IS THERE SUCH A THING AS

∞

?

A PROOF THAT THERE ISN'T

① $\infty = \frac{1}{0}$.

② Therefore $0 \times \infty = 1$.

③ So ...

$$
\begin{aligned}
0 &= 0 \times 1 \\
&= 0 \times (0 \times \infty) \\
&= (0 \times 0) \times \infty \\
&= 0 \times \infty \\
&= 1
\end{aligned}
$$

$\therefore$ a contradiction .

A PROOF THAT THERE IS .

The sequence 1, 2, 3, 4, 5, ...

of all positive integers is

END LESS ,

that is,

IN FINITE .

Or, to paraphrase, the number

of positive integers is ∞ .

SOME UNDERLYING ASSUMPTIONS.

(i) $\frac{1}{0} = \infty$.

(ii) ∞ obeys the conventional laws of arithmetic.

(iii) Given a set of mathematical objects (e.g. positive integers) there is something, $\ast$, say, for which one can say

The number of such objects is $\ast$.

DOES $\frac{1}{0} = \infty$?

$$\infty > 1 \Rightarrow \frac{1}{\infty} < 1$$

$$\infty > 10 \Rightarrow \frac{1}{\infty} < 0.1$$

$$\infty > 100 \Rightarrow \frac{1}{\infty} < 0.01$$

$$\vdots$$

$$\frac{1}{\infty} < 0.0000 \ldots\ldots 001 .$$

But also $\infty \geqslant 0$, so

$\frac{1}{\infty} \geqslant 0$.

So $\frac{1}{\infty} = 0$?

Eg. $0.0000 \text{------} 0023605 \text{----}$

$> 0.0000 \ldots\ldots\ldots 001$

AN ALTERNATIVE VIEW

1. ∞ is bigger than every positive integer n.

2. $\delta = \frac{1}{\infty}$ is infinitesimally small <u>but not zero</u>.

3. δ^2 is infinitesimally small <u>even relative to δ</u>, but still not zero.

This leads to a

<u>consistent arithmetical system</u>.

Which is bigger,

$1 + 3\delta + 100\delta^2$ or $1 + 4\delta - 100\delta^2$?

What about $\dfrac{2+\delta}{3+\delta}$ and $\dfrac{2-\delta}{3-2\delta}$?

It's the same as comparing

$(2+\delta)(3-2\delta)$ with $(2-\delta)(3+\delta)$

or $6 - 4\delta + 3\delta - 2\delta^2$ with $6 + 2\delta - 3\delta - \delta^2$

or $6 - \delta - 2\delta^2$ with $6 - \delta - \delta^2$.

So $\dfrac{2+\delta}{3+\delta} < \dfrac{2-\delta}{3-2\delta}$

COMPLEX NUMBERS

Does $\sqrt{-1}$ exist ?

Does it matter ?

Let $i = \sqrt{-1}$. Then

eg. $(1+i)(3-2i) = 3 - 2i + 3i - 2i^2$

$$= 3 + i - 2(-1)$$
$$= 3 + i + 2 = 5 + i$$

All we needed to know was

(i) usual rules of arithmetic apply ;

(ii) $i^2 = -1$.

BACK TO ∞

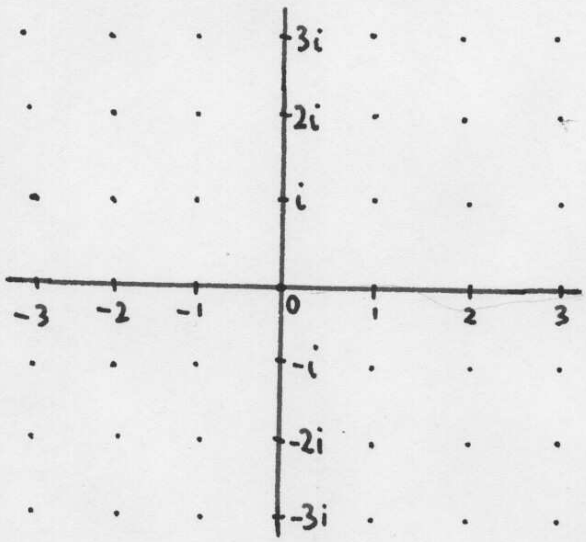
We've been keeping assumption (ii) and dropping assumption (i). The reverse is possible:

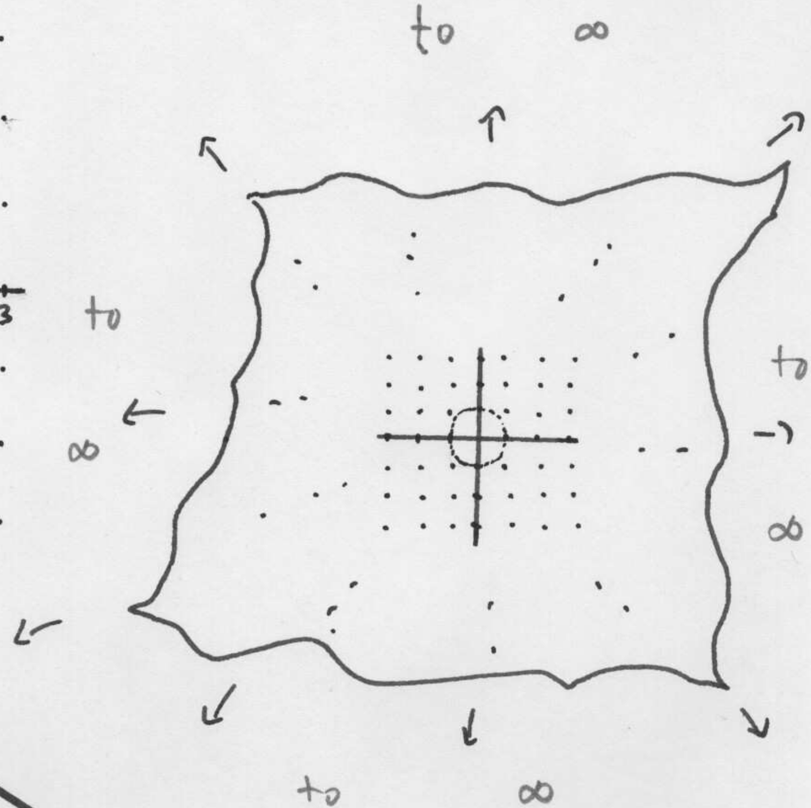
yes, $\frac{1}{0} = \infty$, but

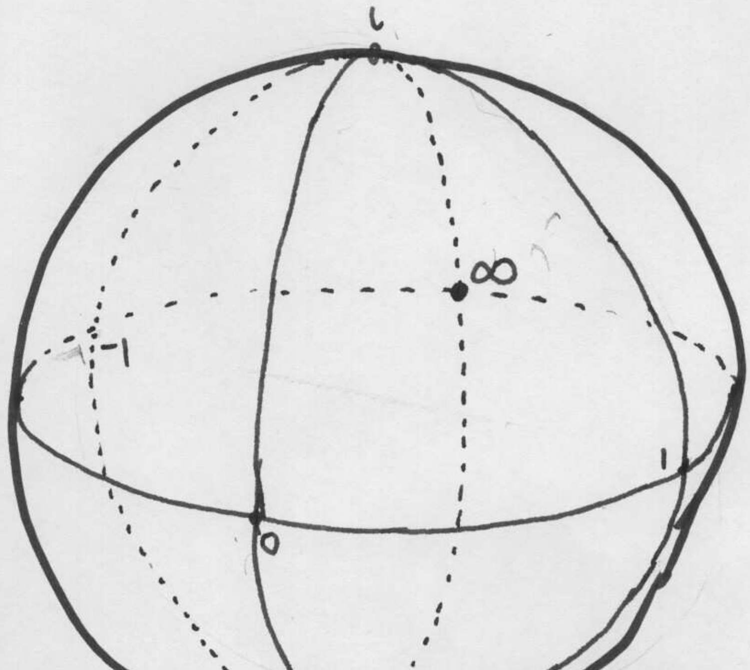
$\infty \times 0$ is UNDEFINED.

Eg., it can be convenient to give a meaning to $\dfrac{x+3}{x-2}$ when $x = 2$.

THE EXTENDED COMPLEX PLANE.







$$z \longrightarrow \frac{1}{z}$$

does what?

$$\frac{1}{1} = 1 \ , \quad \frac{1}{-1} = -1$$

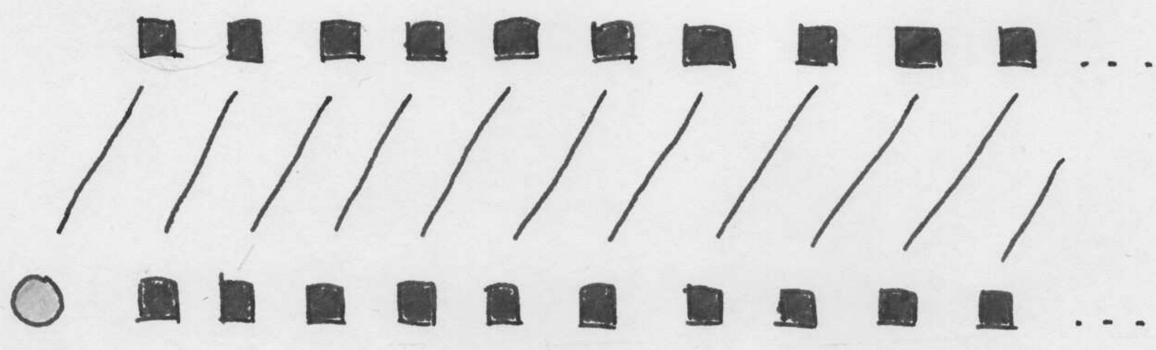
$$\frac{1}{0} = \infty \ , \quad \frac{1}{\infty} = 0$$

$$\frac{1}{i} = -i \ , \quad \frac{1}{-i} = i \ .$$

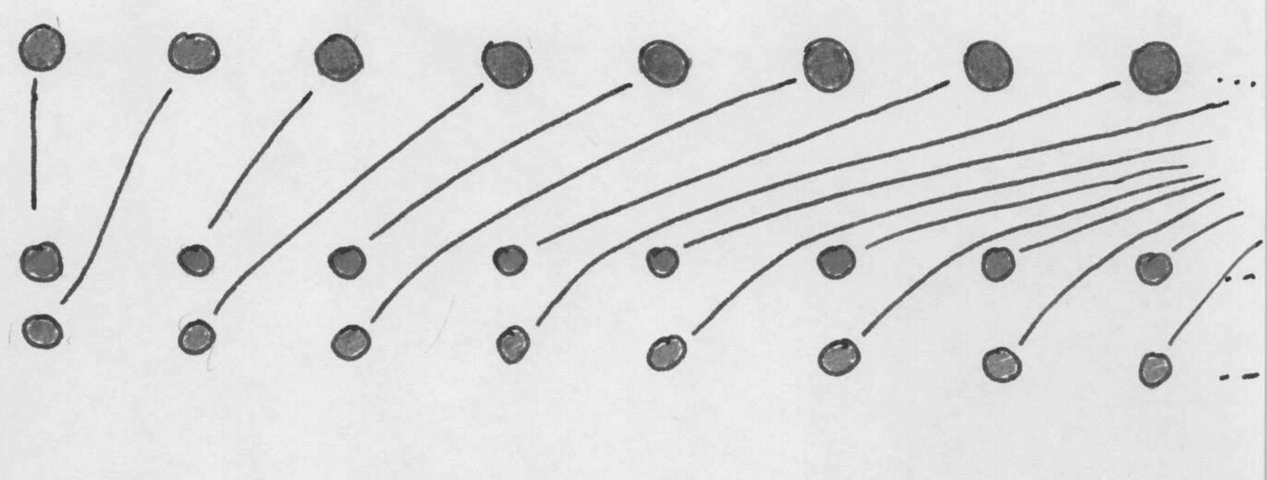
WHAT IS $\infty + 1$?

(i) It's just $\infty + 1$.

(ii)



AND $2 \times \infty$?



<u>CARDINALS</u>

Two sets are the same size if they can be put into one-to-one correspondence.

Cantor showed that the set of real numbers is <u>bigger</u> than the set of positive integers. More generally, given any set, there is always a bigger set.

So - many different "sizes" of ∞.

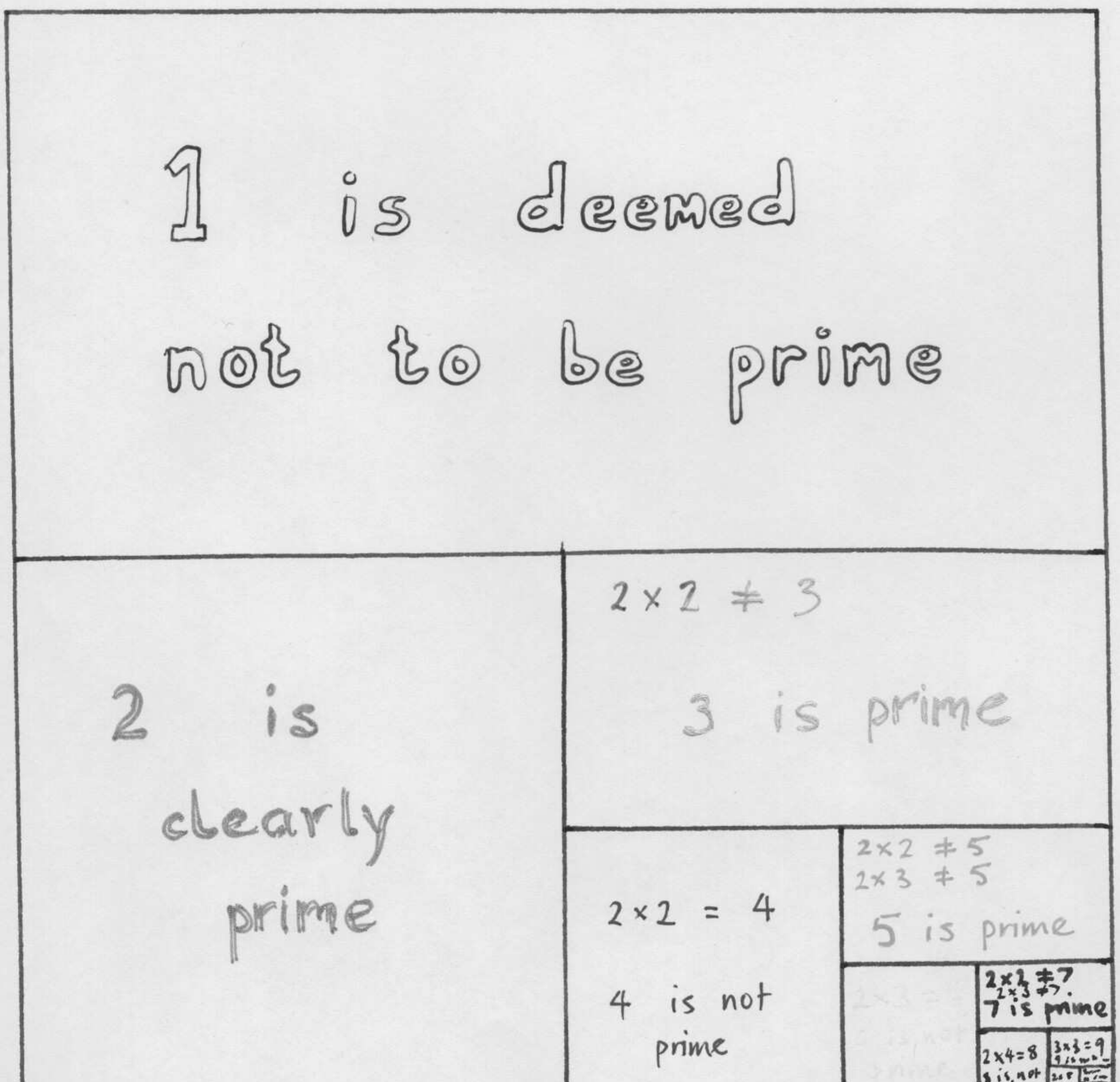
?? Is there a size in between that of the positive integers and that of the real numbers ??

What is an "arbitrary" real number?

HOW GOD DOES MATHEMATICS.

<u>Theorem</u>: There are infinitely many primes.

<u>Proof</u>:



HOW GOD DOES METAMATHEMATICS

1 second A - not a proof that there are infinitely many primes

$\frac{1}{2}$ second B - "

⋮

2^{-m} seconds Ra@ * the $\int 15\, y^2$ is $\sqrt{2}$ of xzzpq inzloverji + 27ω .... draattqM

- not a proof that there are infinitely many primes

⋮

2^{-n} seconds Assume that there are only finitely many primes $p_1, \dots, p_n$ and let $N = p_1 p_2 \cdots p_n$ be their product. Then N is a product of primes, but is divisible by no p_i, a contradiction.

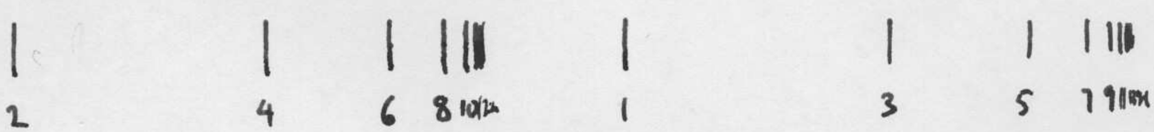
- a proof that there are infinitely many primes!

TWO WAYS OF COUNTING THROUGH THE POSITIVE INTEGERS.

(i) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . .

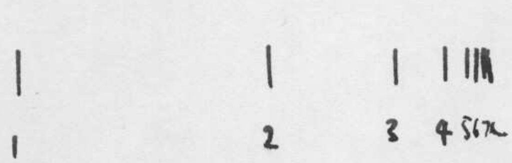
(ii) 2, 4, 6, 8, 10, 12, ... 1, 3, 5, 7, 9, 11, ...

Which takes longer?



2ω

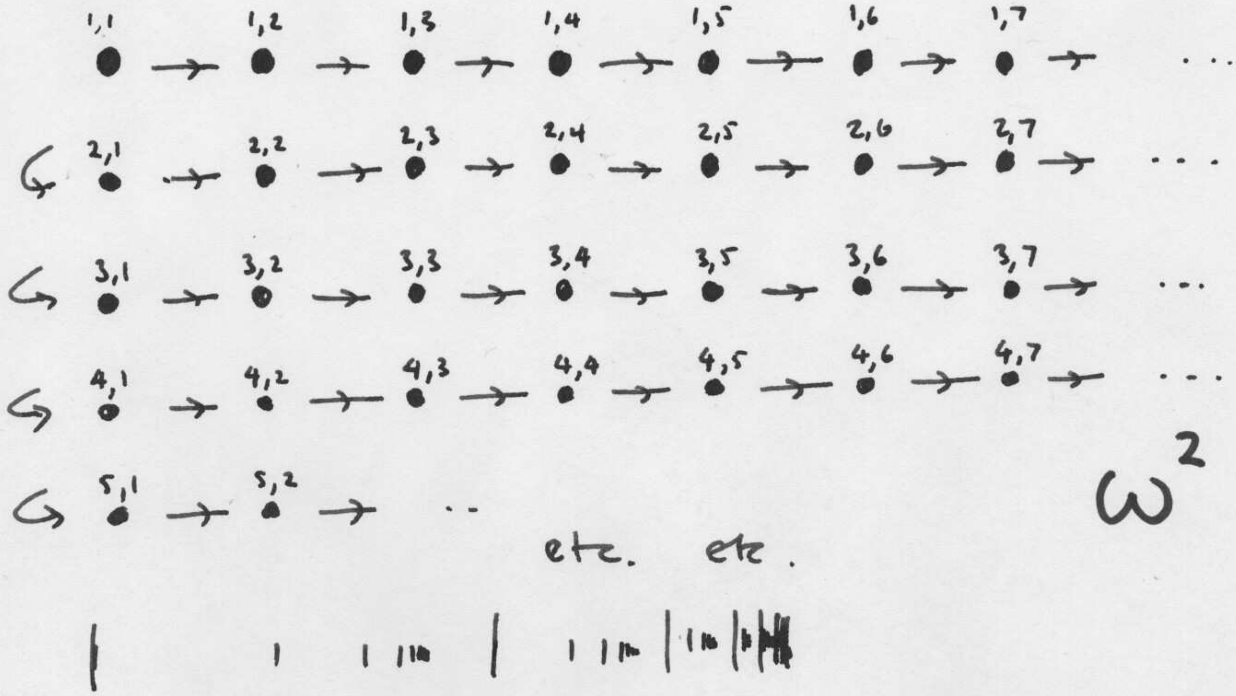
or



ω

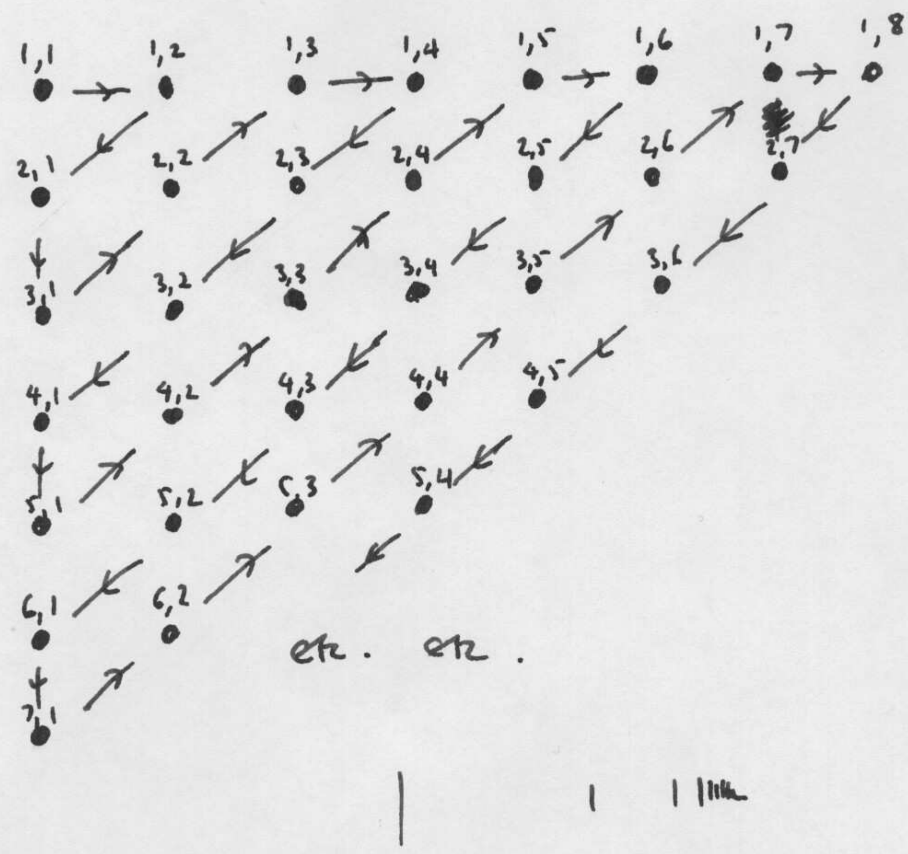
TWO WAYS OF COUNTING POINTS IN THE PLANE

(i)



ω^2

(ii)



ω

ORDINALS

$1, 2, 3, 4, 5, \ldots \quad \omega, \quad \omega+1,$

$\omega+2, \quad \omega+3, \quad \ldots \quad 2\omega, \quad 2\omega+1, \quad 2\omega+2,$

$2\omega+3, \quad \ldots, \quad 3\omega, \quad 3\omega+1, \quad \ldots$

$\ldots \ldots \quad \omega^2, \quad \omega^2+1, \quad \omega^2+2, \ldots$

$\ldots \quad \omega^2+\omega, \quad \omega^2+\omega+1, \ldots \quad \omega^2+2\omega, \quad \omega^2+2\omega+1,$

$\ldots \quad \omega^2+3\omega, \quad \ldots \quad 2\omega^2, \quad \ldots$

$3\omega^2, \quad \ldots \quad \omega^3, \quad \ldots$

$\omega^4, \quad \ldots \quad \ldots \quad \omega^\omega, \quad \ldots$

$\omega^{\omega+1}, \quad \ldots, \quad \omega^{\omega+2}, \quad \ldots, \quad \omega^{2\omega}, \quad \ldots$

$\omega^{\omega^\omega}, \quad \ldots \quad \omega^{\omega^{\omega^\omega}}, \quad \ldots$

$\ldots \quad \left.\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}\right\}\omega$ and on and on and on

WHAT IS THE POINT OF ORDINALS?

X - a set of mathematical objects

$\circledcirc$ - an infinitary operation

INPUT

$(x_1, x_2, x_3, \dots)$

$$\searrow \quad \boxed{\circledcirc} \longrightarrow Y$$

OUTPUT

What can you generate?

Level 1 - all x_i belong to X.

Level 2 - all x_i belong to level 1.

$\vdots$

Level ω - all x_i belong to some level n.

$\vdots$

etc. etc.

INFINITE ARITHMETIC

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2 .$$

Why ?

$$\text{Let } S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} .$$

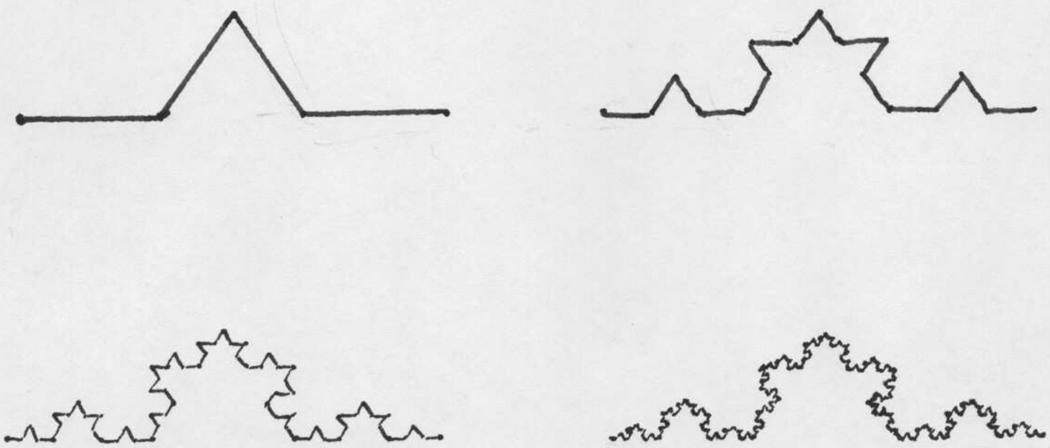
$$= 2 - \frac{1}{2^n} .$$

We'd like to say

(i) $S_\infty > S_n$;

(ii) $S_\infty \leq 2$.

HOW LONG IS THE EDGE OF A SNOWFLAKE?



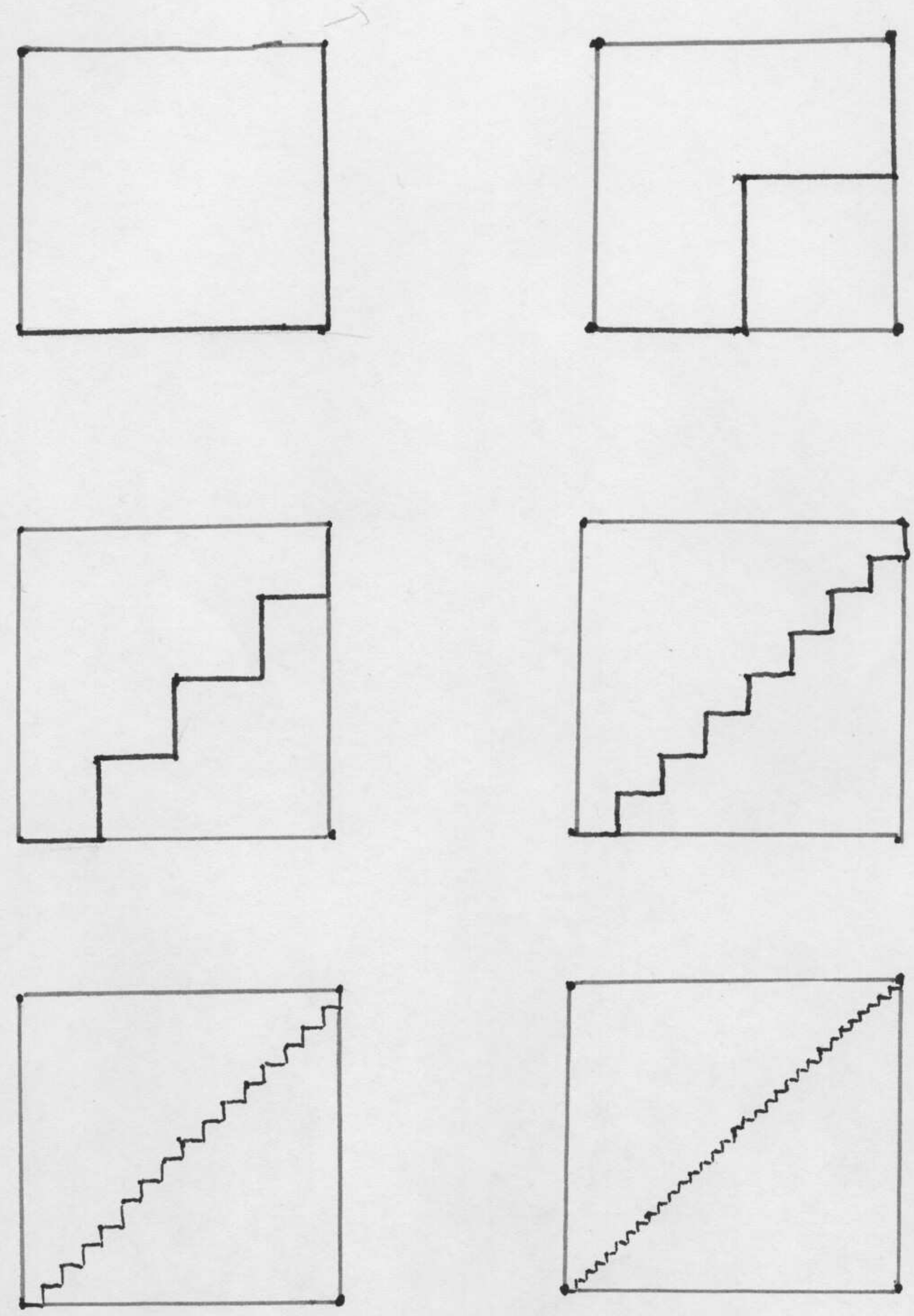
. . .

AND ITS AREA ?

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \dots$$

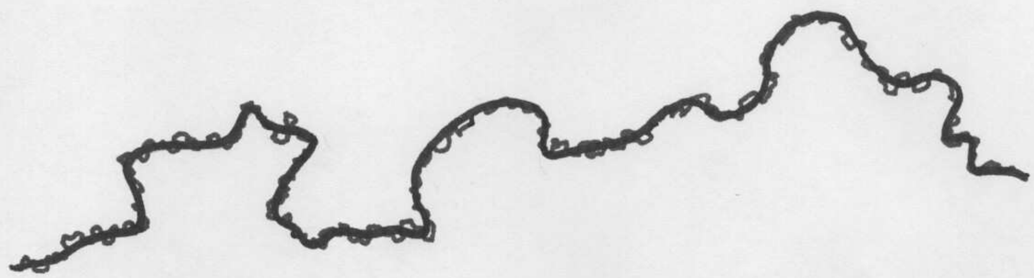
$$= 1\,\frac{4}{5}$$

HOW LONG IS THE DIAGONAL OF A SQUARE?

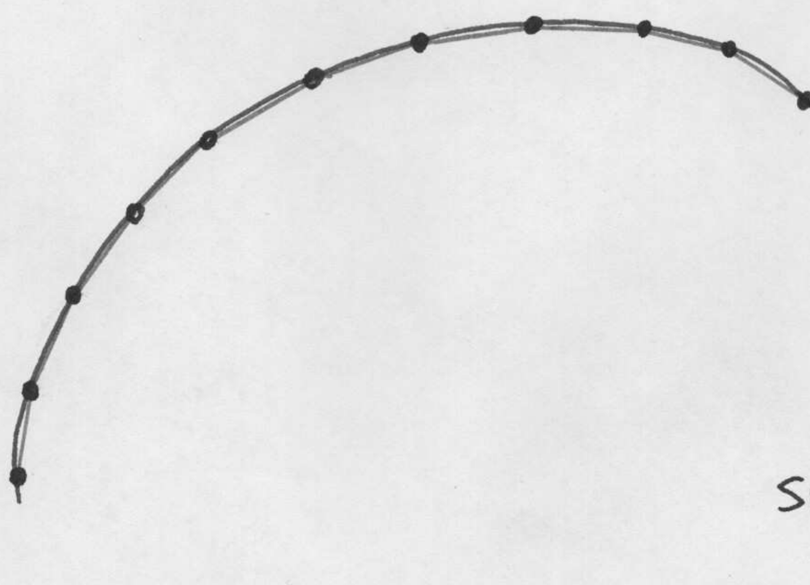


RESOLUTION OF THE PROBLEM.

- Two curves can be close but still have very different lengths.



- Definition of length.



Snowflake ✓

Diagonal ×

∞ - eliminated

USES OF ∞ - A SUMMARY

- Extends the real or complex numbers by adding $\frac{1}{0}$, but not all arithmetical rules applicable.

- An abstract symbol, not equal to $\frac{1}{0}$ but obeying rules of arithmetic.

- A measure of size - but many different levels of ∞.

- A measure of complexity - but <u>many</u> different levels of ∞.

- A fanciful idea that can be eliminated from discourse.

Does it exist?