $f_1\in S_{\sigma,\psi}(U,\mathcal{O})$ eigen for $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$. Let $\mathfrak{m}'=(\varpi,(T_v-a_v,S_v-\psi(v))_{v\notin S})$ be the maximal ideal of $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$, resp. the maximal ideal $\mathfrak{m}=(\varpi,(T_v-a_v,S_v-\psi(v))_{v\notin S},(X_{\pi_w}-x_w)_{w\in S'})$ of $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$, associated to (f,p).

We assume from now on that $\mathfrak{m}'$ is non-Eisenstein; we also say that $\mathfrak{m}$ is non-Eisenstein. We introduce the fundamental objects:

Definition 3.8 *We put $\mathbb{T}=\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ and $M=S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$.*

Remarks: 1) We could also introduce $\mathbb{T}'=\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ but we'll show later that $\mathbb{T}'=\mathbb{T}$.

2) M is a finite free $\mathcal{O}$-module; it is a faithful $\mathbb{T}$-module.

In order to study these objects, it will be important to introduce "horizontal thickenings" à la Taylor-Wiles.

Definition 3.9 *A Taylor-Wiles set is a finite set Q of primes of F, disjoint of S and such that for all $v \in Q$, $Nv \equiv 1 \pmod{p}$ and the Hecke polynomial $P_{v,f}(X) = X^2 - a_v X + Nv^{k_0-1}\theta(v) = X^2 - a_v X + Nv \cdot \psi(v)$ has roots α_v and β_v in $\mathcal{O}$ and distinct modulo $\varpi\mathcal{O}$.*

By Cebotarev theorem, it is easy to see that there exists infinitely many such, mutually disjoint, sets (even after adding more conditions, as we'll see later).

For any $v \in Q$, let $\mathbb{F}_v = \mathcal{O}_F/\mathfrak{p}_v$ be the residue field, and let $\mathbb{F}_v^\times = \Delta_v \times E_v$ with Δ_v the p-Sylow of $\mathbb{F}_v^\times$ (and E_v the maximal subgroup of order prime to p). Let $S_Q = S \sqcup Q$. Let U_Q^-, resp. U_Q be the level group defined by $U_{Q,v}^- = U_{Q,v} = U_v$ for $v \notin Q$ and

$U_{Q,v}^-$, resp. $U_{Q,v}$, is the Iwahori subgroup of $U_v = GL_2(\mathcal{O}_{F_v})$, resp., the subgroup of the Iwahori subgroup consisting of the elements $g \in U_{Q,v}^-$ whose diagonal modulo $\mathfrak{p}_v$ is of the form (ze, z) with $z \in \mathbb{F}_v^\times$ and $e \in E_v$. Note that the map $U_{Q,v}^- \to \Delta_v$ sending the diagonal (a,d) of g mod.$\mathfrak{p}_v$ to ad^{-1} provides an isomorphism $U_{Q,v}^-/U_{Q,v} \cong \Delta_v$. If one defines $\Delta_Q = \prod_{v\in Q} \Delta_v$, we have simlarly $U_Q^-/U_Q \cong \Delta_Q$ by sending the diagonal (a,d) of g mod. Q to ad^{-1}.

Let $\mathcal{O}[\Delta_Q]$ be the group algebra of Δ_Q and $I_Q = ([\delta]-1)_\delta$ its augmentation ideal. It is local, with maximal ideal $(\varpi) + I_Q$. We denote by $\widetilde{\mathbb{T}}_{\sigma,\psi}(U_Q^-)$ the ring of endomorphisms of $S_{\sigma,\psi}(U_Q^-))$ generated by $\mathbb{T}_{\sigma,\psi}(U_Q^-)$ and the Iwahori level Hecke operators $X_w = U_Q^- \xi_{\pi_w} U_Q^-$ for $w \in Q$, where ξ_{π_w} is defined as in Sect.3.4 (note that X_w does not depend on the choice of a uniformizing parameter π_w of $F_w^\times$). We put a similar definition for $\widetilde{\mathbb{T}}_{\sigma,\psi}(U_Q)$ except that the operators $X_{\pi_w} = U_Q \xi_{\pi_w} U_Q$ do depend on the choice of a uniformizing parameter π_w of $F_w^\times$). Actually this ambiguity gives rise to a structure of $\mathcal{O}[\Delta_Q]$-algebra on $\widetilde{\mathbb{T}}_{\sigma,\psi}(U_Q)$ given by the normal action of U_Q^-: $[\delta]$ acts by $U_Q u_\delta^- U_Q$ for any $u_\delta^- \in U_Q^-$ lifting δ.

In the same manner as we defined the ideal $\mathfrak{m}$ of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$, we define the ideal $\mathfrak{m}_Q$ of $\widetilde{\mathbb{T}}_{\sigma,\psi}(U_Q^-, \mathcal{O})$ and $\widetilde{\mathbb{T}}_{\sigma,\psi}(U_Q, \mathcal{O})$ by $\mathfrak{m}_Q = (\varpi, T_v - a_v, S_v - \psi(v))_{v \notin S} + (X_v - x_v)_{v \in S'} + (X_w - \alpha_w)_{w \in Q}$. It is clear that $\mathfrak{m}_Q \neq (1)$, because from f one can construct a non-zero form on U_Q^- eigen for the X_w's with eigenvalues α_w, namely $f | \prod_{w\in Q}(X_w - \beta_w)$. Recall indeed, that X_w, acting in Iwahori level at w, is annihilated by $X^2 - T_w X + N(w) S_w$. Hence, the f-part of $S_{\sigma,\psi}(U_Q^-, \mathbb{C})$ (which is the $2^{|Q|}$-dimensional subspace spanned by $f(g\eta_I)$ $(I \subset Q$, $\eta_I = \prod_{w \in I} \eta_w$, $\eta_w = \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$, is annihilated by $(X_w - \beta_w)(X_w - \alpha_w)$

for any $w \in Q$.

Lemma 3.10 *1) The trace* $\sum_{\delta\in\Delta_Q}[\delta] : S_{\sigma,\psi}(U_Q,\mathcal{O}) \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})$ *induces an isomorphism*

$$S_{\sigma,\psi}(U_Q,\mathcal{O})/I_{\Delta_Q}S_{\sigma,\psi}(U_Q,\mathcal{O}) \cong S_{\sigma,\psi}(U_Q^-,\mathcal{O}),$$

2) $S_{\sigma,\psi}(U_Q,\mathcal{O})$ *is free of rank one over* $\mathcal{O}[\Delta_Q]$.

Proof: Let $Funct(X_Q,W)$ be the module of W-valued functions on the finite set $X_Q = G\backslash G_f/U_QZ_f$. In Cor.3.4, we have shown that the map $f \mapsto (f(g_x))_{x\in X_Q}$ provides an isomorphism of $\mathcal{O}$-modules $(*)$ $\quad S_{\sigma,\psi}(U_Q,\mathcal{O}) \cong Funct(X_Q,W)$. The action of U_Q^- on X_Q by right translations induces an action of Δ_Q which is free (see details below) and such that $X_Q/\Delta_Q = X_Q^-$. Moreover, it is clear that $(*)$ is Δ_Q-linear.

Therefore, to prove 1), it is enough to notice that $\sum_\delta[\delta]$ induces the augmentation isomorphism $\mathcal{O}[\Delta_Q]/I_Q \cong \mathcal{O}$. For 2), take a set of representatives of X_Q^- in X_Q. It is an $\mathcal{O}[\Delta_Q]$-basis of $Funct(X_Q^-,W)$.

[Details on the freeness of the action : the relation $x\cdot\delta = x$, for $x = \overline{g_f}$, can be translated as $g_fu_\delta^- = \gamma g_fuz$ for $u \in U_Q$ and $z \in Z_f$, we have $\gamma \in G\cap g_fU_Q^-g_f^{-1}Z_f$ which is contained in $F^\times \cap U_Q^-$. Hence γ is a unit $\epsilon \in \mathcal{O}_F^\times$. So, $u = g_f^{-1}\gamma g_fu_\delta^-z_f^{-1} = \epsilon u_\delta^-z^{-1}$ belongs to U_Q.]

The key level lowering lemma (Taylor-Wiles control lemma) takes the following form

Proposition 3.11 *The linear map* $S_{\sigma,\psi}(U,\mathcal{O}) \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})$ *defined by* $g \mapsto g|\prod_{w\in Q}(X_w-\beta_w)$ *defines an isomorphism* $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \cong S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$.

Proof: Recall we assumed $\alpha_w \not\equiv \beta_w \pmod{\varpi}$ and $Nw \equiv 1 \pmod{p}$. It is a classical fact (following easily from the existence of Galois representations and Cebotarev density theorem) that the subalgebra $\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ of $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ generated by the Hecke operators outside S is also generated by the Hecke operators outside a bigger finite set; let's take S_Q; we see that the maximal ideal $\mathfrak{m}' = \mathfrak{m}\cap\mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ is generated by $T_v - a_v$ and $S_v - \psi(v)$ for $v \notin S_Q$ (that is, the ideal does not change by removing the generators for $v \in Q$). On the other hand, let $\mathbb{T}'_{\sigma,\psi}(U_Q^-,\mathcal{O})$ be the subalgebra of $\mathbb{T}_{\sigma,\psi}(U_Q^-,\mathcal{O})$ generated by the Hecke operators outside S_Q and let $\mathfrak{m}'_Q = \mathfrak{m}_Q\cap\mathbb{T}'_{\sigma,\psi}(U_Q^-,\mathcal{O})$; there is a natural surjective homomorphism $\Psi_Q : \mathbb{T}'_{\sigma,\psi}(U_Q^-,\mathcal{O}) \to \mathbb{T}'_{\sigma,\psi}(U,\mathcal{O})$ sending $\mathfrak{m}'_Q$ onto $\mathfrak{m}'$.

We proceed by induction on the cardinality of Q; therefore we assume in the sequel that $Q = \{w\}$. Let $Y_w : S_{\sigma,\psi}(U, \mathcal{O}) \to S_{\sigma,\psi}(U_Q^-, \mathcal{O})$ given by $Y_w\phi(g) = \phi(g \begin{pmatrix} 1 & 0 \\ 0 & \pi_w \end{pmatrix})$. Let us consider the natural homomorphism

$$S_{\sigma,\psi}(U, \mathcal{O})^2 \to S_{\sigma,\psi}(U_Q^-, \mathcal{O}), \quad (\phi_1, \phi_2) \mapsto \phi_1 + Y_w\phi_2$$

Note that it follows from the Jacquet-Langlands correspondence that it is injective over $\mathbb{C}$, hence over $\mathcal{O}$ (q-expansion for instance).

It is compatible with Ψ_Q, hence we can localize at $\mathfrak{m}'$ and $\mathfrak{m}'_Q$:
$S_{\sigma,\psi}(U, \mathcal{O})^2_{\mathfrak{m}'} \to S_{\sigma,\psi}(U_Q^-, \mathcal{O})_{\mathfrak{m}'_Q}$.

Moreover, for $(\phi_1, \phi_2) \in S_{\sigma,\psi}(U, \mathcal{O})^2_{\mathfrak{m}'}$, we have

$$(*) \quad X_w\phi_1 = T_w\phi_1 - Y_w\phi_1, \quad X_wY_w\phi_2 = Nw\psi(w)\phi_2.$$

This shows that the left-hand side is stable by X_w. We can therefore localize at $\mathfrak{m}_Q$, and get
$(S_{\sigma,\psi}(U, \mathcal{O})^2)_{\mathfrak{m}_Q} \to S_{\sigma,\psi}(U_Q^-, \mathcal{O})_{\mathfrak{m}_Q}$.

I. Let us prove that this map is an isomorphism.

From the theory of new forms and from the compatibility between local and global Langlands correspondence (Carayol's theorem), both sides are free of the same rank over $\mathcal{O}$: if a form is eigen for X_w with eigenvalue congruent to α_w mod. ϖ, it cannot be w-new otherwise the corresponding local component π_w would be special at w and we would have by local-global compatibility: $\alpha_w/\beta_w \equiv Nw^{\pm 1}$ mod. ϖ, which is impossible by assumption.

To prove the surjectivity over $\mathcal{O}$, it is enough to show the injectivity modulo ϖ (that is, the cokernel is $\mathcal{O}$-flat). As already noticed, the reduction modulo ϖ is given by the same map $(S_{\sigma,\psi}(U, \mathbb{F})^2)_{\mathfrak{m}_Q} \to S_{\sigma,\psi}(U_Q^-, \mathbb{F})_{\mathfrak{m}_Q}$.

In Lemma 7.5 of [13], Kisin showed that one can prove this injectivity without using duality (which is the main tool in [21] and in [17] to raise the level from a form of level prime to w to a w-new form).

Indeed, let us assume that we have ϕ_1, ϕ_2 such that $(**)\, \phi_1 = Y_w\phi_2$. Apply the trace (of degree $Nw+1$), from level U_Q^- to level U to the relation deduced from (*): $T_w\phi_1 - Y_w\phi_1 = Nw\psi(w)\phi_2$;

We get: $(Nw + 1)T_w\phi_1 - T_w\phi_1 = Nw\psi(w)(Nw + 1)\phi_2$
because $T_w = Tr \circ X_w = Tr \circ Y_w$ in level prime to w.
This yields $NwT_w\phi_1 = Nw\psi(w)(Nw + 1)\phi_2$, or

$$(1)\, T_w\phi_1 = \psi(w)(Nw + 1)\phi_2.$$

Similarly, applying the trace to (**), we get $(2)\, (Nw + 1)\phi_1 = T_w\phi_2$.

So, by multiplying (1) by T_w, we have $(T_w^2 - \psi(w)(Nw+1)^2)\phi_1 = 0$.

Since $Nw \equiv 1 \pmod p$, we have $(T_w^2 - 4\psi(w))\phi_1 = 0$, But $T_w^2 - 4\psi(w) \notin \mathfrak{m}$ because the roots of $X^2 - T_w X + Nw\psi(w)$ are distinct mod.$\mathfrak{m}$. Therefore, $\phi_1 = 0$. Hence, from (1) we also have $\phi_2 = 0$.

II. Let us show the proposition.

It is enough to show that $\phi \mapsto (X_w - \beta_w)\phi$ induces an isomorphism $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \to (S_{\sigma,\psi}(U,\mathcal{O})^2)_{\mathfrak{m}_Q}$. By the formulas $(*)$ above, the endomorphism X_w of the $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$-module $S_{\sigma,\psi}(U,\mathcal{O})^2$ has the matrix

$$\begin{pmatrix} T_w & Nw\psi(w) \\ -1 & 0 \end{pmatrix}$$

By Hensel's lemma, in the completion $\mathbb{T}_{\mathfrak{m}}$ of $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ at $\mathfrak{m}$, the polynomial $X^2 - T_w X + Nw\psi(w)$ has two distinct roots A_w and B_w lifting $\overline{\alpha}_w$ and $\overline{\beta}_w$. Since $A_w - B_w \notin \mathfrak{m}$, it is invertible on $S_{\sigma,\psi}(U,\mathcal{O})^2_{\mathfrak{m}}$, hence one has $1 = (B_w - A_w)^{-1}(X_w - A_w) - (B_w - A_w)^{-1}(X_w - B_w)$ which decomposes this space as the sum of $S_A = (X_w - B_w)(S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}})$ and $S_B = (X_w - A_w)(S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}})$. S_B is annihilated by $X_w - B_w$ which is invertible in $\mathbb{T}_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$, hence $\mathbb{T}_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q} \cdot S_B = 0$. The first is annihilated by the element $X_w - A_w$ of $\mathfrak{m}_Q$, so $\mathbb{T}_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q} \cdot S_A = S_A$. In conclusion, $(X_w - B_w)(S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}) = S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$, as desired. Since $B_w \equiv \beta_w \pmod{\mathfrak{m}}$, the same holds for $X_w - \beta_w$ instead of $X_w - B_w$.

Definition 3.12 *For any Taylor-Wiles set Q, we define the Q-thickenings of $(\mathbb{T}, M)$ as $\mathbb{T}_Q = \mathbb{T}_{\sigma,\psi}(U_Q,\mathcal{O})_{\mathfrak{m}_Q}$ and $M_Q = S_{\sigma,\psi}(U_Q,\mathcal{O})_{\mathfrak{m}_Q}$.*

Corollary 3.13 *M_Q is finite free over $\mathcal{O}[\Delta_Q]$ and $M_Q/I_Q M_Q \cong M$ as $\mathcal{O}$-modules.*

4 Hilbert Moduli varieties

Let $G = \mathrm{Res}^F_{\mathbb{Q}}(GL_2)$ and G^* the subgroup "of matrices with determinant in $\mathbb{Q}^\times$".

4.1 Hilbert-Blumenthal abelian varieties (HBAV)

An HBAV is an abelian scheme $A \to S$ with a ring homomorphim $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ such that $\mathrm{Lie}(A/S)$ is locally free of rank one over $\mathcal{O}_A \otimes \mathcal{O}_F$. Its dual (A^t, ι^t) is also naturally an HBAV. Let $\mathcal{P}(A) = \{\lambda \in \mathrm{Hom}(A, A^t); \lambda = \lambda^t, \iota^t(a) \circ \lambda^t = \lambda \circ \iota(a)\}$ its polarization module. Consider $\mu : A \times_S A \to A$ the addition, and $p_i : A \times_S A \to A$ the two projections. Given a relatively ample sheaf L on A, the sheaf $\psi(L) = \mu^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$ is invertible on $X \times_S X$ and defines a point of $\mathrm{Pic}^0_{\mathrm{X/S}}(X)$. Recall that $\mathrm{Pic}^0_{\mathrm{X/S}}(X) = \mathrm{Hom}_S(X, X^t)$, hence $\psi(L)$ gives rise to $\Lambda(L) : A \to A^t$. A polarisation is an element $\lambda \in \mathcal{P}(X)$ given at each geometric point of s by the construction above: $\lambda_s = \Lambda(L_s)$, (L_s may not globalize over S a priori). Let $\mathcal{P}_+(A)$ be the subset of $\mathcal{P}(A)$ consisting of polarizations. There is an $\mathcal{O}_F$-linear embedding $i : \mathcal{P}(X) \to F$ such that $i(\mathcal{P}_+(A)) = i(\mathcal{P}(A)) \cap F^+$.

4.2 Hilbert-Shimura moduli varietes

Here, $B = M_2(F)$. Let $V \subset G_f^*$ be a neat compact open subgroup. Let $\mathfrak{c}$ be a fractional ideal of F and $g_f \in G_f$ such that $\det(g_f)$ is an idele of $\mathfrak{c}$; since $\mathfrak{c}$ will only come in through its stric ideal class, we can and will assume it is prime to the level M of V.

It is easy to see that the set $Sh^*_{V,\mathfrak{c}} = Sh^*_{g_f V g_f^{-1}}$ is in bijection with the set of isogeny classes of $(A, \iota, \overline{\lambda}, \overline{\alpha})$ where $(A, \iota,)$ is a complex HBAV, $\lambda : A \to A^t$ is a $\mathfrak{c}$-polarisation, (that is, such that $i(\mathcal{P}(A) = \mathfrak{c} \subset F)$ and $\overline{\lambda} = \mathbb{Q}^+ \cdot \lambda$ denotes its class, and $\overline{\alpha}$ is the class modulo V of an isomorphism $\alpha : F_f^2 \cong H_1(A, \mathbb{Q}) \otimes \mathbb{Q}_f$ such that $e_\lambda(\alpha(x), \alpha(y)) = c_\alpha \cdot \mathrm{Tr}_{F/\mathbb{Q}}(x_1 y_2 - x_2 y_1)$ with $c_\alpha \in \mathbb{Q}_f^\times$.

Mumford's Geometric Invariant Theory shows that if V is neat, these complex d-folds are algebraic and actually descend as smooth quasi-projective schemes defined over $\mathbb{Q}$, with finite transition maps defined over $\mathbb{Q}$ and Hecke correspondences defined over $\mathbb{Q}$ as well.

Moreover, the connected components admit integral models which can be described as follows. Let's assume for simplicity that $V \subset GL_2(\widehat{\mathcal{O}}_F)$ and let $M \geq 1$ be the level of V. Let us fix a fractional ideal $\mathfrak{c}$ of F. Consider the following moduli problems over $\mathbb{Z}[\frac{1}{MDN(\mathfrak{c})}, \zeta_M]$ of isomorphism classes

$$\mathcal{M}^1_{V,\mathfrak{c},j} : S \mapsto \{(A, \iota, \lambda, \overline{\alpha_M})_{/S}\}/\sim, \quad \text{resp.} \quad \mathcal{M}_{V,\mathfrak{c},j} : S \mapsto \{(A, \iota, \lambda \circ \mathcal{O}_F^+, \overline{\alpha_M})_{/S}\}/\sim$$

where (A, ι) is an HBAV over S, λ is a $\mathfrak{c}$-polarization (that is, a polarization such that $i(\mathcal{P}(A)) = \mathfrak{c} \subset F$) and $\overline{\alpha_M}$ is the class modulo $V/V(M)$ of an isomorphism $\alpha_M : (\mathcal{O}_F/M)^2 \to A[M]$ such that $e_\lambda(\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \zeta_M^j$

Proposition 4.1 *1) Over $\mathbb{Z}[\frac{1}{MDN(\mathfrak{c})}, \zeta_M]$, $\mathcal{M}^1_{V,\mathfrak{c},j}$ is representable by a fine moduli scheme $M^1_{V,\mathfrak{c},j}$, which is quasi-projective smooth. The problem $\mathcal{M}_{V,\mathfrak{c},j}$ is the quotient of $\mathcal{M}^1_{V,\mathfrak{c},j}$ by a finite group which is a quotient of $\mathcal{O}_F^+$, acting* *by $\epsilon : \lambda \mapsto \lambda \circ \epsilon$; in particular, it admits a coarse moduli scheme $M_{V,\mathfrak{c},j}$ over $\mathbb{Z}[\frac{1}{MDN(\mathfrak{c})}, \zeta_M]$.*

2) Over $\mathbb{Q}(\zeta_M)$, the problem $\mathcal{M}^1_{V,\mathfrak{c},j} \otimes \mathbb{Q}(\zeta_M)$ is isomorphic to the problem of isogeny classes of $(A, \iota, \mathbb{Q}^+\lambda, \overline{\alpha})_{/S}$ (where λ is a $\bar{\mathfrak{c}}$-polarization, where $\bar{\mathfrak{c}}$ is the strict class of $\mathfrak{c}$).

3) The scheme $M^1_{V,\mathfrak{c},j} \otimes \mathbb{Q}(\zeta_M)$ defines the j-component of $Sh^*_{V,\mathfrak{c}} \otimes \mathbb{Q}(\zeta_M)$. The $\mathbb{Q}$-variety $Sh^*_{V,\mathfrak{c}}$ has good reduction outside $MDN(\mathfrak{c})$.

For $n + 2v = mt$, $m \geq 0$ and $n \geq 0$, one can define locally constant étale sheaves $\mathcal{L}(n, v; \mathcal{O}_K/\mathfrak{p}^n))$ over the $\mathbb{Q}$-schemes Sh^*_V.

Let $\pi : A \to Sh^*_{V,\mathfrak{c}}$ be the universal (up to isogeny) abelian variety attached to $Sh^*_{V,\mathfrak{c}}$. Let K be a number field containing the Galois closure of F. Let p be an arbitrary rational prime and $\mathfrak{p}$ a prime in K dividing p. One considers the representation of $GL_2(\mathcal{O}_F/p^n\mathcal{O}_F)$ on $L(n, v; \mathcal{O}_K/p^n\mathcal{O}_K)$ as before, and one takes its $\mathfrak{p}$-part. Let $\overline{s}$ be a geometric point of $Sh^*_{V,\mathfrak{c}}$ and $\pi_1(Sh^*_{V,\mathfrak{c}}, \overline{s}) \to GL_2(\mathcal{O}_F/p^n\mathcal{O}_F)$ the representation associated to $R^1\pi_*\mathbb{Z}/p^n\mathbb{Z}$. The composition of these two representations defines the étale sheaf $\mathcal{L}(n, v; \mathcal{O}_K/p^n\mathcal{O}_K)$.

Now, let U be a subgroup of G_f, $U_i = b_i U b_i^{-1}$ (the b_i's are indexed by strict ideal classes $\bar{\mathfrak{c}}_i$, as in Sect 1), let $V_i = U_i \cap G^*_f$; conjugation by b_i give rise to a finite covering $Sh_{V,\mathfrak{c}_i} = Sh^*_{V_i} \to Sh_U$; these coverings allow to construct by descent an étale sheaf $\mathcal{L}(n, v; \mathcal{O}_K/p^n\mathcal{O}_K)$ on Sh_U. It gives rise to the étale cohomology groups $H^\bullet_?(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n, v; \mathcal{O}_{K_\mathfrak{p}}))$ and $H^\bullet_?(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n, v; K_\mathfrak{p}))$ for $? = c, \emptyset$.

5 Galois representations

5.1 Some history

references for the construction:

- Shimura, Deligne (for $F = \mathbb{Q}$),
- Brylinski-Labesse (representation of degree 2^d)
- Ohta, Carayol (Shimura curves case)
- Hida (Λ-adic version of Shimura-Deligne representations for Hida families)
- Wiles (general p-ordinary case), Taylor (general case), both use approximation by the Shimura curve case.

- Blasius-Rogawski (general case by Langlands functoriality).

For all this, see a survey in [19].

Local behaviour at p:

- In the ordinary case: Deligne (mod.p case, lettre à Serre), Wiles, Hida,
- Faltings (Hodge-Tate property),
- Scholl, Faltings (crystalline), T. Saito (pst),
- Buzzard-Diamond-Jarvis , T. Gee, M. Schein (Serre weights).

The first construction is due to Shimura (in the case $F = \mathbb{Q}$ and weight $k = 2$) and Deligne (when $F = \mathbb{Q}$, $k \geq 2$). For arbitrary F, Brylinski and Labesse [2] constructed the Galois representation of degree 2^d associated to a Hilbert modular form.

5.2 Brylinski-Labesse representation

We keep the notations of the Hilbert-Shimura varieties.

We denote by $H_!^\bullet(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n, v; K_\mathfrak{p}))$ the image of $H_c^\bullet(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n, v; K_\mathfrak{p}))$ in $H^\bullet(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n, v; K_\mathfrak{p}))$. These are Galois and Hecke modules (both actions commute, because the Hecke operators are defined over $\mathbb{Q}$) The $\mathbb{Q}$-action of $T_\mathfrak{q}$ is first defined on $Sh^*_{V_i}$ and descends to Sh_U. On $Sh^*_{V_i}$, it is defined by $\pi_{1*} \circ [\xi] \circ \pi_2^*$ where $[\xi] : \pi_2^*\mathcal{L} \to \pi_1^*\mathcal{L}$ is defined by plethysms from the standard representation of $GL_2(\mathcal{O}_{F,p})$. For this representation, in order to define $[\xi]$ over, say, Sh^*_V, one considers the diagram of universal abelian varieties (with the universal $\mathfrak{q}$-isogeny) over the corresponding Shimura varieties:

$$\begin{array}{ccccccc} & & & (A \xrightarrow{\Pi} A') & & & \\ & & \swarrow & \downarrow & \searrow & & \\ A & & & Sh_{V^0(\xi)} & & & A' \\ \downarrow & \swarrow & & & & \searrow & \downarrow \\ Sh^*_V & & & & & & Sh^*_V \end{array} .$$

Let $H^1(A/Sh^*_V) = H^1_{et}(A/Sh^*_V, \mathcal{O}_{K_\mathfrak{p}})$; by definition, we have $\mathcal{L} = H^1(A'/Sh^*_V)$, hence, $\pi_2^*\mathcal{L} = H^1(A'/Sh^*_{V^0(\xi)})$, $\pi_1^*\mathcal{L} = H^1(A/Sh^*_{V^0(\xi)})$ and $[\xi] = \Pi^*$.

Let f be any cusp eigenform on G of level U and weight $k = n + 2t$, where $n_\tau \geq 0$ for all τ; let $v_\tau \geq 0$ and $v_\tau = 0$ for at least one τ, such that $n + 2v = mt$; let $\theta : F_f^\times / F^\times (F_f^\times \cap U) \to \mathbb{C}^\times$ be (the finite part of) its central character: $\theta(\mathfrak{q})$ is the eigenvalue of $(T_\mathfrak{q})^2 - T_{\mathfrak{q}^2}/N(\mathfrak{q})^m$. Let $k_0 = \max(k_\tau)$. Let $\mathfrak{q}$ be a prime of F unramified for U and $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ be roots of the Hecke polynomial $X^2 - a_\mathfrak{q}X + N(\mathfrak{q})^{k_0-1}\theta(\mathfrak{q})$ for any sufficiently big number field K (at least containing the Galois closure of F and the Hecke eigenvalues of f), for any $\mathfrak{p}$ of K, we have by the trace formula (see [2]):

Proposition 5.1 *There exists a representation $\rho^{BL} : \mathrm{Gal}(\overline{F}/F) \to GL_{2^d}(K_\mathfrak{p})$ which is unramified outside a finite set of primes Σ of F and such that for any rational prime q prime to Σ, and totally split in F, and for S_q the set of prime ideals of F above q, the characteristic polynomial of the geometric Frobenius $\rho^{BL}(Frob_\mathfrak{q})$ has roots $\alpha_{I_1}\beta_{I_2}$ where $S_q = I_1 \sqcup I_2$ runs over all partitions of the set S_q.*

Comments:

1) Actually, ρ^{BL} is Galois and Hecke direct factor in the middle degree étale cohomology group $H^d_!(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n,v;K_\mathfrak{p}))$ (and also in $IH^d(Sh_U \times \overline{\mathbb{Q}}, \mathcal{L}(n,v;K_\mathfrak{p})))$; hence it is unramified outside MDp.

2) Yoshida [23] has defined a notion of tensor induction and shown that ρ^{BL} extends to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and can be written as $\bigotimes -\mathrm{Ind}_{G_\mathbb{Q}}^{G_F} \rho^{BRT}$ where ρ^{BRT} : $G_F = \mathrm{Gal}(\overline{F}/F) \to GL_2(K_\mathfrak{p})$ is the Blasius-Rogawski-Taylor Galois representation associated to a Hilbert eigenform f which will be described below.

5.3 Ohta-Carayol representation

M. Ohta (1982) (and Carayol, 1986) constructed the degree 2 Galois representation associated to a cusp eigenform on a Shimura curve.

Theorem 2 *Let f be cusp eigen on a Shimura curve of level M, of weight (k,w). Then for any sufficiently big number field containing F and the eigenvalues of f, and for any prime $\mathfrak{p}$ of K, there exists a continuous*

$\rho_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{F}/F) \to GL_2(K_\mathfrak{p})$ which is unramified outside Mp and such for $\mathfrak{q}$ prime of F not dividing Mp, the characteristic polynomial of the geometric Frobenius is the Hecke polynomial $X^2 - a_\mathfrak{q}X + N\mathfrak{q}^{k_0-1}\theta(\mathfrak{q})$.

We sketch Ohta's construction when $k = 2t$. Let $X = Sh_{U/F}$ be the Shimura canonical model of the Shimura curve of level group U (say, of level M); it is proper smooth over F, and each of its connected components is geometrically connected. Let K be a number field containing the Galois closure of F and the eigenvalues of f. We consider $H = H^1(X \times \overline{F}, K_\mathfrak{p})$. By Hodge theory, we have a decomposition $H \otimes_{K_\mathfrak{p}} \mathbb{C} = S_{2t,t}(U) \oplus \overline{S_{2t,t}(U)}$ which is

$\mathbb{T}_{2t,t}(U,K)$-linear. Assume that f is new of level U. Since the eigensystem of f occurs with multiplicity one in $S_{2t,t}(U)$, the idempotent 1_f of $\mathbb{T}_{2t,t}(U,K)$ associated to f cuts a vector space $V_f = 1_f \cdot H$ which is two-dimensional over $K_{\mathfrak{p}}$. It is endowed with a $K_{\mathfrak{p}}$-linear action of $\mathrm{Gal}(\overline{F}/F)$. By proper smooth base change, it is unramified outside Mp. For $\mathfrak{q}$ prime not dividing Mp, the Hecke correspondence $T_{\mathfrak{q}}$ is given by $\pi_i : X^0(\mathfrak{q}) \to X$, $i = 1, 2$. Assuming U is neat, X has a smooth model over $\mathcal{O}_{F_{\mathfrak{q}}}$ and $X^0(\mathfrak{q})$ has a semistable model over $\mathcal{O}_{F_{\mathfrak{q}}}$ with two irreducible components X^e and X^m with $\pi_1 : X^m \cong X$ and $\pi_1 : X^e \to X$ purely inseparable of degree $N\mathfrak{q}$, while the inverse holds for π_2. Let $P_{f,\mathfrak{q}}(X) = X^2 - a_{\mathfrak{q}}X + N\mathfrak{q}^{k_0-1}\theta(\mathfrak{q})$ be the Hecke polynomial of f at $\mathfrak{q}$. It is the image of the universal Hecke polynomial $X^2 - T_{\mathfrak{q}}X + N\mathfrak{q}S_{\mathfrak{q}}$ by 1_f. Ohta's original proof uses Shimura's theory of special points in order to prove the Eichler-Shimura's relation $P_{f,\mathfrak{q}}(\rho^{OC}(Frob_{\mathfrak{q}})) = 0$. Today, a more geometric proof makes use of Deligne-Carayol strange models and of the Rapoport-Zink local model. Then, using Poincaré duality, one can deduce that the characteristic polynomial of the geometric Frob at $\mathfrak{q}$ on V_f is exactly $P_{f,\mathfrak{q}}(X)$.

5.4 R. Taylor's construction

Let $f \in S_{k,w}(U)$, $k \geq 2t$, and w as above, a Hilbert cusp eigenform. There is a tensor root for Brylinski-Labesse Galois representation. But although Brylinski-Labesse representation is geometric (de Rham with explicit Hodge-Tate weights), Taylor's construction [17] does not provide this information for ρ_{f,ι_p}. However, Blasius-Rogawski's construction [1] does provide this information (after extension to an auxiliary imaginary quadratic field, though).

Theorem 3 *Let f Hilbert cusp eigen of weight (k,w) of level M. Then for any sufficiently big number field containing F and the eigenvalues of f, and for any prime $\mathfrak{p}$ of K, there exists a continuous*

$\rho_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{F}/F) \to GL_2(K_{\mathfrak{p}})$ *which is unramified outside Mp and such for $\mathfrak{q}$ prime of F not dividing Mp, the characteristic polynomial of the geometric Frobenius is the Hecke polynomial $X^2 - a_{\mathfrak{q}}X + N\mathfrak{q}^{k_0-1}\theta(\mathfrak{q})$.*

Remark: If $f|S_v = \psi(v)f$, for all v prime to M, we have $\det \rho_{f,\mathfrak{p}} = \psi^{gal}\chi_{cyc}$, where ψ^{gal} is characterized by $\psi^{gal}(Fr_v) = \psi(v)$ (here Fr_v is the geometric Frobenius). Actually, we'll write ψ instead of ψ^{gal}.

Sketch of proof: If F has odd degree or if f is special at some finite place, one applies Jacquet-Langlands to transfer the Hecke eigensystem to a quaternion algebra which is definite at all but one place, and apply Ohta-Carayol.

If not, Taylor's method consists in constructing congruences mod. p^n (n arbitrarily large) with newforms which are special at some place $\mathfrak{q}_n$. Then, the method of pseudo-representations allows to construct, by p-adic limit, a pseudo-representation which gives rise to the desired representation.

For this, one assumes that the degree of F is even, go to a totally ramified quaternion algebra by JL, and consider again the map

$$S_{k,w}(U,\mathcal{O})^2 \to S_{k,w}(U_w^-,\mathcal{O}), \quad (\phi_1,\phi_2) \mapsto \phi_1 + Y_w\phi_2$$

Actually we decompose

$$S_{k,w}(U_w^-,L) = S_{k,w}(U_w^-,L)^{w-old} \oplus S_{k,w}(U_w^-,L)^{w-new}$$

hence for the Hecke algebras (outside Mw), the restrictions give rise to an injective ring homomorphism:

$$\mathbb{T}(U_w^-,\mathcal{O}) \to \mathbb{T}(U_w^-,\mathcal{O})^{w-old} \times \mathbb{T}(U_w^-,\mathcal{O})^{w-new}$$

which becomes an isomorphism after $\otimes_{\mathcal{O}} L$:

$$\mathbb{T}(U_w^-,L) \cong \mathbb{T}(U_w^-,L)^{w-old} \times \mathbb{T}(U_w^-,L)^{w-new}.$$

Note that $\mathbb{T}(U_w^-,\mathcal{O})^{w-old}$ coincides with the Hecke algebra outside Mw of level U: $\mathbb{T}(U,\mathcal{O})$. Let $\theta_f : \mathbb{T}(U,\mathcal{O}) \to \mathcal{O}$ be the character given by the eigenform f. Let

$$\mathfrak{c}^w = \mathbb{T}(U_w^-,\mathcal{O}) \cap (\mathbb{T}(U_w^-,L)^{w-old} \times \{0\}), \quad \mathfrak{d}^w = \mathbb{T}(U_w^-,\mathcal{O}) \cap (\{0\} \times \mathbb{T}(U_w^-,L)^{w-new}),$$

Note that the inclusions

$$\mathbb{T}(U_w^-,\mathcal{O})^{w-old} \subset \mathbb{T}(U_w^-,\mathcal{O})^{w-old} \times \mathbb{T}(U_w^-,\mathcal{O})^{w-new} \supset \mathbb{T}(U_w^-,\mathcal{O})^{w-new}$$

induce

$$\frac{\mathbb{T}(U_w^-,\mathcal{O})^{w-old} \times \mathbb{T}(U_w^-,\mathcal{O})^{w-new}}{\mathbb{T}(U_w^-,\mathcal{O})} \cong \frac{\mathbb{T}(U_w^-,\mathcal{O})^{w-old}}{\mathfrak{c}^w} \cong \frac{\mathbb{T}(U_w^-,\mathcal{O})^{w-new}}{\mathfrak{d}^w}.$$

We view $\mathfrak{c}^w$ as an ideal of $\mathbb{T}(U_w^-,\mathcal{O})^{w-old}$ and we form $\eta_f^w = \theta_f(\mathfrak{c}^w)$; it is an ideal in $\mathcal{O}$. The character θ_f defines a character $\mathbb{T}(U_w^-,\mathcal{O})^{w-old}/\mathfrak{c}^w \to \mathcal{O}/\eta_f^w$; this right-hand-side is the largest quotient of $\mathcal{O}$ such that the character $\mathbb{T}(U_w^-,\mathcal{O}) \to \mathcal{O}$ defined by θ_f via the first projection factors through $\mathbb{T}(U_w^-,\mathcal{O})^{w-new}$ (by the second projection). It follows (?) from the Jacquet-Langlands isomorphism (or Lemma 4 of [17]), that

Lemma 5.2 *there is a finite set T_f of primes of F depending on f but not on w, such that if $w \notin T_f$, we have $\eta_f^w = \{x \in \mathcal{O}; xf \in S_{k,w}(U_w^-, \mathcal{O})\}$ (that is, η_f is the denominator of $\mathcal{O} \cdot f$ in $L \cdot f \cap S_{k,w}(U_w^-, \mathcal{O})$.*

Instead of copying Taylor's proof, let us explain why this statement is easy in the Hilbert modular case, (where we can normalize f by its first Fourier coefficient being one). In that case, q-expansion provides a perfect duality between Hilbert forms and their Hecke algebras: in this proof (only), we consider $\mathbb{T} = \mathbb{T}(U_w^-, \mathcal{O})^{all}$ (and its quotients $\mathbb{T}^{w-old}$ and $\mathbb{T}^{w-new}$) as the algebra generated by the Hecke operators at ALL places (and we assume, as before, that f is a NEWform of level U); we have a perfect $\mathbb{T}$-bilinear pairing $\mathbb{T}(\mathcal{O}) \times S(\mathcal{O}) \to \mathcal{O}$. In particular, we have the commutative diagram

$$\begin{array}{ccc} S(\mathcal{O}) \cap S^{w-old}(L) \oplus S(\mathcal{O}) \cap S^{w-new}(L) & \subset & S(\mathcal{O}) \\ || & & || \\ \mathrm{Hom}(\mathbb{T}^{w-old}, \mathcal{O}) \oplus \mathrm{Hom}(\mathbb{T}^{w-new}, \mathcal{O}) & \subset & \mathrm{Hom}(\mathbb{T}, \mathcal{O}) \end{array}$$

Moreover, if we put $S(\mathcal{O})^{w-old} = \mathbb{T}^{w-old} \cdot S(\mathcal{O})$, $\mathfrak{c}^w = \mathbb{T} \cap (\mathbb{T}^{w-old} \times \{0\})$ and $S(\mathcal{O})^{w-new} = \mathbb{T}^{w-new} \cdot S(\mathcal{O})$, we also the commutative diagram

$$\begin{array}{ccc} S(\mathcal{O}) & \subset & S(\mathcal{O})^{w-old} \oplus S^{w-new}(\mathcal{O}) \\ || & & || \\ \mathrm{Hom}(\mathbb{T}, \mathcal{O}) & \subset & \mathrm{Hom}(\mathfrak{c}^w, \mathcal{O}) \oplus \mathrm{Hom}(\mathfrak{d}^w, \mathcal{O}) \end{array}$$

Hence the inclusion $S(\mathcal{O}) \cap S^{w-old}(L) \subset S(\mathcal{O})^{w-old}$ can be rewritten as

$$(2) \quad \mathrm{Hom}(\mathbb{T}^{w-old}, \mathcal{O}) \subset \mathrm{Hom}(\mathfrak{c}^w, \mathcal{O}).$$

Let $1_f \in \mathbb{T}^{w-old} \otimes_{\mathcal{O}} L$ be the projector which gives a section to $\theta_f \otimes Id_L$. We see from (2) that the inclusion $Lf \cap S(\mathcal{O}) \subset \mathcal{O} \cdot f$ can be rewritten as

$$\mathcal{O} \cdot \theta_f \subset 1_f \cdot \mathrm{Hom}(\mathfrak{c}^w, \mathcal{O}) = (\eta_f^w)^{-1}.\theta_f \quad QED.$$

The next lemma is

Lemma 5.3 $\theta_f(T_w^2 - S_w(Nw+1)^2)) \cdot f \in S_{k,w}(U^w \mathcal{O})$

The proof for this lemma is by the calculations of [17], end of Sect.1, or the calculations $(*)$ (due to Kisin) in the proof of Prop.3.8 above [this point to be checked as Exercise].

Lemmas 5.2 and 5.3 obviously imply

Corollary 5.4 *There is a finite set T_f of primes of F such that if $w \notin T_f$ and $Nw \not\equiv -1 \pmod p$,*

$$ord_\varpi(\eta_f^w) \geq ord_\varpi(\theta_f(T_w^2 - S_w(Nw+1)^2)).$$

On the other hand, it is easy to show, using Cebotarev theorem for the Brylinski-Labesse representation mod. ϖ^n, that for any n, there exists w such that ϖ^n divides $\theta_f(T_w^2 - S_w(Nw+1)^2)$. Indeed for a form f with roots α_w and β_w for the Hecke polynomial $P_{f,w}(X)$ at w, we can rewrite this quantity as $(Nw(\alpha_w+\beta_w)^2 - \theta_f(S_w)(1+Nw)^2) = \theta_f(S_w)(Nw(1+\frac{\alpha_w}{\beta_w})(1+\frac{\beta_w}{\alpha_w}) - (1+Nw)^2)$. If we consider w of degree one over F, prime to D_f and totally decomposed in the extension of F compositum of the field of $\mathrm{Ker}\,\rho^{BL} \mathrm{mod.}\,\varpi^n$ and $F(\zeta_{p^n})$, the eigenvalues of $\rho^{BL}(Frob_w)$ are congruent to 1, hence also their quotients, among which is α_w/β_w; hence $\theta_f(T_w^2 - S_w(Nw+1)^2) \equiv \theta(S_w)(2^2-2^2) \equiv 0 \bmod \varpi^n$.

From this remark, we find a sequence of places w_i and of w_i-new eigenforms f_i which are congruent mod. ϖ^i to f. This gives a Cauchy sequence of pseudorepresentations with the right characteristic polynomial. By taking the limit, we obtain for f itself a pseudo-representation which is unramified outside Mp, with the correct characteristic polynomial. It gives rise to the desired representation.

To state the Hodge-Tate property, we make the simplifying assumption that p splits totally in F. Recall that we have fixed a p-adic embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Hence if p splits totally in F, given a prime $\mathfrak{q}$ in F above p, there exists a unique $\tau_\mathfrak{q} \in J_F$ such that $\mathfrak{q} = \{x \in F; |\tau_\mathfrak{q}(x)|_\mathfrak{q} < 1\}$. For $v = \sum_{\tau \in J_F} v_\tau \tau \in \mathbb{Z}[J_F]$, let us put $v_\mathfrak{q} = v_{\tau_\mathfrak{q}}$.

Theorem 4 *If f is a Hilbert eigenform of weight (k, w), the representation $\rho_{f,\mathfrak{p}}^{BRT} : \mathrm{Gal}(\overline{F}/F) \to GL_2(K_\mathfrak{p})$ is Hodge-Tate at any prime $\mathfrak{q}$ of F dividing p, with Hodge-Tate weights $v_\mathfrak{q}$ and $w_\mathfrak{q}$, where $k = n + 2t$, $n + 2v = mt$, $w = v + k - t$.*

This follows from Blasius-Rogawski's construction, which is of motivic nature. Actually, if $k = 2t$ and f is in the principal series at all finite places of F, the proof of this result is still missing.

6 Local deformations

6.1 The case where $v \in \Sigma$.

We assume that $\overline{\rho}_v : G_v \to GL_2(\mathbb{F})$ is an extension of an unramified character $\overline{\gamma}$ of the local Galois group G_v by $\overline{\gamma}(1)$. Let $\gamma : G_v \to \mathcal{O}^\times$ be a fixed unramified character which lifts $\overline{\gamma}$. If $\overline{\rho}_v$ is indecomposable, it follows easily from Schlessinger's criterion that the functor $\overline{D}_v^{\psi}$ of deformations of $\overline{\rho}_v$ which are extensions of the character γ by $\gamma(1)$, is prorepresentable. In case $\overline{\rho}_v$ is decomposable, $\overline{D}_v^{\psi,\square}$ of its framed deformations is still prorepresentable. Let $\overline{R}_v^{\psi,\square}$ be the universal ring of framed deformations.

Proposition 6.1 *$\overline{R}_v^{\psi,\square}$ is flat over $\mathcal{O}$, its generic fiber $\overline{R}_v^{\psi,\square}[1/p]$ is irreducible and formally smooth of dimension 3. In particular, $\overline{R}_v^{\psi,\square}$ is a domain.*

Proof: For simplicity, we only treat the case when $\overline{\rho}_v$ is indecomposable. See V. Pilloni's notes [15] for the general case. Let $\chi : G_v \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character of G_v. In the indecomposable case, it is not necessary to introduce Kisin's projective scheme $\mathcal{L}^{\gamma,\gamma\chi,\square} \to \mathrm{Spec}\,\overline{R}_v^{\psi,\square}$ (see [11] or [15]). However, for pedagogical reasons, we prefer to define and use it even in our case.

We consider the functor $L^{\gamma,\gamma\chi,\square} : AR_{\mathcal{O}} \to SETS$ sending a local artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$ to the set of pairs (ρ_v, L_A) where $\rho_v : G_v \to GL_2(A)$ is a lifting of $\overline{\rho}_v$ with $\rho_v \sim \begin{pmatrix} \gamma\chi & * \\ 0 & \gamma \end{pmatrix}$, and L_A is a line of the space of ρ_v which is direct factor, stable by G_v, with action given by $\gamma\chi$.

Exercise: Even if $\overline{\rho}_v$ is decomposable and $\overline{\chi} = 1$, the functor $L^{\gamma,\gamma\chi,\square}$ is representable by a projective morphism $\Theta : \mathcal{L}^{\gamma,\gamma\chi,\square} \to \mathrm{Spec}\,\overline{R}_v^{\psi,\square}$.

Here, anyway, since we assume that $\overline{\rho}_v$ is indecomposable, we see that for any lifting $\rho_v : G_v \to GL_2(A)$ of $\overline{\rho}_v$ in $\overline{D}_v^{\psi,\square}(A)$, there exists one and only one G_v-stable line L_A in A^2 on which G_v acts by $\gamma\chi$. In other words, one has $\mathcal{L}^{\gamma,\gamma\chi,\square} \overset{\Theta}{\cong} \mathrm{Spec}\,\overline{R}_v^{\psi,\square}$.

Let us prove that $\mathcal{L}^{\gamma,\gamma\chi,\square}$ is formally smooth over $\mathcal{O}$. After twisting, we can assume that $\gamma = 1$. Let $A' \to A$ be a surjection between objects of $AR_{\mathcal{O}}$. A point $\eta \in L^{1,\chi,\square}(A)$ gives rise to a class $c(\eta) \in \mathrm{Ext}^1_{G_v}(A, A(1)) = H^1(G_v, A \otimes \mathbb{Z}_p(1))$. If we can lift this class to $c' \in \mathrm{Ext}^1_{G_v}(A', A'(1))$, the corresponding representation defines a point $\eta' \in L^{1,\chi,\square}(A')$ which lifts η. In other words, the formal smoothness will follow if we show the surjectivity of $H^1(G_v, A' \otimes \mathbb{Z}_p(1)) \to H^1(G_v, A \otimes \mathbb{Z}_p(1))$. For this it is enough to show that for any finite $\mathbb{Z}_p$-algebra, the canonical morphism $H^1(G_v, \mathbb{Z}_p(1)) \otimes A \to H^1(G_v, A \otimes \mathbb{Z}_p(1))$ is an isomorphism. By decomposing A as a product of rings $\mathbb{Z}/p^r\mathbb{Z}$, one is reduced to show that the cokernel $H^2(G_v, \mathbb{Z}_p(1))[p^r]$ vanishes. But, by Tate local duality, one sees that $H^2(G_v, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ is torsion-free as desired. In particular, $\mathcal{L}^{\gamma,\gamma\chi,\square}$ is flat over $\mathcal{O}$. In our case, we conclude that $\overline{R}_v^{\psi,\square}$ is a domain, flat over $\mathcal{O}$.

Let $L = Fr(\mathcal{O})$. As explained in Kisin's notes [10], the formal completion of this scheme at a closed point ξ represents the functor of deformations of ξ on AR_L. . Since $\chi : G_v \to L^\times$ is not trivial, it is easy to see that $\widehat{\mathcal{L}^{1,\chi,\square}[1/p]}_\xi$ is a $\widehat{PGL}_2$-torsor, hence is smooth of dimension three. Therefore $\overline{R}_v^{\psi,\square}$ is a power series ring in three variables over $\mathcal{O}$.

6.2 The case where $v \in V_p$

From now on, we assume that p splits completely in F, so that for any $\mathfrak{p} \in V_p$, we have $F_\mathfrak{p} = \mathbb{Q}_p$. For any weight (k, w) as above, one can define $R^{\psi,k,w,cris}$ as the largest torsion-free quotient of $R_\mathfrak{p}^{\psi,\square}$ such that for any finite extension $L'/L = Fr(\mathcal{O})$ and for a point $\xi : R_v^{\psi,\square} \to L'$ factors through $R^{\psi,k,w,cris}$ if and only if the framed deformation $\rho_\xi : G_\mathfrak{p} \to GL_2(L')$ associated to ξ is crystalline of weights $v_\mathfrak{p}$ and $w_\mathfrak{p}$. Actually, we'll be concerned only by the case where $k \geq 2$ is an integer, and the weight is $(kt, (k-1)t$, so that $v_\mathfrak{p} = 0$ and $w_p = k - 1$ for any $\mathfrak{p} \in V_p$. We put

For each $\mathfrak{p} \in V_p$, $\overline{R}_\mathfrak{p}^{\psi,\square} = R^{\psi,kt,(k-1)t,cris}$.

Remarks: 1) In Kisin's work [12], more general cases are studied: $\overline{R}_\mathfrak{p}^{\psi,\square} = R^{\psi,k,\tau,\overline{\rho}}$. whose characteristic zero points correspond to representations which are potentially semistable of type τ. We take here τ to be trivial, and the representations to be crystalline.

2) It is important to notice that if $k \geq p$, the ring $\overline{R}_\mathfrak{p}^{\psi,\square}$ is a quotient of a deformation ring but is not itself a deformation ring. Nevertheless, the completion of Spec $\overline{R}_\mathfrak{p}^{\psi,\square}$ at a characteristic zero point ξ is the universal deformation ring for the framed deformations of ρ_ξ which are crystalline of weights 0 and $k - 1$. See [12] for more details on this ring.

However, if $p < k$, Fontaine-Laffaille theory shows that $\overline{R}_\mathfrak{p}^{\psi,\square}$ is the universal deformation ring for framed deformations with Fontaine-Laffaille type 0 and $k - 1$. Fontaine-Laffaille theory also shows that this scheme is formally smooth of relative dimension four over $\mathcal{O}$.

Proposition 6.2 *For any $k \geq 2$, the ring $\overline{R}_\mathfrak{p}^{\psi,\square}$ is flat over $\mathcal{O}$, reduced, and its irreducible components are formally smooth over $\mathcal{O}$ of relative dimension 4.*

The proof of this result is based on the study of a projective morphism $\Theta : \mathcal{L}^{\psi,\square} \to \text{Spec } \overline{R}_\mathfrak{p}^{\psi,\square}$ with connected fibers, defined in terms of p-adic Hodge theory. The theory of Breuil-Kisin's modules allows to study the irreducible components of $\mathcal{L}^{\psi,\Box}$. The two main points are that its generic fiber $\mathcal{L}^{\psi,\Box}[1/p]$ is a finite union of irreducible components which are formally smooth, and to show, on the other hand, that $\Theta[1/p]$ is an isomorphism.

7 Global Deformations

7.1 The modular residual representation

Let F be a totally real field in which p is totally decomposed. Let V_p be the set of primes above p in F; let Σ be a finite set of primes disjoint of V_p and of even cardinality (later, it will be the set of ramification of a quaternion algebra B totally definite over F). We put $\Sigma_p = \Sigma \cup V_p$. Let S' be a finite set of primes disjoint of Σ_p. Put $S = \Sigma \cup V_p \cup S'$. Let $\mathcal{O}$ be a (big) discrete valuation ring finite flat over $\mathbb{Z}_p$ with fraction field L, uniformizing parameter ϖ and residue field $\mathbb{F}$. Let $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F,S}$ be its quotient by the normal subgroup generated by the inertia subgroups at primes dividing S, and let $\overline{\rho} : G_{F,S} \to GL_2(\mathbb{F})$ be a Galois representation unramified outside S such that

- $\overline{\rho}$ is unramified outside V_p, with odd determinant,
- the restriction of $\overline{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.
- there exists a Hilbert cusp eigenform f of conductor Σ, of diagonal weight (k, w) with $k = k_0 t$ and $w = (k_0 - 1)t$ for an integer $k_0 \geq 2$, such that $\overline{\rho} = \rho_{f,\mathfrak{p}} \,\mathrm{mod}.\varpi$.

Comments:

1) Σ will be the set of places where we allow unipotent monodromy for the deformations of the unramified $\overline{\rho}$. On the other hand, S' will consist of auxiliary primes where no ramification will occur. A prime v is called auxiliary if $Nv \not\equiv 1 \pmod{p}$ and $\overline{\rho}(Frob_v)$ satisfies

$$(AUX) \quad Nv \cdot tr\,\overline{\rho}(Frob_v)^2 - (1 + Nv)^2 \det\overline{\rho}(Frob_v) \in \mathbb{F}^\times.$$

A sufficient condition for this is that $Nv \equiv -1 \pmod{p}$ and the eigenvalues α_v and β_v of $Frob_v$ satisfy $\alpha_v \beta_v^{-1} \not\equiv -1 \,\mathrm{mod}.\varpi$.

2) The actual situation of these notes (that is, the crystalline case, in [12]) is that given an ellipic cusp eigenform f_0 of weight $k_0 \geq 2$ and level prime to p, one chooses a solvable totally real field F in which p split completely and such that the base change $f = BC(f_0)$ of f_0 has exact level Σ (the choice of F kills the ramification outside Σ); S' is introduced to make the level neat on the

Hecke side, but will not introduce ramification at primes in S' because of the assumption (AUX). More precisely, this assumption for a prime v prevents the possibility of a congruence mod. ϖ between the given v-old form $BC(f_0)$ and a v-new form. The existence of F is due to R. Taylor. Also, because of this situation, the only relevant weights for Hilbert modular forms will be diagonal: $k = k_0 t$, $v = 0$, $w = (k_0 - 1)t$, and the character ψ will be $\chi_{cyc}^{k_0-1}$.

Let B be the quaternion algebra which ramifies exactly at all archimedean places and at all places in Σ and let R be a maximal order of B. We transport f by the Jacquet-Langlands correspondence to a quaternionic eigenform, still denoted by f, in $S^B_{k,w}(U^{max}, \mathcal{O})$, where $U^{max} = \widehat{R}^\times$. We consider also $U = U_{S'} \times U^{S'}$ where $U_{S'}$ is the product of the strict Iwahori subgroups at $v \in S'$, and $U^{S'}$ is unramified. We assume that U is ad-neat (see Sect.3.3). For each $v \in S'$, we denote by x_v and y_v the two roots of the Hecke polynomial $P_{f,v}(X) = X^2 - a_v X + Nv^{k_0-1}$. The form $f_1 = f|\prod_{v\in S'}(X_v - y_v)$ is eigen for the T_v, S_v (with eigenvalues a_v resp. $\psi(v)$) for $v \notin S$ and also for X_v for $v \in S'$, with eigenvalue x_v. We consider the maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$ generated by ϖ and $T_v - a_v, S_v - \psi(v)$ for $v \notin S$ and by $X_v - x_v$ for $v \in S'$. This ideal is not trivial since it is the kernel of the reduction modulo ϖ of the character of action of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$ on f_1. Hence, $\mathbb{T} = \mathbb{T}_{\sigma,\psi}(U, \mathcal{O})_\mathfrak{m}$ and $M = S_{\sigma,\psi}(U, \mathcal{O})_\mathfrak{m}$ are not trivial. We also introduce the maximal ideal $\mathfrak{m}' = (\varpi, T_v - a_v, S_v - \psi(v))_{v\notin S}$ and $\mathbb{T}' = \mathbb{T}'_{\sigma,\psi}(U, \mathcal{O})_{\mathfrak{m}'}$.

Note that $\mathfrak{m}$ (or $\mathfrak{m}'$) is non Eisenstein. Indeed, by the irreducibility assumption, the restriction of $\overline{\rho}$ to any abelian extension of F is still irreducible (exercise). By Cebotarev property, this prohibits the existence of an abelian extension F'/F such that $Tr\overline{\rho}(Fr_v) = Tr(1 \oplus 1) \pmod{\varpi}$ for v totally split in F'. We introduce the "universal modular deformation" of $\overline{\rho}$

$$\rho_{\mathfrak{m}'} : G_{F,S} \to GL_2(\mathbb{T}')$$

It is unramified outside Σ_p with characteristic polynomial of Fr_v given by $X^2 - T_v X + Nv\psi(v)$ for all $v \notin S$.

Construction: We first notice that $\mathbb{T}'$ is reduced (it is generated by a commutative family of normal operators); hence it is a subalgebra of the product of the fields of eigenvalues of all eigenforms g occuring in M. Therefore is a Galois representation defined over $\mathbb{T}' \otimes L'$ for a sufficiently big p-adic field. The corresponding pseudo-representation is defined over $\mathbb{T}'$ itself. Since $\mathbb{T}'$ is local noetherian complete, and that the residual pseud-representation comes from an irreducible representation, a classical result of Nyssen and Rouquier implies that this pseudo-representation does come from a representation defined over $\mathbb{T}'$ itself.

Proposition 7.1 *We have $\mathbb{T} = \mathbb{T}'$. In particular, $\mathbb{T}$ is reduced.*

Proof: It is enough to show that $X_v \in \mathbb{T}'$ for all $v \in S'$. Any eigenform g occuring in M is actually v-old at every $v \in S'$ otherwise the two eigenvalues of $\rho_{g,\mathfrak{p}}(Fr_v)$ would satisfy $y_v^{(g)} = Nv^{\pm 1}x_v^{(g)}$. By reduction modulo ϖ, this contradicts the condition $Nv(x_v + y_v)^2 - (Nv+1)^2 x_v y_v \neq 0$ in $\mathbb{F}$. We then consider $char(\rho_{\mathfrak{m}'}(Fr_v) = X^2 - t_v + Nv\psi(v) \in \mathbb{T}'[X]$. Either this polynomial has two distinct roots mod. $\mathfrak{m}'$ (after increasing $\mathcal{O}$ if necessary) and by Hensel's lemma, its roots are in $\mathbb{T}'$, so $X_v \in \mathbb{T}'$. Or this polynomial has a double root in $\mathbb{F}$, and this root lifts to $\mathbb{T}'$. In all cases, we see that $X_v \in T'$.

Recall that we still write ψ for the Galois character $\psi^{gal} : G_{F,S} \to \mathcal{O}^\times$ associated to ψ.

7.2 Universal deformation rings

Let $\psi\chi_{cyc} : G_{F,S} \to \mathcal{O}^\times$ a p-adic lifting of $\det\overline{\rho}$ (it will be $\det\rho_F$). Let $R^{\psi}_{F,S}$ be the universal ring of S-ramified deformations of $\overline{\rho}$ with determinant $\psi\chi_{cyc}$. Actually, they are unramified outside $S - S'$. Thus, this ring classifies (certain) liftings to $GL_2(A)$ of $\overline{\rho}$ mod. $\widehat{GL_2}(A)$, or more precisely, mod. $\widehat{PGL_2}(A)$ since GL_2 acts by the adjoint action.

Let $R^{\psi,\square}_{F,S}$ be the universal ring of S-ramified framed deformations of $\overline{\rho}$ with determinant $\psi\chi_{cyc}$. This means it represents the functor on $CNL_{\mathcal{O}}$

$$A \mapsto \{(\rho, (B_v)_{v\in\Sigma_p})\}$$

where ρ is a lifting of $\overline{\rho} : G_{F,S} \to GL_2(\mathbb{F})$, and B_v is a basis of the space V_A of ρ, lifting the canonical basis of the space $\mathbb{F}^2$ of $\overline{\rho}$ The equivalence relation is given by the action of the formal scheme $\widehat{\mathbb{G}}_m$, by $\lambda \cdot (\rho, (B_v)) = (\rho, (\lambda B_v)_v)$. This formal scheme should be viewed as sending a ring A to the commutant of a lifting ρ, which is here independent of lifting, and equal to $A^\times$. This commutant is the stabilizer of ρ for the adjoint action of $\widehat{GL_2}(A)$. Note that its tangent space is $H^0(G_{F,S}, ad\,\overline{\rho}) = \mathrm{Com}(\overline{\rho})$, namely, by Schur's lemma, the group of homotheties.

Lemma 7.2 *The morphism $R^{\psi}_{F,S} \to R^{\psi,\square}_{F,S}$ defines a torsor on $(\prod_{v\in\Sigma_p} GL_2)/\mathbb{G}_m^{diag}$; hence, it is formally smooth of relative dimension $4|\Sigma_p| - 1$.*

One can therefore write $R^{\psi,\square}_{F,S} = R^{\psi}_{F,S}[[x_1, \ldots, x_j]]$, with $j = 4|\Sigma_p| - 1$.

We call the variables $x_1, \ldots, x_j$ the frame variables.

Explanation: We can rewrite $R^{\psi}_{F,S}$ as the universal ring for tuples $(\rho, (B_v))$ modulo the action of $\widehat{GL_2}^{\Sigma_p}$ by, say, $(\rho, (B_v)) \mapsto (g_1\rho g_1^{-1}, g_1B_1, g_2B_2, \ldots, g_n.B_n)$;

there are $4|\Sigma_p|$ dimensions of choice for (g_v). By definition of framed deformations,one has to mod out this action by the diagonal inclusion of the center $\widehat{Z}(A)$ of $\widehat{GL_2}$, which is one dimensional.

We can similarly consider local deformation problems. Let $v \in \Sigma_p$, and $R_v^{\psi,\square}$ be the universal ring of framed deformations of $\overline{\rho}_v = \overline{\rho}|_{D_v}$ with determinant $\psi\chi_{cyc}$. This problem is representable because it consists of pairs (ρ_v, B_v), a lifting and a basis, modulo action of the commutant $\mathrm{Com}(\rho_v)$ on the basis; this set can be identified to the trivially representable problem $A \mapsto \mathrm{Hom}_{\overline{\rho}}(D_v, GL_2(A))$.

Let $R_{\Sigma_p}^{\psi,\square} = \widehat{\bigotimes}_{v\in\Sigma_p} R_v^{\psi,\square}$, which classifies deformations of $(\overline{\rho}_v)_{v\in\Sigma_p}$. We have a tautological morphism $R_{\Sigma_p}^{\psi,\square} \to R_{F,S}^{\psi,\square}$ (given by the restrictions to D_v's).

Proposition 7.3 *The minimal number of generators of $R_{F,S}^{\psi,\square}$ over $R_{\Sigma_p}^{\psi,\square}$ is less than or equal to*

$$g = h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) + \sum_{v\in\Sigma_p} \delta_v - 1,$$

where $H^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) = \{x \in H^1(G_{F,S}, ad^0\,\overline{\rho});\ res_v(x) = 0, \forall v \in \Sigma_p\}$ and $\delta_v = h^0(D_v, ad\,\overline{\rho}) = \dim(\mathrm{Com}(\overline{\rho}_v))$.

Proof: To show this we consider the diagram of functors:

$$\begin{array}{ccc} D_{F,S}^{\psi,b} & \xrightarrow{\theta_2} & D_{\Sigma_p}^{\psi,b} \\ \downarrow f_2 & & \downarrow f_1 \\ D_{F,S}^{\psi,\square} & \xrightarrow{\theta} & D_{\Sigma_p}^{\psi,\square} \\ \downarrow & & \downarrow \\ D_{F,S}^{\psi} & \xrightarrow{\theta_1} & D_{\Sigma_p}^{\psi} \end{array}$$

where the upper index b on the first line means we fix a lifting and bases, without any equivalence relation. From $D_{\Sigma_p}^{\psi,b}$ to $D_{\Sigma_p}^{\psi,\square}$, one needs to divide by $\mathrm{Com}(\rho_v)^{\Sigma_p}$. From $D_{\Sigma_p}^{\psi,b}$ to $D_{\Sigma_p}^{\psi}$, one needs to divide by $\widehat{GL_2}^{\Sigma_p}$.

The minimum number of generators we look for is the relative dimension of the tangent spaces, that is, we take the tangent spaces of all these functors, and we need to see $\dim \mathrm{Ker}\, d\theta \leq g$. It is clear that $\dim \mathrm{Ker}\, d\theta_1 = h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho})$. Since $D_{F,S}^{\psi,b}$ is also a $\widehat{GL_2}^{\Sigma_p}$-torsor over $D_{F,S}^{\psi}$, we see that $\dim \mathrm{Ker}\, d\theta_2 = \dim \mathrm{Ker}\, d\theta_1$. We have therefore

$$\dim \mathrm{Ker}\, d(f_1 \circ \theta_2) \leq h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) + \sum_{v\in\Sigma_p} \delta_v.$$

But $f_1 \circ \theta_2 = \theta \circ f_2$ and f_2 is surjective, so that $\dim \mathrm{Ker}\, d(\theta \circ f_2) = \dim \mathrm{Ker}\, d\theta + \dim \mathrm{Ker}\, df_2$. We have $\dim \mathrm{Ker}\, df_2 = 1$, hence $\dim \mathrm{Ker}\, d\theta \leq h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) + \sum_v \delta_v - 1$.

Let's introduce the Selmer group and the dual Selmer group $H^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho})$ and $H^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1))$. The group $H^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho})$ is defined as the subgroup of $H^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho})$ of classes whose restriction at the decomposition group G_v for each $v \in S$ belongs to L_v, with $L_v = 0$ if $v \in \Sigma_p$, and $L_v = H^1(G_v, ad^0(\overline{\rho}))$ for all $v \in S - \Sigma_p$. Note that by oddness, $L_v = H^1(G_v, ad^0(\overline{\rho})) = 0$ for $v \in S_\infty$. By Tate local duality, the orthogonal $L_v^\perp \subset H^1(G_v, ad^0(\overline{\rho})(1))$ is $L_v^\perp = H^1(G_v, ad^0(\overline{\rho})(1))$ if $v \in \Sigma_p$, and $L_v^\perp = 0$ for all $v \in S - \Sigma_p$ or $v \in S_\infty$. Therefore, the dual Selmer group (see for instance Sect.2 of [5]) is the subgroup of $H^1(G_{F,S}, ad^0\,\overline{\rho}(1))$ of classes such that if $v \in S - \Sigma_p$ or if $v \in S_\infty$, $res_v(x) = 0$ (but no condition if $v \in \Sigma_p$).

Corollary 7.4 *With the same notations, we also have* $g \leq h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}) + |\Sigma_p| - 1 - [F : \mathbb{Q}]$.

Proof: The Poitou-Tate Euler characteristic formula (as interpreted by Wiles, see [5] Th.2.14) says that

$$h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) - h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1)) = h^0(G_{F,S}, ad^0\,\overline{\rho}) - h^0(G_{F,S}, ad^0\,\overline{\rho}(1)) +$$

$$+ \sum_{v \in \Sigma_p} \ell_v - \sum_{v \in \Sigma_p} h^0(G_v, ad^0(\overline{\rho})) + \sum_{v|\infty} \ell_v - \sum_{v|\infty} h^0(G_v, ad^0(\overline{\rho}))$$

with $\ell_v = \dim L_v$. But $h^0(G_{F,S}, ad^0\,\overline{\rho}) = h^0(G_{F,S}, ad^0\,\overline{\rho}(1)) = 0$ by irreducibility of $\overline{\rho}|_{G_{(F(\zeta_p)}}$. Besides, we have $\sum_{v \in \Sigma_p} \ell_v = \sum_{v \in S_\infty} \ell_v = 0$, and $h^0(G_v, ad^0\,\overline{\rho}) = 1$ for any $v|\infty$ and $h^0(G_v, ad^0\,\overline{\rho}) = \delta_v - 1$ for any $v \in \Sigma_p$. Hence,

$$h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) - h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1)) = -[F : \mathbb{Q}] + |\Sigma_p| - \sum_{v \in \Sigma_p} \delta_v.$$

From the proposition, we know $g \leq h^1_{\Sigma_p}(G_{F,S}, ad^0\,\overline{\rho}) + \sum_v \delta_v - 1$. Replacing by the equality above, we get the corollary.

7.3 Global deformations with local conditions

Let us introduce also (quotient) universal rings of framed deformations (or liftings) with local conditions (for $\rho|_{D_v}$, $v \in \Sigma_p$). We denote them by a bar:

$$\begin{array}{ccc} R^{\psi,\square}_{\Sigma_p} & \rightarrow & R^{\psi,\square}_{F,S} \\ \downarrow & & \downarrow \\ \overline{R}^{\psi,\square}_{\Sigma_p} & \rightarrow & \overline{R}^{\psi,\square}_{F,S} \end{array}$$

By definition, this will be a cartesian diagram, hence the bound established in the previous section applies: the number g defined in the previous section is also the minimum number of generators of $\overline{R}^{\psi,\square}_{F,S}$ over $\overline{R}^{\psi,\square}_{\Sigma_p}$; hence it is bounded by the bound of the corollary above.

Definition 7.5 *For each $v \in \Sigma_p$, we define $\overline{R}^{\psi,\square}_v$ as in Sect.6.2 and we put $\overline{R}^{\psi,\square}_{\Sigma_p} = \widehat{\bigotimes}_{v\in\Sigma_p}\overline{R}^{\psi,\square}_v$; this ring has no ϖ-torsion, and is of relative dimension $\mathbf{d} = 3|\Sigma_p| + [F : \mathbb{Q}]$ over $\mathcal{O}$. Let $\overline{R}^{\psi,\square}_{F,S} = R^{\psi,\square}_{F,S} \otimes_{R^{\psi,\square}_{\Sigma_p}} \overline{R}^{\psi,\square}_{\Sigma_p}$.*

Note that the number $\mathbf{d}$ (local deformation dimension) will play a crucial role in the following arguments (as well as the Taylor-Wiles number h and the frame number j). In particular, it follows from Def.6.6 and from Cor.6.3 that $\overline{R}^{\psi,\square}_{F,S}$ is topologically generated over $\mathcal{O}$ by

$$h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1)) + 4|\Sigma_p| - 1.$$

It remains to clarify the mysterious global term $h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1))$. By the method of Taylor-Wiles systems, one can interpret it as a "local term". Its dimension will be the common cardinality of the Taylor-Wiles sets. We will call this number the Taylor-Wiles number, denoted by h.

7.4 Taylor-Wiles systems

Definition 7.6 *Let $f \in S_{\sigma,\psi}(U, \mathcal{O})$ be a cusp eigenform. A finite set of finite places of F disjoint to $S = \Sigma^p \cup S'$ is called a Taylor-Wiles set if for any $v \in Q$, we have $Nv \equiv 1 \pmod p$, and the two roots α_v, β_v of the Hecke polynomial $P_{f,v}(X)$ are distinct mod. ϖ. As before, we introduce the finite p-group Δ_Q as the product of the p-Sylows of $\mathbb{F}_v^\times$. We put $S_Q = S \cup Q$ and $\Sigma_{p,Q} = \Sigma_p \cup Q$.*

Let $R^{\psi,\square}_{F,S_Q}$ be the deformation ring defined as before but where we allow arbitrary ramification at $v \in Q$. If we define $L_v = H^1(G_v, ad^0\overline{\rho})$ for any $v \in Q$, we can apply the Poitou-Tate formula of Euler characteristics to get:

$$\dim_{\mathbb{F}} \mathrm{Tang}(R^{\psi,\square}_{F,S_Q} \otimes \mathbb{F}) = h^1_{\Sigma_{p,Q}}(G_{F,S}, ad^0\,\overline{\rho}) =$$

$$h^1_{\Sigma^\vee_{p,Q}}(G_{F,S}, ad^0\,\overline{\rho}) - [F:\mathbb{Q}] + |\Sigma_p| - \sum_v \delta_v + |Q|.$$

Moreover there is an action of $\mathcal{O}[\Delta_Q]$ on R^ψ_{F,S_Q} and $R^{\psi,\square}_{F,S_Q}$. To see this, recall the

Lemma 7.7 *For $v \in Q$, any deformation of $\overline{\rho}_v$ with det=$\psi\chi_{cyc}$, is equivalent to $\begin{pmatrix} \chi_1\delta & 0 \\ 0 & \chi_2\delta^{-1} \end{pmatrix}$, with unramified characters and $\delta : \Delta_Q \to A^\times$.*

Proof: For any $v \in Q$, take a basis which diagonalizes $\rho(Frob_v)$ over A and use for $\tau \in I_v^{(p)}$ the p-Sylow of the tame inertia at v: $\rho(Fr_v\tau Fr_v^{-1}) = \rho(\tau)^{Nv}$ and $\rho(\tau) = D(1_2 + X)$, with $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where D is diagonal and $D \equiv 1_2 \mathrm{mod}.\mathfrak{m}_A$; from this and from $\alpha_v/\beta_v \not\equiv Nv\,\mathrm{mod}.\mathfrak{m}_A$, one sees by induction on r that $b, c \in\in \mathfrak{m}_A^r$ for any r, so $b = c = 0$ and $D = diag(\delta, \delta^{-1})$ for a character of $I_v^{(p)}$, which factors through Δ_v.

We define the universal morphism $\delta^{univ} : \Delta_Q \to (R^\psi_{F,S_Q})^\times$. This gives rise to a structure of $\mathcal{O}[\Delta_Q]$-module.

By a classical argument of Wiles (see Darmon-Diamond-Taylor), using Cebotarev density one can show:

Proposition 7.8 *There exists a sequence (Q_n) of Taylor-Wiles sets of constant cardinality $h = h^1_{\Sigma_p^\vee}(G_{F,S}, ad^0\,\overline{\rho}(1))$ s.t. for any $v \in Q_n$, $Nv \equiv 1 \pmod{p^n}$ and $H^1_{\Sigma^\vee_{p,Q_n}}(G_{F,S}, ad^0\,\overline{\rho}(1)) = 0$.*

If we introduce local conditions at Σ_p, we shall obtain $\overline{R}^\psi_{F,S_{Q_n}}$ and $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$. We have thus proved, modulo the local calculations, that the minimal number of generators of $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$ over $\mathcal{O}$ is less than $h+j$.

Indeed the bound is $0 + 4|\Sigma_p| - 1 + |Q_n|$.

Before exploiting this information, we need to come back to the Hecke side.

8 Hecke algebras and Hecke modules

We come back to the situation in Subsection 7.1. We have defined the universal modular deformation $\rho_\mathfrak{m} : G_{F,S} \to GL_2(\mathbb{T})$ of $\overline{\rho}$. Its determinant is $\psi\chi_{cyc}$, hence it gives rise to a unique algebra homomorphism $R^\psi_{F,S} \to \mathbb{T}$. By Prop.7.1, this homomorphism is surjective. Similarly, $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\mathfrak{m}_{Q_n}}$ is surjective. Moreover both members are $\mathcal{O}[\Delta_{Q_n}]$-modules, and by Carayol's compatibility theorem for Langlands correspondences, the map is $\mathcal{O}[\Delta_{Q_n}]$-linear. We define also $M_{Q_n} = S_{\sigma,\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}_{Q_n}}$ (see the definition of U_{Q_n} in Sect.3.4).

To construct a Taylor-Wiles-Kisin system, we need an $\overline{R}^{\psi,\square}_{F,S}$-module $M^{\square}$ and a sequence of $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$-modules $M^{\square}_{Q_n}$.

Definition 8.1 *We put* $M^{\square} = R^{\psi,\square}_{F,S} \otimes_{R^{\psi,\square}_{F,S}} M$ *and*

$M^{\square}_{Q_n} = R^{\psi,\square}_{F,S_{Q_n}} \otimes_{R^{\psi,\square}_{F,S_{Q_n}}} M_{Q_n}$. *These are called the framed Taylor-Wiles modules.*

By local-global compatibility, $M^{\square}_{Q_n}$ is actually a $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$-module. At p, it is a result of T. Saito; at $v \in \Sigma$, it is (an easy case of) Carayol's theorem.

Lemma 8.2 $M^{\square}_{Q_n}$ *is free of constant finite rank over* $\mathcal{O}[\Delta_{Q_n}][[y_{h+1}, \ldots, y_{h+j}]]$ *and* $M^{\square}_{Q_n}/(y_1, \ldots, y_{h+j})M^{\square}_{Q_n} \cong M^{\square}$.

Notation: We shall denote this rank by s; it is the rank of M over $\mathcal{O}$.
Proof: It follows immediately from Lemma 3.10 that $S_{\sigma,\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}Q_n}$ is free of finite rank over $\mathcal{O}[\Delta_{Q_n}]$ and that its quotient by the augmentation ideal of $\mathcal{O}[\Delta_{Q_n}]$ is $S_{\sigma,\psi}(U, \mathcal{O})_{\mathfrak{m}}$. By Lemma 7.2, we have $\overline{R}^{\psi,\square}_{F,S_{Q_n}} = \overline{R}^{\psi}_{F,S_{Q_n}}[[y_{h+1}, \ldots, y_{h+j}]]$; this implies the finiteness and freeness in the statement. Then, it follows from the definition of the structure of $\mathcal{O}[\Delta_{Q_n}]$-module of $R^{\psi}_{F,S_{Q_n}}$ that

$$R^{\psi}_{F,S_{Q_n}}/(y_1, \ldots, y_h)R^{\psi}_{F,S_{Q_n}} = R^{\psi}_{F,S}.$$

Hence, we have $M^{\square}_{Q_n}/(y_1, \ldots, y_{h+j})M^{\square}_{Q_n} \cong M^{\square}$. This implies the constancy of the rank.

9 Taylor-Wiles-Kisin method

For each $n \geq 1$, one can choose a surjective $\mathcal{O}$-algebra homomorphism $\mathcal{O}[[y_1, \ldots, y_h]] \to \mathcal{O}[\Delta_{Q_n}]$. As above, after choosing "frame variables" $y_{h+1}, \ldots, y_{h+j}$ such that $R^{\psi,\square}_{F,S_{Q_n}} = R^{\psi}_{F,S_{Q_n}}[[y_{h+1}, \ldots, y_{h+j}]]$, we get

$$\mathcal{O}[[y_1, \ldots, y_{h+j}]] \to R^{\psi,\square}_{F,S_{Q_n}} \to \overline{R}^{\psi,\square}_{F,S_{Q_n}}$$

As mentioned in the Remark following Prop.7.8, this last module can be generated by at most $h+j$ variables over $\mathcal{O}$; since $\overline{R}^{\psi,\square}_{\Sigma_p}$ can be generated by $\mathbf{d}$ variables over $\mathcal{O}$ (Def.7.1), we see that there exists a surjective local $\mathcal{O}$-algebras homomorphism

$$\overline{R}^{\psi,\square}_{\Sigma_p}[[x_1,\ldots,x_g]] \to \overline{R}^{\psi,\square}_{F,S_{Q_n}}$$

with $g = h+j-\mathbf{d}$. We used the fact that the dimension of the tangent space mod. ϖ of $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$ over $\mathcal{O}$ is the sum of the dimension of the tangent space of $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$ over $\overline{R}^{\psi,\square}_{\Sigma_p}$ and that of $\overline{R}^{\psi,\square}_{\Sigma_p}$ over $\mathcal{O}$.

[Indeed, if $A \to B \to C$ are three local rings with same residue field k, we have an exact sequence

$$0 \to t_{C/B} \otimes_B k \to t_{C/A} \otimes_C k \to t_{B/A} \otimes_B k \to 0$$

because this sequence can be rewritten as

$$0 \to \mathrm{Der}_B(C,k) \to \mathrm{Der}_A(C,k) \to \mathrm{Der}_A(B,k) \to 0]$$

Recall that the modules $M^{\square}_{Q_n}$ are finite free over $\mathcal{O}[\Delta_{Q_n}][[y_{h+1},\ldots,y_{h+j}]]$ of constant rank. As in the original Taylor-Wiles method, these modules will be used to create a module limproj $M^{\square}_{Q_n}$ (for a suitable projective system) which will be finite free over $\mathcal{O}[[y_1,\ldots,y_{h+j}]]$. This will guarantee that after taking a projective limit (for the same projective system), the map $\mathcal{O}[[y_1,\ldots,y_{h+j}]] \to$ limproj $\overline{R}^{\psi,\square}_{F,S_{Q_n}}$ is injective.

Let $\mathfrak{c}_n$ be the ideal of $\Lambda = \mathcal{O}[[y_1,\ldots,y_{h+j}]]$ generated by

$$\varpi^n, (1+y_1)^{p^n}-1, \ldots, (1+y_h)^{p^n}-1, y_{h+1}^{p^n}, \ldots, y_{h+j}^{p^n}$$

. One can easily show that the sequence $(\mathfrak{c}_n)$ is cofinal to the sequence $(\mathfrak{m}_\Lambda^n)_n$ with $\mathfrak{m}_\Lambda = (\varpi, y_1, \ldots, y_{h+j})$. Indeed, $\mathfrak{m}_\Lambda^{(h+j)np^n} \subset \mathfrak{c}_n \subset \mathfrak{m}_\Lambda^n$. In particular the quotient rings $\Lambda/\mathfrak{c}_n$ and the quotient modules $M^{\square}_{Q_n}/\mathfrak{c}_n M^{\square}_{Q_n}$ are finite. Indeed for $\mathfrak{c}_0 = \mathfrak{m}_\Lambda$, it comes from the "horizontal control", then we proceed by (finite) induction on k and we get that $M^{\square}_{Q_n}/\mathfrak{m}_\Lambda^k M^{\square}_{Q_n}$ is finite, which is enough by cofinality.

These finite quotients provide the finite objects on which one can use Dirichlet's pigeon hole principle. More precisely, let $S = \overline{R}^{\psi,\square}_{\Sigma_p}[[X_1,\ldots,X_g]]$; note that S is $\mathcal{O}$-flat of relative dimension $d+j$ by Def.7.1. We also put $\overline{R}_{Q_n} = \overline{R}^{\psi,\square}_{F,S_{Q_n}}$ and $\overline{R} = \overline{R}^{\psi,\square}_{F,S}$. We have the following situation

$$\begin{array}{ccccc} S & \to & \overline{R}_{Q_n} & \to & \mathrm{End}_\Lambda(M^\Box_{Q_n}), & M^\Box_{Q_n} \\ & \nearrow & \downarrow & & & \downarrow \\ \Lambda & & \overline{R} & \to & \mathrm{End}_\Lambda(M^\Box), & M^\Box \end{array}$$

where the top left horizontal arrow is surjective and the vertical ones are quotient by the ideal $(y_1, \ldots, y_h)$ of Λ.

Lemma 9.1 *After replacing (Q_n) by a subsequence, there exist S-and Λ-linear homomorphisms*

$$f_n : M^\Box_{Q_{n+1}}/\mathfrak{c}_{n+1}M^\Box_{Q_{n+1}} \to M^\Box_{Q_n}/\mathfrak{c}_n M_{Q_n^\Box}$$

such that f_n modulo $(y_1, \ldots, y_h)$ induces the identity on $M^\Box/\mathfrak{c}_n M^\Box$.

Before explaining the construction, which involves the notion of glueing data of Kisin, let's mention a corollary. Let us consider the inverse limit $M^\Box_\infty = \mathrm{limproj}\, M^\Box_{Q_n}/\mathfrak{c}_n M^\Box_{Q_n}$ of the f_n's, which is both an S-module and a Λ-module.

Corollary 9.2 *$M^\Box_\infty$ is finite free over Λ.*

Proof: It follows immediately from Lemma 7.5 above.
Proof of Lemma: For any local ring A, we denote by $\mathfrak{m}_A^{(r)}$ the ideal generated by the r-th powers of elements of . For $m \geq 0$, let $r_m = smp^m(h+j)$ (with $s = \mathrm{rk}_{\mathcal{O}[[y_{h+1},\cdots,y_{h+j}]]}M$). Even without the presence of s as a factor of r_m we know that $\mathfrak{m}_\Lambda^{(r_m)} \subset \mathfrak{c}_m$. More importantly (here the s is needed), we also have for any $m, n \geq 0$,

$$\mathfrak{m}_{\overline{R}_n}^{(r_m)} \cdot M^\Box_{Q_n} \subset \mathfrak{c}_m \cdot M^\Box_{Q_n}.$$

Indeed for $a \in \mathfrak{m}_{\overline{R}_n}$, a induces a nilpotent endomorphism $\overline{a}$ of the s-dimensional $\mathbb{F}$-vector space $M^\Box/(\varpi, y_{h+1}, \ldots, y_{h+j})M^\Box$. Therefore, $\overline{a}^s = 0$ and $a^s \cdot M$ is contained in $(\varpi, y_{h+1}, \ldots, y_{h+j}) \cdot M^\Box$; this implies that for any $n \geq 0$, $a^s \cdot M^\Box_{Q_n} \subset (\varpi, y_1, \ldots, y_{h+j}) \cdot M^\Box_{Q_n}$, and

$$a^{sp^m(h+j)} \cdot M^\Box_{Q_n} \subset (\varpi, y_1^{p^m}, \ldots, y_{h+j}^{p^m}) \cdot M^\Box_{Q_n}$$

because $(\sum_{i=1}^r a_i\xi_i)^{rp^m} \in (\xi_1^{p^m}, , \xi_r^{p^m})$. But $(\varpi, y_1^{p^m}, \ldots, y_{h+j}^{p^m}) = (\varpi, (1+y_1)^{p^m} - 1, \ldots, (1+y_h)^{p^m} - 1, y_{h+1}^{p^m}, \ldots, y_{h+j}^{p^m})$, hence the result. This leads to the definition:
. A glueing data of level m is a pair (D, L), or rather, a sextuple $(D, u_m, v_m, w_m, L, x_m)$ where

- D is a complete noetherian local Λ-algebra such that $\mathfrak{m}_D^{(r_m)} = 0$,
- $u_m : \Lambda \to D$ a local algebra homomorphism,
- $v_m : D \to S/\mathfrak{c}_n S + \mathfrak{m}_S^{(r_n)}$ a local Λ-algebra homomorphism,
- $w_m : S \to D$ a sujective local algebra homomorphism,
- L is a D-module which, via u_m, is free of finite rank over $\Lambda/\mathfrak{c}_m$, together with an S-linear surjection $x_m : L \to M^\square/\mathfrak{c}_m M^\square$.

A morphism between two glueing data $(D_1, L_1) \to (D_2, L_2)$ is given by $D_1 \to D_2$ (local algebra homomorphism) and $L_1 \to L_2$; the first map is S-linear via the w_m's, and the second is D_1-linear and is compatible to the projections $L_i \to M^\square/\mathfrak{c}_m M^\square$.

The point for the construction is that the set of isomorphism classes of glueing data of level m is finite. Indeed, D is a quotient of the finite ring $S/\mathfrak{m}_S^{(r_m)}$, the ring $\Lambda/\mathfrak{c}_m$ is finite and $L \cong (\Lambda/\mathfrak{c}_m)^s$.

Let us consider for level 1 a sequence of glueing data: $D_{1,n} = \overline{R}_n/\mathfrak{c}_1\overline{R}_n + \mathfrak{m}_{\overline{R}_n}^{(r_1)}$ and $L_{1,n} = M_{Q_n}^\square/\mathfrak{c}_1 M_{Q_n}^\square$ with u_1 given by the structure of $\Lambda/\mathfrak{c}_1$-algebra of $D_{1,n}$, v_1 given by $\overline{R}_n \to \overline{R}$, and w_1 given by $S \to \overline{R}_n$. By Dirichlet's principle, one is reached infinitely many times. Take such an infinite subset I_1 of integers, and start again with $D_{n,2} = \overline{R}_n/\mathfrak{c}_2\overline{R}_n + \mathfrak{m}_{\overline{R}_n}^{(r_2)}$,... by diagonal extraction, one gets a sequence (D_m, L_m) of glueing data of level m which are compatible to level change:

$D_{m+1}/\mathfrak{c}_m D_{m+1} + \mathfrak{m}_{D_{m+1}}^{(r_m)} \cong D_m$, and

$L_{m+1}/\mathfrak{c}_m L_m \cong L_m$,

these two maps defining an isomorphism of glueing data. This proves the lemma.

Before recalling Kisin's version of the Taylor-Wiles' method, we need to recall few facts on Hilbert-Samuel multiplicities (left as exercises):

Let A be a noetherian local ring, and M a finitely generated A-module; there exists a (unique) polynomial $P(X) \in \mathbb{Q}[X]$ such that for any sufficiently large n, we have $\ell_A(M/\mathfrak{m}_A^n M) = P(n)$. It is called the Hilbert-Samuel polynomial of (M, A). Its degree is the dimension of the support of M, which is less than $d = \dim \mathrm{A}$. We nevertheless write it as $P(X) = \frac{e(M,A)}{d!} X^d + \ldots ...$; the number $e(M, A)$ is an integer, called the Hilbert-Samuel multiplicity of (M, A) (abbreviated as HS multiplicity). In case $M = A$, we simply write $e(A)$. If $\mathrm{Ann}\, M \neq 0$, one has $e(M, A) = 0$; however, if M is faithful over A, $e(M, A) \neq 0$. Once A is fixed, this function is additive on exact sequences.

This implies that if M is a faithful finitely generated A-module which is generically of rank 1, $e(M, A) = e(A)$. Indeed, choose a non-zero vector $m \in M$, note that $\mathrm{Ann}\,(M/A \cdot m) \neq 0$ and apply the additivity to the exact sequence

$$0 \to A \to M \to M/A \cdot m \to 0.$$

Proposition 9.3 *(Kisin's version of the Taylor-Wiles' method) The following four statements are equivalent*

- *(i) $M_\infty^\square$ is a faithful S-module,*
- *(ii) $M_\infty^\square$ is a faithful S-module of rank one at each generic point,*
- *(iii) We have the Kisin's numerical inequality for Hilbert-Samuel multiplicities:*
$$e(S/\varpi S) \leq e(M_\infty^\square/\varpi M_\infty^\square, S/\varpi S),$$
- *(iv) The equality holds in the inequality above.*

Comment: The numerical criterion (iii) must be interpreted as follows: the HS multiplicity measures the size of a local noetherian ring; its value on the Galois-theoretic deformation ring is bounded by the value on an automorph-theoretic deformation ring (Hecke ring); as we shall see, this is the missing bound for an $R = T$ theorem, as we already know that R surjects to T.

Before proving this key proposition, let us state the Main Theorem of Modularity which is, as we shall see, a direct corollary thereof.

Theorem 5 *Let $\rho : G_{F,S} \to GL_2(\mathcal{O})$ be an arbitrary lifting of the residual modular reprepresentation $\overline{\rho}$ fixed above. Assume that for any $v \in \Sigma$, ρ_v is an extension of γ by $\gamma(1)$ and that for any $v \in V_p$, ρ_v is crystalline of weights 0 and $k-1$, and assume that the numerical criterion holds,*

then, ρ is modular: there exists a cusp eigenform g in $S_{\sigma,\psi}(U, \mathcal{O})$ such that $\rho \cong \rho_{g,p}$.

Proof of Proposition: As a preparation, let $\mathbb{T}_\infty^\square$ be the image of S in $\mathrm{End}_\Lambda(M_\infty^\square)$. It is a finite torsion-free local Λ-algebra. Therefore, any irreducible component has relative dimension $h+j$ over $\mathcal{O}$. On the other hand, we know that any irreducible component of S is of dimension $\mathbf{d}+(h+j-\mathbf{d}) = h+j$ over $\mathcal{O}$. It follows that the irreducible components of $\mathbb{T}$ form a subset of those of S.

[Note that if $\mathrm{Spec}(T) \subset \mathrm{Spec}(S)$ is a closed subscheme in an integral irreducible scheme, we have either $\dim(T) < \dim(S)$ or $T = S$, so that if we knew that S is a domain, we would have $S = \mathbb{T}_\infty^\square$.]

We first prove that (i) implies (ii). Condition (i) means that $S = \mathbb{T}_\infty^\square$. We have $M_\infty^\square/(y_1, \ldots, y_{h+j})M_\infty^\square = M^\square/(y_{h+1}, \ldots, y_{h+j})M^\square = S_{\sigma,\psi}(U, \mathcal{O})_\mathfrak{m}$. This last module is generically of rank one over $\mathbb{T}$ (that is, after inverting ϖ), by Corollary 3.5. Hence by Nakayama's lemma, we have a generically surjective S-linear map $S \to M_\infty^\square$; indeed, for the kernel $\mathfrak{a}$ of $\mathbb{T}_\infty^\square \to \mathbb{T}$ and for any prime $\mathfrak{p}$ of height 0 in $\mathbb{T}_\infty^\square$, we see that $\mathfrak{p} + \mathfrak{a}$ does not contain ϖ so that $(\mathbb{T}_\infty)_\mathfrak{p} \to (M_\infty)_\mathfrak{p}$ is surjective; thus, the generic rank of $M_\infty^\square$ is at most 1. By faithfulness, it is at least one: equality holds.

(ii) implies (iii): choose a vector $m \in M_\infty^\square$ which is not divisible by ϖ and $\overline{m}$ its image in $M_\infty^\square/\varpi M_\infty^\square$. Then, $M_\infty^\square/\varpi M_\infty^\square$ we have a short exact sequence:

$$0 \to S/\varpi S \to M_\infty^\square/\varpi M_\infty^\square \to (M_\infty^\square/\varpi M_\infty^\square)/(S/\varpi S) \cdot \overline{m} \to 0$$

because $M_\infty^\square/S \cdot m$ has no ϖ-torsion. Then apply the additivity of HS multiplicity.

(iii) obviously implies (iv).

Let us show (iv) implies (i): Let $\phi : S \to \mathbb{T}_\infty^\square$ and $I = \operatorname{Ker}\phi$. Since S has no ϖ-torsion, we have an exact sequence

$$0 \to I/\varpi I \to S/\varpi S \to \mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square \to 0$$

it is enough to prove that $e(I/\varpi I, S/\varpi S) = 0$ (because it implies that $I/\varpi I$ is torsion over $S/\varpi S$). We have

$$(**) \quad e(S/\varpi S) = e(I/\varpi I, S/\varpi S) + e(\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square, S/\varpi S)$$

We now observe that $e(\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square, S/\varpi S) = e(\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square)$, because S and $\mathbb{T}_\infty^\square$ have same dimension. We also have $e(\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square) = e(M_\infty^\square/\varpi M_\infty^\square, \mathbb{T}_\infty/\varpi \mathbb{T}_\infty^\square)$ because $M_\infty^\square/\varpi M_\infty^\square$ is faithful over $\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square$ and of generic rank one; as in the proof of $(ii) \Rightarrow (iii)$, this implies the numerical criterion with $\mathbb{T}_\infty^\square/\varpi \mathbb{T}_\infty^\square$ instead of $S/\varpi S$. But Condition (iv) reads: $e(M_\infty^\square/\varpi M_\infty^\square, S/\varpi S) = e(S/\varpi S)$. Thus, we can rewrite $(**)$ as $e(S/\varpi S) = e(I/\varpi I, S/\varpi S) + e(S/\varpi S)$. Hence, $e(I/\varpi I, S/\varpi S) = 0$. As seen before, this shows that ϕ is an isomorphism, and $M_\infty^\square$ is faithful over S.$\square$

Let us now deduce the Main Theorem from the Proposition. Let $\rho : G_{F,S} \to GL_2(E')$ be a deformation of $\overline{\rho}$ with the right local conditions at all v's in Σ_p. By considering the canonical basis of $(E')^2$, we obtain a point $\overline{R}_{F,S}^{\psi,\square} \to E'$. Let $x : S \to E'$ be the composition of this point with $S \to \overline{R}_\infty \to \overline{R}_{F,S}^{\psi,\square}$. If we assume the numerical criterion, we have $S = \mathbb{T}_\infty^\square$. This numerical inequality is precisely the Breuil-Mézard conjecture, established by Kisin [13] (using previous works by Breuil and Berger and Colmez's local Langlands correspondence). Observe that $M_\infty^\square \otimes_{\mathbb{T}_\infty^\square} E' = S_{\sigma,\psi}(U,\mathcal{O})_\mathfrak{m} \otimes_\mathbb{T} E'$. Therefore, ρ is the Galois representation attached to any non zero element of this module (it is necessarily an eigenform with the correct eigenvalues). This module is not zero because it is not so modulo ϖ.

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