THE WORK OF HACON AND M$^{\text{C}}$KERNAN

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Contents

1. Introduction 1
2. Birational Geometry 4
3. Minimal Models of Surfaces 4
4. Line Bundles and the Kleiman Criterion 6
5. The Canonical Line Bundle 7
6. The Cone Theorem 9
7. The Contraction Theorem 9
8. Minimal Models in Higher Dimensions 11
9. The Minimal Model Program with Scaling 15
10. Finite Generation of Adjoint Rings 16
11. What Remains to Do: Dichotomy and Trichotomy 18
References 19

1. Introduction

This note is a sketch of the theory of minimal models of algebraic varieties and the recent advances of Hacon, M$^{\text{c}}$Kernan and their collaborators [HM06, BCHM09, HM09].

The classification theory of algebraic varieties—nonsingular, say, or mildly singular—aims, in the first instance, to establish the following basic dichotomy, which I state before defining some key words. The goal is to divide, up to surgery (called 'birational equivalence' in algebraic geometry), all proper algebraic varieties into two basic classes:

(1) Varieties X with nef canonical line bundle.
(2) Fibre spaces $X \to Z$ such that the *anti*-canonical line bundle K_X^* is ample on all general fibres X_z.

Recall that a line bundle L on an algebraic variety X is *nef* if:

$$c_1(L)[C] = \int_C c_1(L) \geq 0$$

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for all proper (i.e., compact) algebraic curves $C \subset X$. The first Chern class $c_1(L)$ of a line bundle is represented by the curvature form

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log h$$

of a Hermitian metric h on L: thus, nef is a cohomological version of 'non-negatively curved.' On a proper algebraic manifold X, the most basic line bundle is the canonical line bundle

$$K_X = \bigwedge^{\text{top}} T^*_X,$$

and to say that K_X is nef is another way to say that X is, in some weak sense, *nonpositively* curved. The sign change can be confusing but it makes sense: K_X is the top exterior power of the *dual* of the tangent sheaf: K_X nef means that the Ricci curvature integrated over all proper algebraic curves is ≤ 0.

To understand the second basic class, recall that a line bundle L on a proper algebrac variety X is *ample* if for some integer $N > 0$ the global sections of $L^{\otimes N}$ define an embedding of X in projective space. It follows from this that L has a Hermitian metric with *strictly positive* curvature form. Thus, the fibres X_z of the fibration $f\colon X \to Z$ in class (2) are the opposite of class (1): here the *dual* of the canonical line bundle is ample, and the Ricci curvature integrated over all proper algebraic curves is > 0. Because of the presence of these $X_z \subset X$, X can never hope to end up in class (1). I should also make the point that, in algebraic geometry, we say that a morphism $f\colon X \to Z$ is a fibration, or that X is a fibre space, simply to mean that $\dim Z < \dim X$. This implies that fibres X_z and $X_{z'}$ over general points z, $z' \in Z$ are diffeomorphic. However, in general, there can be rather singular fibres, all fibres need not have the same dimension, and even over the set of regular values f need not be an *algebraic* fibre bundle.

The *minimal model program* is similar to the geometrisation program in 3-dimensional topology. Starting with a nonsingular projective variety Y, the intent is to perform controlled birational surgeries on Y until we reach a variety X in one of the two basic classes. As I discuss below, these surgeries modify an extremal ray R with $K \cdot R < 0$, not directly a geometric locus on which the curvature is > 0. There are two types of surgeries: divisorial contractions and flips. Divisorial contractions contract a codimension 1 locus and have been known to exist for quite some time. By contrast, flips modify the variety along a locus of codimension ≥ 2: their existence has been a conjecture for about 25 years and is now a theorem of Hacon and M$^{\text{c}}$Kernan [HM09]. At the moment, it is still a conjecture that the program terminates: we don't know that a sequence of flips must stop: hence, we can not yet establish the basic dichotomy.

Hacon, M$^{\mathrm{c}}$Kernan et al. [BCHM09] have proved termination under special assumptions that allow to run the program in some useful cases and construct a model in one of the two basic classes (I state the precise conditions below).

The theory of minimal models of surfaces was understood by Castelnuovo and Enriques circa 1900, and their fine classification was complete by 1915 [CE15]. It was subsequently extended to the complex analytic setting by Kodaira in the 1960s. Today, many mathematicians know and use the classification of surfaces. This paper is an introduction to the higher dimensional theory.

I attempt to address a general mathematical audience, and I try only to assume the most basic familiarity with the notion of an algebraic variety over the complex numbers, roughly at the level of Miles Reid's *Undergraduate algebraic geometry* [Rei88]. My account is considerably lighter than [Kol87], which I recommend as a serious introduction to algebraic geometry and minimal model theory, and [Wil87] and [Mor87]. My main purpose is to lead you as quickly as possible, with just enough theory and examples, to a position where you can appreciate the recent advances. Surveys of a similar weight, discussing other aspects of this story, are Reid's *Tendencious survey* [Rei87a], the preface of [CR00], Reid's *Old person's view* [Rei00] and his *Update on 3-folds* [Rei02], my own *What is a flip?* [Cor04], and Kollár's *What is a minimal model?* [Kol07].

The best proper introduction to the theory is the book of Kollár and Mori [KM98]; and, if you really want to study the recent advances, you will find some further background in the book [Cor07].

I do not even try to do justice to those who contributed to the subject (you know who you are, please accept my apologies!) or its history, recent and distant. Indeed, my 'historical' remarks are—as is common for a practising mathematician—mostly the works of my fantasy and part of my way to imagine a story[1].

It was a great honour, and pleasure, to be invited—together with S. Mori—to give a talk at the Clay Institute on the occasion of the research award to Hacon and M$^{\mathrm{c}}$Kernan. This note is based on Mori's talk, as well as my notes for my own talk[2].

[1]The book of Beauville [Bea78] has many well-researched notes on the history of the classification of surfaces.

[2]I thank F. Catanese, T. Coates, P. Hacking, J. Kollár, J. M$^{\mathrm{c}}$Kernan, S. Mori, and M. Reid for comments on earlier versions. I am also grateful to the referee for several useful comments.

Convention 1.1. I work over the field $\mathbb{C}$ of complex numbers. Most algebraic varieties in this note are projective and nonsingular. 'Nonsingular' here is synonymous with 'manifold.' If you prefer, you can substitute 'complex projective algebraic variety' with 'complex projective analytic variety:' these notions are indeed equivalent. For a complex algebraic variety, 'proper' is synonymous with 'compact.'

2. Birational Geometry

A key feature of algebraic geometry in dimension ≥ 2, and a source of enduring interest, is *birational geometry.* For instance, consider the origin $P = (0,0) \in \mathbb{C}^2 = X$ with coordinates x, y. The *blow up* of $P \in X$ is the surface:

$$Y = \{xm_1 - ym_0 = 0\} \subset \mathbb{C}^2 \times \mathbb{P}^1,$$

where $m_0 : m_1$ are homogeneous coordinates on $\mathbb{P}^1$, together with the natural projection $f\colon Y \to X$. It is immediate from the equation that $f^{-1}(P) = E \cong \mathbb{P}^1$, and f identifies $Y \setminus E$ with $X \setminus \{P\}$. The function $m = m_1/m_0$ is well defined on the chart $\{m_0 \neq 0\}$, which is identified with $\{y = mx\} \subset \mathbb{C}^3$. Thus, the exceptional set E is the set of tangent directions at $P \in X$, with the point at infinity corresponding to the vertical line $\{x = 0\}$. We can use the above construction as a local (complex analytic) model for the blowing up $f\colon E \subset Y \to P \in X$ of a nonsingular point on an arbitrary nonsingular surface. The normal bundle of E in X is $\mathcal{O}(-1)$.

Convention 2.1. A *rational map* is a map given by rational functions; I denote rational maps by broken arrows, e.g. $\varphi\colon Y \dashrightarrow X$, to signify that they are not everywhere defined.

A *morphism* is a map defined everywhere; I denote it by a solid arrow, e.g. $f\colon Y \to X$.

Definition 2.2. A rational map $\varphi\colon Y \dashrightarrow X$ is *birational* if has an inverse which is a rational function. Equivalently, there are algebraic subvarieties $E \subset Y$ and $F \subset X$—the exceptional sets—such that $\varphi\colon Y \setminus E \to X \setminus F$ is an isomorphism of algebraic varieties.

Two algebraic varieties are *birationally equivalent* if there is a birational map between them.

3. Minimal Models of Surfaces

We want to classify nonsingular projective algebraic surfaces[3] *up to birational equivalence.* The *contraction theorem* of Castelnuovo and

[3]Below we also say something about the classification of compact complex analytic surfaces

Enriques states that, if Y is a nonsingular surface, and $E \subset Y$ a nonsingular curve isomorphic to $\mathbb{P}^1$ and with normal bundle $\mathcal{O}(-1)$—called a -1-curve—then there is a birational morphism $f\colon E \subset Y \to P \in X$ contracting E to a nonsingular point $P \in X$; in fact, this process is nothing but the inverse of blowing up $P \in X$, and the curve E is then the inverse image of P.

The following is the *traditional statement* of the minimal model theorem for surfaces, as can be found, for instance, in [Bea78]:

Theorem 3.1. *Let Y be a nonsingular projective surface, and let*

$$Y = Y_0 \to Y_1 \to \cdots \to Y_i \to Y_{i+1} \to \cdots \to Y_r = X$$

be the composite of any sequence of contractions of -1-curves until none can be found (since $b_2 = \operatorname{rank} H_2(Y_i; \mathbb{Z})$ drops with each contraction, any such sequence must stop). Then, one of the following two alternatives holds:

- X **is a minimal model:** *If Z is any nonsingular projective surface birationally equivalent to Y, then the induced birational map $Z \dashrightarrow X$ is a* morphism.
- X **is a ruled surface or $\mathbb{P}^2$:** *where ruled means there is a morphism $f\colon X \to B$ to a curve and every fibre is $\mathbb{P}^1$.*

By Noether's theorem, a ruled surface $S \to B$ is always birational to $\mathbb{P}^1 \times B$.

Put differently, the traditional statement establishes the following dichotomy. Let $\mathcal{B}(Y)$ be the ordered set with elements pairs (Z, φ) of a nonsingular projective surface Z and a birational map $\varphi\colon Z \dashrightarrow Y$, where $Z_1 > Z_2$ if the composite map $\varphi_2^{-1} \circ \varphi_1\colon Z_1 \dashrightarrow Z_2$ is a morphism. Then: *Either* $\mathcal{B}(Y)$ has a smallest element, the minimal model X of Y; *or* Y is birational to $\mathbb{P}^1 \times B$ or $\mathbb{P}^2$.

Example 3.2. A nonsingular quadric surface, say for instance the surface $(xz + y^2 = 1) \subset \mathbb{C}^3$, is birationally equivalent to the xy-plane by the birational transformation

$$z = \frac{1 - y^2}{x}.$$

A nonsingular quadric surfaces in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$: thus, we have just constructed a birational map $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, and it is easy to check that neither φ nor its inverse is a morphism. On the other hand, neither $\mathbb{P}^1 \times \mathbb{P}^1$ nor $\mathbb{P}^2$ contain -1-curves. With a little more work, this shows that $\mathcal{B}(\mathbb{P}^2)$ does not have a smallest element.

The statement is due to Castelnuovo around 1900. The result does not generalise to higher dimensions in this form: birational geometry in dimension > 2 is inherently more complicated, and we must learn to forget about minimal models that are literally 'smallest'—*see* for instance the discussion in [Mor87, § 9], but also [Kol07]. It took almost 100 years before the work of Mori [Mor82] opened the way to a higher dimensional theory. An accessible introduction to minimal models of surfaces from the perspective of Mori theory can be found in [Rei97, Chapter D]; it is interesting to compare Reid's treatment to that of Beauville [Bea78] (which goes back to Kodaira).

4. Line Bundles and the Kleiman Criterion

Line bundles are essential to the study of algebraic varieties. Given a line bundle L on X, an $r+1$-dimensional vector subspace $V \subset \Gamma(X, L)$ determines a rational map:

$$\varphi_V \colon X \dashrightarrow \mathbb{P}(V^*) = \mathbb{P}^r$$

defined outside the set Z of common zeros of the elements of V:

$$\varphi_V(x) = H_x = \{\sigma \in V \mid \sigma(x) = 0\}$$

(if $x \notin Z$, then $H_x \subset V$ is a hyperplane). In more concrete terms, choose a basis $\sigma_0, \dots, \sigma_r$ of V, and set $\varphi_V(x) = \big(\sigma_0(x) : \dots : \sigma_r(x)\big)$; although $\sigma_0, \dots, \sigma_r$ are not functions on X, their ratios σ_i/σ_j are rational functions, hence we get a well-defined point of $\mathbb{P}^r$. When $Z = \emptyset$, that is, when for all $P \in X$ there is a section $\sigma \in V$ that is non-zero at P, we say that V is *base point free*; in this case φ_V is a morphism.

We can pair line bundles against algebraic curves $C \subset X$:

$$L \cdot C = \deg L_{|C} = c_1(L)[C] = \int_C c_1(L_{|C}).$$

Definition 4.1. A line bundle L on an algebraic variety X is *nef* if $L \cdot C \geq 0$ for all proper algebraic curves $C \subset X$.

Recall that the first Chern class $c_1(L)$ is represented by the curvature form:

$$\omega = \frac{i}{2\pi} \partial\overline{\partial} \log h \geq 0$$

of a Hermitian metric h on L. Thus, if L has a Hermitian metric with semipositive curvature, then L is nef: nef is a weak version of positivity for line bundles.

The line bundle L is *eventually free* if $V = \Gamma(X, L^{\otimes N})$ is base-point free for some integer $N > 0$; if that is the case, then L is nef: for every curve $C \subset X$ there exists a section $\sigma \in \Gamma(X, L^{\otimes N})$ that does not vanish identically on C, so $L \cdot C = (1/N) \deg(\sigma_{|C}) \geq 0$.

We say that L is *ample* if $V = \Gamma(X, L^{\otimes N})$ defines an embedding

$$\varphi\colon X \hookrightarrow \mathbb{P}(V^*)$$

for some integer $N > 0$. Although there is no simple-minded 'numerical' characterisation of eventual freedom, there is a remarkable numerical criterion for ampleness. In order to state it, I need more terminology.

Two line bundles L_1, L_2 on X are *numerically equivalent* if

$$L_1 \cdot C = L_2 \cdot C$$

for all proper algebraic curves $C \subset X$, and we denote by $N^1(X, \mathbb{R})$ the real vector space spanned by line bundles, modulo numerical equivalence (in $N^1(X, \mathbb{R})$, the notation $r_1 L_1 + r_2 L_2$ means $L_1^{\otimes r_1} \otimes L_2^{\otimes r_2}$). The dual vector space $N_1(X, \mathbb{R})$ contains the convex cone $\operatorname{NE} X$ generated by (numerical equivalence classes of) algebraic curves $C \subset X$.

Definition 4.2. The *Kleiman–Mori cone* of X is the closure $\overline{\operatorname{NE}}\, X$, of the cone $\operatorname{NE} X \subset N_1(X, \mathbb{R})$.

Theorem 4.3 (Kleiman criterion for ampleness)**.** *Let X be a proper algebraic manifold. A line bundle L on X is ample if and only if $L > 0$ on $\overline{\operatorname{NE}}\, X \setminus \{0\}$.*

Remark 4.4. If X is a proper algebraic manifold, denote by $\operatorname{Amp} X$ the ample cone of X, that is, the open convex subcone of $N^1(X; \mathbb{R})$ generated by the classes of ample line bundles; and by $\operatorname{Nef} X$ the nef cone of X, that is, the closed convex subcone of $N^1(X; \mathbb{R})$ generated by the classes of nef line bundles. In terms of the dual cones, the Kleiman criterion is equivalent to:

$$\overline{\operatorname{Amp}}\, X = \operatorname{Nef} X,$$

that is, the nef cone is the closure of the ample cone; equivalently, the ample cone is the interior of the nef cone.

5. The Canonical Line Bundle

In general, a nonsingular variety X is only entitled to own one nontrivial line bundle (and its tensor powers): the *canonical line bundle*

$$K_X = \bigwedge^{\dim X} T_X^*.$$

One sees that the space of N-canonical differentials $\Gamma(X, K_X^{\otimes N})$ is a birational invariant for proper algebraic manifolds, and so is the N-th *plurigenus*

$$p_N(X) = \dim \Gamma(X, K_X^{\otimes N}).$$

It follows that the images of the N-canonical maps:

$$\varphi_{\Gamma(X,K_X^{\otimes N})}\colon X \dashrightarrow \mathbb{P}^{p_N(X)-1},$$

as well as the graded ring, called the *canonical ring* of X:

$$R(X,K_X)=\bigoplus_{N\geq 0}\Gamma(X,K_X^{\otimes N})$$

have intrinsic meaning in terms of the birational equivalence class to which X belongs. As I explain below, an important consequence of [BCHM09] is that this ring is finitely generated.

By 1915, Castelnuovo and Enriques had taken the classification of surfaces far beyond the statement of existence of minimal models. In particular, they started the view that we should classify proper algebraic manifolds according to the plurigenera $p_N(X)$ and their asymptotic behaviour for large N. Today, we say that κ is the *Kodaira dimension* of X if there are constants $a, b > 0$ and a positive integer $m \in \mathbb{N}$ such that

$$aN^\kappa < p_N(X) < bN^\kappa \quad \text{for all } N \text{ large and divisible by } m.$$

If all $p_N = 0$, we say by convention that $\kappa = -\infty$.

Castelnuovo and Enriques proved, for example, that a nonsingular proper complex *algebraic* surface X has $\kappa \geq 1$ if and only if $p_{12}(X) \geq 2$. (If X is nonsingular compact *complex analytic*, then $\kappa \geq 1$ if and only if $p_{42}(X) \geq 2$.)

From this perspective, two natural classes of varieties are singled out, showing the two extreme behaviours of the canonical line bundle:

Definition 5.1.
- A proper algebraic manifold X is of *general type* if $\kappa(X) = \dim X$, that is, $\varphi_{\Gamma(X,K_X^{\otimes N})}$ is birational on its image for some $N > 0$.
- A proper algebraic manifold X is *Fano* if the *anti-canonical* line bundle K_X^* is ample.

Note the striking difference between the two definitions: the first in terms of birational invariants of X, the second biregular. The space of anticanonical differentials is not a birational invariant: this is why I am forced to define Fano varieties in biregular terms. (In § 9 below, we study the closely related class of proper algebraic manifolds X such that $K_X \notin \overline{\mathrm{Eff}}\, X$, *see* definition 9.2, theorem 9.3, and remark 9.4.)

6. The Cone Theorem

Mori's cone theorem is a very general, striking, and, at first, completely unforeseen statement on the structure of the Kleiman–Mori cone.

Definition 6.1. If $C \subset \mathbb{R}^n$ is a convex cone, and C contains no lines, a ray $R = \mathbb{R}_+[v] \subset C$ is *extremal* if

$$v_1, v_2 \in C \text{ and } v_1 + v_2 \in R \quad \text{implies} \quad v_1, v_2 \in R.$$

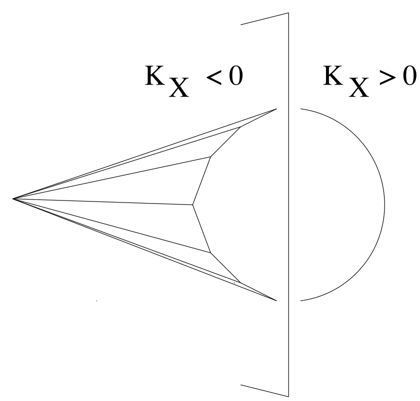

Figure 1. The Kleiman–Mori cone

Theorem 6.2 (Mori, Kawamata, Kollár, Reid, Shokurov,...)**.** *Let X be a projective algebraic manifold. Then*

- *The Kleiman–Mori cone $\overline{\mathrm{NE}}\,X$ is locally finitely rationally generated in the half-space $\{z \mid K_X \cdot z < 0\}$.*
- *If $R \subset \overline{\mathrm{NE}}\,X$ is an extremal ray with $K_X \cdot R < 0$, then there is a morphism $f\colon \mathbb{P}^1 \to X$ such that $R = \mathbb{R}_+[f(\mathbb{P}^1)]$.*

Figure 1 is a picture of the cone theorem; here I refrain from commenting on its meaning, which is discussed in many places.

7. The Contraction Theorem

Theorem 7.1 (Mori, Kawamata, Reid,...)**.** *Let Y be a projective algebraic manifold, and $R \subset \overline{\mathrm{NE}}\,Y$ an extremal ray with $K_Y \cdot R < 0$. There is a morphism $f_R\colon Y \to Z$, the* contraction of R, *characterised by the following two properties:*

1. *f_R 'contracts' R: If $C \subset Y$ is an algebraic curve and $[C] \in R$, then $f(C)$ is a point in Z;*
2. *If $g\colon Y \to W$ also contracts R, then g factors through f_R.*

This statement is a higher dimensional analog of Castelnuovo and Enriques' contraction theorem for -1-curves on nonsingular surfaces. Note the key difference: the input of Castelnuovo–Enriques is a specific locus on Y, a -1-curve, while the input of the contraction theorem is an abstract notion—an extremal ray—about which we have no explicit information.

Theorem 7.2. *Let $f_R\colon Y \to Z$ be the contraction of an extremal ray $R \subset \overline{\mathrm{NE}}\, Y$ as in theorem 7.1 above. Then, f_R is of one of the following types:*

- **divisorial contraction:** *f_R is birational, and contracts an irreducible subvariety $E \subset Y$ of codimension 1 (a divisor);*
- **small contraction:** *f_R is birational, and the exceptional set has codimension ≥ 2;*
- **Mori fibration:** *f_R is not birational, that is, $\dim Z < \dim Y$ and $-K_Y$ is ample on the fibres $f_R^{-1}(z)$, $z \in Z$; in particular, all nonsingular fibres are Fano manifolds.*

The notion of extremal ray and attendant contraction is a generalisation both of the contraction of a -1-curve on a nonsingular surface—a divisorial contraction—*and* of a Fano variety with rank $N^1(X, \mathbb{R}) = 1$—the case $Z = \{\text{point}\}$ of a Mori fibration.

There are no small contractions from a nonsingular 3-fold (but there are plenty from a $\mathbb{Q}$-factorial 3-fold with terminal singularities, *see* the discussion below), but it is easy to find an example in dimension 4:

Example 7.3. Consider a nonsingular projective 4-fold Y containing a surface $S \subset Y$ isomorphic to $\mathbb{P}^2$ and with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The surface S can always be contracted in the category of complex analytic varieties. When S can be contracted in the category of complex projective algebraic varieties, one has often also to contract 'far away' copies of S disjoint from S: in that case, the contraction of S is a small contraction.

Back to Surfaces. If X is a nonsingular projective surface, then:

- A divisorial contraction is the same thing as the contraction of a -1-curve;
- there are no small contractions;
- $f_R\colon X \to Y$ is a Mori fibration if and only if: either every fibre is isomorphic to $\mathbb{P}^1$, or $X = \mathbb{P}^2$ and Y is a point.

Thus, from the perspective of the cone and contraction theorem, we can restate the minimal model theorem for surfaces as follows:

Theorem 7.4. *Let Y be a nonsingular projective surface. Then, every sequence*

$$Y = Y_0 \to Y_1 \to \cdots \to Y_i \to Y_{i+1} \to \cdots$$

of divisorial contractions ends at a surface $Y_r = X$ for which one of the following two alternatives holds:

- X **is a 'minimal' model:** *there are no extremal rays left, that is, by the cone theorem, K_X is nef.*
- **There is a Mori fibration** $f_R \colon X \to Y$**:** *i.e., X has an extremal ray and the attendant contraction is a Mori fibration.*

It would be possible to prove this statement with 1900s methods, and I suspect that Castelnuovo and Enriques would have found it obvious; but even Beauville [Bea78] did not find it useful to state it in this way.

As I will explain, this statement, with one or two important qualifications but otherwise basically unchanged, *does* generalise to higher dimensions, provided that we *define*:

Definition 7.5. A projective variety X is a *minimal model* if X has terminal singularities (these are discussed in § 8 below; for instance, X could be nonsingular) and K_X is nef.

Minimal models are so called not because they are 'smallest' in a literal sense (though they are 'as small as they can be,' *see* [Kol07] for an accessible discussion of precisely this point), but because they generalise the traditional minimal models of surfaces to higher dimensions. To be fair, it is not because they were obsessed with minimal models that are literally 'smallest' that algebraic geometers failed, for almost a century, to make progress in higher dimensions. It required real inspiration to discover the cone theorem, and technical virtuosity to prove it. But it is interesting to note that, with the benefit of hindsight, it is the classical 'Italian' perspective that, today, looks hopelessly technical, whereas the modern view in terms of the sign of the canonical line bundle makes immediate contact with the archetypal classification of geometries into those with positive, zero and negative curvature.

8. Minimal Models in Higher Dimensions

Example 8.1. Consider a nonsingular projective 3-fold Y containing a surface $E \subset Y$, isomorphic to $\mathbb{P}^2$, and with normal bundle $N_Y E \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. Then $E \subset Y$ can be contracted to a *singular* point $P \in Y_1$, such that a small analytic neighbourhood of $P \in Y_1$ is isomorphic to a neighbourhood of the origin in the quotient of $\mathbb{C}^3$ by the action of $\mathbb{Z}/2\mathbb{Z}$ sending x, y, z to $-x, -y, -z$. This contraction is a divisorial contraction.

If Y is nonsingular and $f_R\colon Y \to Y_1$ a divisorial contraction, then we have just seen that Y_1 can be singular. To generalise the theory of minimal models of surfaces to higher dimensions, we must therefore allow some singularities. The smallest class large enough to accommodate all the operations of the minimal model program is $\mathbb{Q}$-factorial varieties with terminal singularities.

$\mathbb{Q}$-factorial. An integral *Weil divisor* on a *normal* algebraic variety X is a formal finite linear combination

$$D = \sum n_i D_i$$

where $n_i \in \mathbb{Z}$ and $D_i \subset X$ is a subvariety of codimension 1. Because X is normal, if $\Gamma \subset X$ is irreducible of codimension 1, then the local ring $\mathcal{O}_{X,\Gamma}$ of X along Γ is a *discrete valuation ring*; this makes it possible to define the divisor of a rational function $f \in Q(X)$:

$$\operatorname{div}_X f = \sum_{\Gamma} (\operatorname{mult}_\Gamma f)\, \Gamma$$

(where the sum is over *all* the irreducible subvarieties of codimension 1; it turns out that the sum is finite). We say that D_1 is *linearly equivalent to* D_2, written $D_1 \sim D_2$, if $D_1 - D_2$ is the divisor of a rational function.

When X is nonsingular, every Weil divisor is *locally* in the Zariski topology the divisor of a rational function.

Definition 8.2. A Weil divisor D is *Cartier* if D is *locally* in the Zariski topology the divisor of a rational function; D is $\mathbb{Q}$-*Cartier* if an integer multiple rD of D is Cartier.
A variety X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier.

I warn you that $\mathbb{Q}$-factorial is a local property in the Zariski but not in the analytic topology.

If X is a variety, a Weil divisor D always gives rise to a *cycle class* $[D] \in H_{2\dim X-2}(X, \mathbb{Z})$ (you can imagine physically triangulating the support of D). If X is $\mathbb{Q}$-factorial, then this class lies in the image of the Poincaré map $P\colon H^2(X;\mathbb{Q}) \to H_{2\dim X-2}(X;\mathbb{Q})$[4]: indeed, the Cartier divisor rD has a cycle class $\operatorname{cl}(rD) \in H^2(X;\mathbb{Z})$, and

$$[D] = P\Big(\frac{1}{r}\operatorname{cl}(rD)\Big).$$

(The converse is also true if X has terminal singularities, *see* [Kol91, Proposition 2.1.7].) Thus, when X is $\mathbb{Q}$-factorial, it makes sense to

[4]The Poincaré map is cap product with the fundamental class $[X] \in H_{2\dim X}(X;\mathbb{Z})$.

take the intersection product $D \cdot C \in \mathbb{Q}$ of a Weil divisor and a proper algebraic curve $C \subset X$.

Divisors and Line Bundles. I recall the relation between divisors and line bundles. When X is normal, we associate a *divisorial sheaf* $\mathcal{O}(D)$ to every Weil divisor D, depending only on the linear equivalence class of D, as follows:

$$\Gamma(\mathcal{O}(D), U) = \big\{ f \in Q(X) \mid \operatorname{div}_U f + D_{|U} \geq 0 \big\}$$

where $U \subset X$ is a Zariski open subset.

Zariski teaches us how to take the divisor $K_X = \operatorname{div}_X \omega$ of a meromorphic differential. This depends on the choice of ω, but the linear equivalence class does not. Although there is no canonical line bundle, $\mathcal{O}(K_X)$ is a well-defined divisorial sheaf.

If D is Cartier, then $\mathcal{O}(D)$ is an invertible sheaf, that is, a line bundle, and $\operatorname{cl}(D) = c_1(\mathcal{O}_X(D)) \in H^2(X; \mathbb{Z})$. (OK I admit it: like all algebraic geometers, I do not distinguish between a line bundle and its sheaf of sections.) Thus, when X is nonsingular, divisor and line bundle are interchangeable notions.

When X is $\mathbb{Q}$-factorial, we can think of a divisorial sheaf $\mathcal{O}(D)$ as a '$\mathbb{Q}$-line bundle,' in the sense that $\mathcal{O}(rD)$ is a line bundle, for some integer $r > 0$. For many purposes, this is just as good as a line bundle.

Terminal Singularities. I do not write down the definition of terminal singularities here, since it is not very enlightening. Instead, I refer you to [Rei87b] for more information. If X has terminal singularities, then the following two key properties hold:

- The canonical divisor is $\mathbb{Q}$-Cartier. Equivalently, the sheaf $\mathcal{O}_X(K_X)$ is a $\mathbb{Q}$-line bundle; in particular, $c_1(K_X)$ makes sense as an element of $H^2(X; \mathbb{Q})$.
- The space $\Gamma\big(X, \mathcal{O}(NK_X)\big)$ of pluricanonical differentials is a birational invariant for all integers $N > 0$.

In fact, if K_X is $\mathbb{Q}$-Cartier and ample, then the second of these two properties implies that X has terminal singularities. (This is how Reid arrived at the definition of terminal singularities.) Having terminal singularities is a local property in the analytic topology.

The Minimal Model Program.

Fact 8.3. *The cone theorem 6.2, the contraction theorem 7.1, and the classification 7.2 of extremal contractions hold word-for-word for projective $\mathbb{Q}$-factorial varieties with terminal singularities.*

If Y is projective $\mathbb{Q}$-factorial terminal, and $f_R \colon Y \to Y_1$ is a divisorial contraction, then Y_1 is also projective $\mathbb{Q}$-factorial terminal.

So far, so good: Starting with Y projective and $\mathbb{Q}$-factorial terminal, we ask the question: is K_Y nef? If not, then $\overline{\mathrm{NE}}\, Y$ has an extremal ray R with $K_Y \cdot R < 0$, and a contraction $f_R \colon Y \to Z$. If f_R is divisorial, then we just do the contraction, and go on asking the same question of Z instead of Y.

Thus, the problem now is: What to do with small contractions? This is a very serious issue: indeed, I now argue that, if Y is $\mathbb{Q}$-factorial terminal and $f_R \colon Y \to Z$ is a small contraction, then $c_1(K_Z)$ can not possibly make sense as an element of $H^2(Z;\mathbb{R})$. Indeed, if it did, then it would follow that

$$c_1(K_Y) = f^* c_1(K_Z)$$

and then, for any curve $C \in R$, by the projection formula:

$$K_Y \cdot C = c_1(K_Y) \cap [C] = c_1(K_Z) \cdot f_*(C) = 0$$

a contradiction, since $K_Y \cdot C$ is supposed to be strictly negative! Hence, if we meet a small contraction $f_R \colon Y \to Z$, we can not just do the contraction and go on with Z: the singularities of Z are such that it makes no sense even to ask the question: is K_Z nef? There is no way that the cone theorem can make sense on Z.

Definition 8.4. If $f_R \colon Y \to Z$ is a small contraction of an extremal ray $R \subset \overline{\mathrm{NE}}\, Y$, the *flip* of f_R is a new small birational morphism $g_R \colon Y' \to Z$ such that $K_{Y'}$ is $\mathbb{Q}$-Cartier and is ample along the fibres of g_R.

Though not immediate from the definition, the flip is unique if it exists, and, if Y is projective $\mathbb{Q}$-factorial terminal, then Y' is again projective $\mathbb{Q}$-factorial terminal. At this point I can state the first main result of Hacon and M^{c}Kernan [HM09].

Theorem 8.5 (Existence of flips)**.** *Flips of small contractions exist.*

A sketch of the proof, and a discussion of major contributions by Shokurov and Siu, can be found in the introduction to [Cor07]. ([Mor88] proved existence of flips in dimension 3.)

If $Y \to Y_1$ is a divisorial contraction, then $H^2(Y_1;\mathbb{R})$ has rank one less than $H^2(Y;\mathbb{R})$. It follows that any sequence of divisorial contractions must stop. Termination of flips, on the other hand, is an important open problem:

Conjecture 8.6 (Termination of flips)**.** *There is no infinite sequence of flips.*

This conjecture holds in dimension ≤ 4 [KMM87, Theorem 5-1-15] but, in general, it seems to be hard. From what I just said, conjecture 8.6 would imply:

Conjecture 8.7. *Let Y be a projective $\mathbb{Q}$-factorial terminal variety. Then any sequence*

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_i \dashrightarrow Y_{i+1} \dashrightarrow$$

of divisorial contractions and flips ends at a variety $Y_r = X$ for which one of the following two alternatives holds:

X is a minimal model: *that is, K_X is nef.*
There is a Mori fibration: $f_R \colon X \to Z$.

9. The Minimal Model Program with Scaling

Running the minimal model program starting with a projective $\mathbb{Q}$-factorial terminal variety Y, we may come to a point where we have several extremal rays to choose from. The minimal model program with scaling is a variant that narrows the choice down in a coherent way and results in a program with a better chance of terminating.

The starting point is a pair (Y, A) of a $\mathbb{Q}$-factorial terminal variety Y and a divisor A on Y such that $K_Y + tA$ is nef for $t \gg 0$. (This holds, for instance, if A is ample.)

In the case of surfaces, the minimal model program with scaling is simply the minimal model program for Y where we contract first the -1-curves that have smallest intersection number with A.

Definition 9.1. The *nef threshold* $t = t(Y, A)$ is the smallest real number ≥ 0 such that $K_Y + tA$ is nef.

More precisely, the minimal model program with scaling is any sequence of divisorial contraction and flips

$$Y = Y_0 \dashrightarrow \cdots \dashrightarrow Y_i \dashrightarrow Y_{i+1} \dashrightarrow$$

specified inductively as follows: A_i is the transform of A_{i-1} on Y_i[5], $t_i = t(Y_i, A_i)$, and $Y_i \dashrightarrow Y_{i+1}$ is a divisorial contraction or flip of an extremal ray $R_i \subset \overline{\mathrm{NE}}\, Y_i$ with

$$(K_{Y_i} + t_i A_i) \cdot R_i = 0 \quad \text{and} \quad K_{Y_i} \cdot R_i < 0.$$

It is important to understand that a minimal model program with scaling is just a minimal model program with self-imposed restriction of choice of extremal contractions.

In order to state the main result of [BCHM09], I need one more concept.

[5]If $Y_{i-1} \dashrightarrow Y_i$ is a divisorial contraction, A_i is the push forward of A_{i-1}; if it is a flip, then Y_{i-1} and Y_i have the same divisors.

Definition 9.2. A Weil divisor $D = \sum d_i D_i$ on an algebraic variety X is *effective* if all $d_i \geq 0$. We denote by $\operatorname{Eff} X \subset N^1(X, \mathbb{R})$ the convex cone generated by effective divisors. The *pseudo-effective cone* is the closure $\overline{\operatorname{Eff}}\, X$ of $\operatorname{Eff} X$; a Weil divisor D is *pseudo-effective* if $D \in \overline{\operatorname{Eff}}\, X$.

If we believe the minimal model conjecture, then it follows that if K_Y is pseudo-effective, then Y is birational to a minimal model; if, on the other hand, K_Y is not pseudo-effective, then Y is birational to the total space of a Mori fibration.

This is one of the main results of [BCHM09]:

Theorem 9.3 (Partial minimal model theorem)**.** *Suppose Y is projective $\mathbb{Q}$-factorial terminal.*

(1) *If Y is of general type, then any minimal model program with scaling, starting from Y, ends at a minimal model X.*
(2) *If K_Y is not pseudo-effective, then any minimal model program with scaling, starting from Y, ends at a Mori fibration $f\colon X \to Z$.*

Remark 9.4. If X is projective $\mathbb{Q}$-factorial terminal, then $K_X \notin \overline{\operatorname{Eff}}\, X$ if and only if *adjunction terminates* on X, that is, for all divisors D on X, there exists $\lambda_0 = \lambda_0(D) > 0$ such that $D + \lambda K_X \notin \operatorname{Eff} X$ for all $\lambda \geq \lambda_0$. Termination of adjunction was a key tool in the 'Italian' treatment of surfaces.

This theorem does not establish the basic dichotomy in the classification of algebraic varieties; but it does show that, under general and useful assumptions, the expected models exist.

10. Finite Generation of Adjoint Rings

One of the most remarkable consequences of the work of [BCHM09] is the finite generation of adjoint rings. These include, as a special case, canonical rings.

Definition 10.1. Let X be a projective algebraic manifold of dimension n. A Weil divisor $D = \sum D_i \subset X$ is *normal crossing* if for every point $x \in D$ there exist analytic co-ordinate functions $z_1, \dots, z_n$ centred at $x \in X$ such that, locally at x, D is given by the equation $z_1 \cdots z_k = 0$, for some $1 \leq k \leq n$.

Normal crossing divisors are important in algebraic geometry, for the following reason. Consider a nonsingular quasi-projective variety U. Then, by Hironaka's resolution theorem, there exists a compactification

$U \subset X$ such that the complement $D = X \setminus U$ is a normal crossing divisor. It is a fact then that the space:

$$H^0(X, K_X + D)$$

of differentials on X with poles along D depends only on U, not on X. The same is true of the *adjoint ring*

$$R(X, K_X + D) = \bigoplus_{n \geq 0} H^0\big(X, n(K + D)\big)$$

(and the same is true for $H^p\big(X, \Omega^q(\log D)\big)$: this is the starting point of Deligne's construction of the functorial mixed Hodge structure on the cohomology of U; but that is another story...).

It is useful to generalise this construction in two directions. First, there is no need for D to be an integral divisor. Indeed, we can consider any divisor B of the form:

$$B = \sum b_i B_i \quad \text{where } b_i \in \mathbb{Q} \text{ and } 0 < b_i \leq 1$$

—such a divisor is called a *boundary*—whose *support* $\sum B_i$ is normal crossing. (To make sense of the adjoint ring, we can e.g. sum $H^0(X, n(K_X + B))$ only over n divisible enough that nB is integral. Note, however, that this is not necessary: the definition of $\mathcal{O}_X(D)$ in § 8 makes perfect sense even when D is a $\mathbb{Q}$-divisor.) Working with pairs (X, B) of a variety and a boundary $\mathbb{Q}$-divisor allows to interpolate between the absolute case $B = 0$ and the quasi-projective case $U = X \setminus D$. Every self-respecting mathematical theory needs the certification that comes with appropriate categorical foundations, and so a theory of pairs is a must.

Second, we can generalise to rings graded by any finitely generated semigroup.

Definition 10.2. We say that a divisor D on X is *big* if the map $\varphi_{\Gamma(X,nD)} \colon X \dashrightarrow \mathbb{P}\Gamma(X, nD)$ is birational onto its image for $n \gg 0$.

Theorem 10.3. [BCHM09] *For X a projective algebraic manifold, let $B_1, \dots, B_r \subset X$ be boundaries, and assume that each B_i individually has normal crossing support. Assume in addition:*

(1) $B_i = \sum b_{ij} B_{ij}$ *with all* $b_{ij} < 1$*;*
(2) *the* B_i *are big.*

Then, the adjoint ring:

$$R(X; K + B_1, \dots, K + B_r) = \bigoplus_{(n_1, \dots, n_r) \in \mathbb{N}^r} H^0\big(X, \sum n_i (K + B_i)\big)$$

is finitely generated.

Remark 10.4. It is conjectured that both additional assumptions in the statement can be removed. When $r = 1$, that is, there is only one boundary divisor, both assumptions can be removed in dimension ≤ 4 [Fuj08], and the second assumption can be removed in all dimensions [BCHM09].

In [BCHM09], the theorem follows from the partial minimal model theorem 9.3. The 'tendencious' perspective of [Rei87a], which, in fact, has since been the main-stream perspective, is that one should first construct minimal models, and only after that is done one should attempt to show finite generation of the canonical ring. It seems to me that it might now be possible once again to turn this perspective on its head. The recent work of Lazić [Laz09] proves theorem 10.3 by means of a transparent (in principle) argument based on lifting lemmas and induction on the dimension, not relying on any of the details of the minimal minimal program.

11. What Remains to Do: Dichotomy and Trichotomy

In the first instance, it would be desirable to establish the minimal model program—perhaps with scaling—and the basic dichotomy, in full generality.

The classification theory of algebraic varieties aims, in the second instance, to establish the following basic trichotomy.

Conjecture 11.1 (Abundance Conjecture)**.** *If X is a minimal model, then K_X is eventually free.*

In particular the statement implies that, if X is a minimal model, then $K_X \in \operatorname{Eff} X$; combining with the minimal model conjecture, this allows us to restate the basic dichotomy for any Y: either Y is birational to a minimal model, and then $K_Y \in \operatorname{Eff} Y$; or Y is birational to a Mori fibration, and then $K_Y \notin \overline{\operatorname{Eff}}\, Y$.

If the conjecture is true, then when X is a minimal model we have a *canonical morphism*:

$$\varphi_{\Gamma(X,NK_X)} \colon X \to S$$

where $\dim S = \kappa(X)$ and the fibres have $rK = 0$ for some integer $r > 0$. The base S of the canonical morphism is the canonical model of X; we recover it as:

$$S = \operatorname{Proj} R(X, K_X).$$

The canonical morphism allows to divide all varieties into the three classes with $K_X > 0$, $K_X \equiv 0$ and $K_X < 0$, and fibre spaces of these.

Essentially, the basic trichotomy is the philosophical core of the *Iitaka program*, which is surveyed in [Mor87].

The abundance conjecture is known in dimension 2 and 3; the surface case is surveyed in [Rei97] and the 3-fold case in [FA92]. Even the surface case is hard, relying on the detailed classification of elliptic surfaces due to Kodaira (and Bombieri, Mumford, Katsura–Ueno in characteristic p).

The finer effective classification of special classes of surfaces in terms of plurigenera by Castelnuovo and Enriques is extremely useful in applications to diverse areas of mathematics. A similar explicit treatment of 3-fold is the focus of much current research, but is presumably intractable in higher dimensions.

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