$\mathbb{A}^1$-HOMOTOPY AND $\mathbb{A}^1$-ALGEBRAIC TOPOLOGY OVER A FIELD, A SURVEY

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Fabien Morel[1]

Contents

This text is related to the 3 Lectures I gave at PCMI, IAS/Park City Mathematics Institute, July 7-27, 2024 in Park City, Utah [40]. The content of this text is a slight extension of those, in particular I tried to give more precise descriptions of the objects in play.

The reader is assumed to have a good acquaintance with classical homotopy theory, algebraic topology, as well as with some basis in algebraic geometry over a field.

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1. Introduction: the classical (hi-)story

To start, let us recall some standard facts from homotopy theory (see [51][20] for instance, also [47] which is more oriented to stable homotopy theory).

let $\mathcal{H}$ be the homotopy category of C.W.-complexes whose objects are C.W.-complexes (see *loc. cit.*), or quickly said, cellular complexes, and whose morphisms are homotopy classes of continuous maps between those. This category slowly emerged in history as a fundamental one. An isomorhism in that category is what is known as a homotopy equivalence. The classification or understanding of the homotopy types of C.W.-complexes, that is to say the C.W.-complexes up to homotopy equivalence, is a very hard problem.

In the same lines, the computation or understanding of the set of morphisms in $\mathcal{H}$ between two fixed C.W.-complexes is extremely difficult. To cite only one example: the computation of the homotopy classes of maps from an i-sphere S^i to an n-sphere S^n for $i \geq n$ integers, in other words the so called homotopy groups of spheres, is an incredibly mysterious and unsolved problem (althought some general facts are known).

Homotopy groups

For $n \geq 0$ recall that the n-th sphere, that we denote by S^n is the closed subset $S^n \subset \mathbb{R}^{n+1}$ defined by the equation $\Sigma_{i=0}^{n}(x_i)^2 = 1$ where the x_i are the coordinates in $\mathbb{R}^{(n+1)}$ (also clalled the unit sphere). It is a differentiable compact submanifold of $\mathbb{R}^{(n+1)}$ of dimension n.

By choosing for instance $(1, 0, \ldots, 0) \in S^n$ as base point, it becomes a pointed space. For $n \geq 1$, $S^n = \Sigma(S^{(n-1)})$ is the suspension of the $(n-1)$-sphere. As a consequence, for any pointed topological space X, the pointed set $\pi_n(X)$ of pointed homotopy classes of pointed continuous maps $S^n \to X$ is a group, abelian for $n \geq 2$. The first one, $\pi_1(X)$, is called the fundamental group of X, and the $\pi_n(X)$, $n \geq 2$, are the higher homotopy groups of X, which are all abelian.

Let n and i be integers ≥ 1. To study $\pi_i(S^n)$ one may observe by a standard argument that any element $\alpha \in \pi_i(S^n)$ is represented by a differentiable map f. By Sard's theorem, f always admits regular values $x \in S^n$, that is to say such that f is transversal at x, which means that the differential of f is an epimorphism at any point whose image is x. If $i \leq (n-1)$ this means that $f^{(-1)}(x)$ is empty. Thus f maps into $S^n - \{x\} \cong \mathbb{R}^n$ and thus is homotopic to a constant map. It follows that

$$\pi_i(S^n) = 0$$

for $i \leq (n-1)$.

For $i = n$, the same reasonning shows that for any point y_j in $f^{(-1)}(x)$ the differential df_{y_j} has to be an isomorphism between the tangent spaces. So, by the local inversion theorem, the y_j are disjoints and isolated, so finitely many. As S^n is canonically oriented, we may assigne a sign $\epsilon_j \in \{\pm 1\}$ depending on whether df_{y_j}, which is an isomorphism from the tangent space at y_j to the tangent space at x, preserves $(+1)$ or reverses (-1) the orientations.

The number

$$deg(f;x) = \Sigma_{y_j \in f^{(-1)}(x)}\, \epsilon_j \in \mathbb{Z}$$

can be shown to be independent of the choice of x and also on the representent f of $\alpha \in \pi_n(S^n)$. It is called the Brouwer degree of α. It can be shown that

$$deg : \pi_n(S^n) \cong \mathbb{Z}\ ,\ \ \alpha \mapsto deg(\alpha)$$

is a group isomorphism.

Of course it is well known that the $\pi_i(S^n)$ for $i > n$ are harder to compute, and still unknown for i big enough.

Vector bundles and characteristic classes

Let $r \geq 1$ and $BGL_r(\mathbb{R})$ be the classifying space of the topological group $GL_r(\mathbb{R})$; $BGL_r(\mathbb{R})$ is a pointed connected C.W.-complex whose loop space is an h-group isomorphic to $GL_r(\mathbb{R})$ in the category of group objects in the pointed homotopy category. An important result in algebraic topology is that over a C.W.-complex X, real vector bundles of rank r up to isomorphism form a set canonically in bijection with the set of homotopy classes of continuous maps from X to $BGL_r(\mathbb{R})$. This canonical bijection is explicit, taking any continous map $f : X \to BGL_r(\mathbb{R})$ to the pull-back of the universal rank r-vector bundle ξ_r on $BGL_r(\mathbb{R})$ through f.

This theory has a lot of applications, like characteristic classes, and obstruction theory to split off trivial factors. For instance, given an oriented rank n-vector bundle ξ over an n-dimensional C.W.-complex X, there is an natural characteristic class $e(\xi) \in H^n(X;\mathbb{Z})$, the Euler class, whose vanishing is equivalent to the fact that ξ split off a trivial rank 1 vector bundle (or ξ has a nowhere vanishing section), see for instance [29].

Geometric topology

Let $\mathcal{D}iff$ be the categoy of (nice[2]) differentiable manifolds. The obvious functor

$$\mathcal{D}iff \to \mathcal{H}$$

taking a differentiable manifold to its underlying topological space (which admits a C.W.-complex structure) has amazingly fine properties, which led a long time ago to the fact that in big enough dimension (at least 5) compact differentiable manifolds can be classified up to diffeomorphisms essentially by invariants coming from $\mathcal{H}$ through this functor; to

[2] we will not make this precise

be more precise, coming from some rather subtle enhancement of that functor, involving the fundamental group, Poincaré duality for the singular chain complex of the classifying space, the map classifying the normal bundle, etc...

Let us illustrate this fact, by one of the main example of the very fine connection between differential geometry and homotopy theory. Let $M \in \mathcal{D}iff$ be a (nice) compact differentiable manifold of dimension n. Choose a proper embedding $i : M \subset S^{n+N}$ into a big enough sphere. One may always find a tubular neighborhood $\mathcal{T}$ of M in S^{n+N}, that is to say an open subset of S^{n+N} containing M and diffeomorphic to the total space of the normal bundle ν_i of the embedding i (which is of rank N). The Thom-Pontryagin construction consists in collapsing the complement Z of $\mathcal{T}$ in S^{n+N} to a point. Clearly the space S^{n+N}/Z is homeomorphic to the one point compactification of $\mathcal{T}$, which is also homeomorphic to the one point compactification of ν_i which is called the Thom space of ν_i and denoted by $Th(\nu_i)$. This Thom-Pontryagin construction thus defines a continous pointed map

$$S^{n+N} \to Th(\nu_i)$$

Now the classifying map of $\nu_i : M \to BGL_N(\mathbb{R})$, which we mentioned above, induces a continuous pointed map

$$Th(\nu_i) \to Th(\xi_N)$$

between the Thom spaces of ν_i and the Thom space of the universal rank N-bundle ξ_N (note that one has to modify a bit the definition of the Thom space to make it work when the base is non compact). The composition of the Thom-Pontryagin construction $S^{n+N} \to Th(\nu_i)$ and the above map yields a pointed map:

$$S^{n+N} \to Th(\xi_N)$$

This defines an element in $\pi_{n+N}(Th(\xi_N))$, the $n + N$-th homotopy group of $Th(\xi_N)$. If $MGl_\infty(\mathbb{R})$ (denoted MO in the original paper of Thom [48]) denotes the Thom spectrum obtained by using all the $Th(\xi_N)$ and the corresponding maps from the suspension of $Th(\xi_N)$ to $Th(\xi_{N+1})$ [47], this construction define a class

$$[M] \in \pi_n^S(MGl_\infty(\mathbb{R}))$$

the n-th stable homotopy group of the spectrum $MGl_\infty(\mathbb{R})$. This class is called the fundamental class of M in $MGl_\infty(\mathbb{R})$. First this class can be shown to be independent on any choice (even of i), and only depends on M as the notation suggests.

Secondly the main theorem of Thom [48], says that M can be recovered up to unoriented cobordism from the class $[M]$. The ring $\pi_*^S(MGl_\infty(\mathbb{R}) = \pi_*(MO)$ is indeed isomorphic to the unoriented cobordism ring and was entirely computed by Thom.

Amazingly, a very sophisticated refinement of these theorems, called *surgery theory*, relying on very subtle technics of "plumbing", introduced by John Milnor, allows one to classify diffeomorphisms classes of compact differential manifolds of dimension at least 5 in terms of algebraic invariants obtained from homotopy theory, see [13][14]. This beautiful theory, with which I got acquainted thanks to my advisor Jean Lannes, has had[3] a huge influence in my attempt to developp the $\mathbb{A}^1$-homotopy theory with Vladimir Voevodksy [36]. Observe also that the theory of algebraic cobordism was achieved in [24], from the geometric point of view, and that Marc Levine proved that the geometric cobordism coincides with the homotopical one [22], that is the stable $\mathbb{A}^1$-homotopy groups of the algebraic Thom spectrum MGL (introduced by V. Voevodksy).

In the present text, I won't touch stable $\mathbb{A}^1$-homotopy theory at all. I will only address the unstable aspects of the role $\mathbb{A}^1$-homotopy theory may play in algebraic geometry. I will also only address the case where the base scheme is the spectrum of a (perfect) field k.

This text is organized as follows. In the next section I will outline the content of the fundational paper [36], in which one of the main aim was exactly to make available the Thom-Pontryagin construction in algebraic geometry. In the following section, I will explain quickly the content of [32], see also [31], giving the basic structure of the $\mathbb{A}^1$-homotopy sheaves and of the corresponding Postnikov towers. In the last section I will deal with more recent applications, developpments, and open problems without being exhaustive, mostly dealing with smooth (projective) $\mathbb{A}^1$-connected k-schemes.

Acknowledgments. This text is obviously influenced by [36] in section 2 and by [32][31] in section 3. In the last section 4 it is, at several places, influenced by two long collaborations [7] [34] and discussions I had in the past decade(s) with Aravind Asok and Anand Sawant.

Conventions. Everywhere in these notes, k denotes a fixed field, which will be assumed to be perfect in section 3 and 4. Sometimes, at the end of these notes in section 4, k will be even assumed to be algebraically closed. As we will never change the base field, we will drop k from most of the standard notations. For instance, $\mathbb{A}^n$ means the n-th dimensional affine space over k; $\mathbb{P}^n$ means n-th dimensional projective space over k, $\mathbb{G}_m$, GL_n as well. Every object is, unless otherwise stated defined over k.

2. $\mathbb{A}^1$-HOMOTOPY THEORY

In this section we quickly outline the content of the fundational paper [36].

2.1. **Spaces over k.** We let Sm_k denote the category of smooth (separated) k-schemes [18][26][27]. This is the perfect analogue in algebraic geometry over k of the category $\mathcal{D}iff$.

In classical topology, the category $\mathcal{D}iff$ maps obviously to the category $\mathcal{T}op$ of topological spaces and continuous maps by forgetting the structure. In fact, for nice differentiable

[3]for the author

manifolds (as assumed above), the underlying topological space always admits a structure of C.W.-complex. So in that case, the corresponding homotopy category falls in your hands: it is the category $\mathcal{H}$ whose objects are topological spaces admitting a structure of C.W.-complex and homotopy classes of continuous maps between those.

One of the aims of [36] was precisely to define the analogue of the category of "topological spaces", and its associated homotopy category $\mathcal{H}_{\mathbb{A}^1}(k)$ of *spaces over* k to which Sm_k maps naturally. Of course that homotopy category had to satisfy some natural properties. The invariance by $\mathbb{A}^1$, meaning that $\mathbb{A}^1$ should be contractible[4]. Moreover one of the most important other natural property, in fact the only other one, was to make available the Thom-Pontryagin construction.

A way to define a natural category of *spaces* containing your favorite given category of geometric objects is given by Grothendieck theory of sheaves of sets for a given topology. We choosed for reasons exactly related to the Thom-Pontryagin construction mentioned above, to choose the Ninevich topology on the site Sm_k. Then we call a *space* over k a simplicial sheaf of sets in the Nisnevich topology on the category Sm_k. We denote that category by $Space(k)$, thus $Space(k) = \Delta^{op}Shv_{Nis}(Sm_k)$ is the category of simplicial objects in the category $Shv_{Nis}(Sm_k)$ of sheaves of sets in the Nisnevich topology on Sm_k. For recollection on the Nisnevich topology see [36] for instance.

Let us just mention of few things. Call a commutative square in Sm_k:

$$\begin{array}{ccc} V & \subset & Y \\ \downarrow & & \downarrow \\ U & \subset & X \end{array}$$

an *elementary distinguished square* [36], or simply a *Nisnevich square*, if the right vertical morphism $f : Y \to X$ is étale, $U \subset X$ is an open immersion, $V = f^{-1}(U)$ is the inverse image of U in Y, and if the induced morphism (by f)

$$(Y - V)_{red} \to (X - U)_{red}$$

is an isomorphism of schemes. One may show (*loc. cit*) that a functor $\mathcal{F} : (Sm_k)^{op} \to Set$ is a sheaf of sets in the Nisnevich topology if and only if for any elementary distinguished square as above the induced map of sets:

$$F(X) \cong F(Y) \times_{F(V)} F(U)$$

is a bijection.

An example of elementary distinguished square is when $Y \to X$ is an open immersion and $\{U, Y\}$ form an open covering of X. In that case V is the intersection. In particular a sheaf of sets in the Nisnevich topology is a sheaf in the Zariski topology. But there

[4] as most of the cohomological invariants for smooth k-schemes are $\mathbb{A}^1$-invariant

are many elementary distinguished square in which f is not an open immersion, see *loc. cit.*

For any $X \in Sm_k$, the presheaf of sets on Sm_k:

$$Hom_{Sm_k}(-, X) : (Sm_k)^{op} \to Set \ , \ Y \mapsto Hom_{Sm_k}(Y, X)$$

is a sheaf of sets in the Nisnevich topology, called the sheaf represented by X. The induced functor $Sm_k \to Shv_{Nis}(Sm_k)$ so obtained is a full embedding and we will not distinguish between a smooth k-scheme X and its image in $Shv_{Nis}(Sm_k)$, which will be thus simply denoted by X. By considering a sheaf of sets as a constant simplicial object (where all the morphisms in the simplicial structure are the identity), we see that Sm_k is also a full subcategory of $Space(k)$; again we will not distinguish bewteen a smooth k-scheme X and its associated space in $Space(k)$.

Now all the usual constructions in topology are available in $Space(k)$. Quotients, push forward, pull-back, etc.. For instance if $U \subset X$ is an open immersion with $X \in Sm_k$, X/U means the space obtained by collapsing U to the point $Spec(k) = *$. If G is a sheaf of groups, for instance an algebraic group scheme in Sm_k, BG is the classifying space of G obtained by the usual simplicial formulas (see [36] for instance).

2.2. **Homotopy theories of spaces over k.** The $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1}(k)$ is obtained from that the category of spaces by formally inverting the class $W_{\mathbb{A}^1}$ of $\mathbb{A}^1$-weak equivalences in the line of the model categories of Quillen [41]. The class $W_{\mathbb{A}^1}$ of $\mathbb{A}^1$-weak equivalences is formally generated by a standard process in localization of categories from two types of morphisms in the category of spaces. The simplicial weak equivalences, denoted by W_s, and the collection of projections $pr_{\mathcal{X}} : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$, for any space $\mathcal{X}$; see [36].

Recall that for any point $x \in X \in Sm_k$, there is a functor "stalk at x" : $Shv_{Nis}(Sm_k) \to Set$ which takes a sheaf of sets F to its stalk F_x, the colimit over the (opposite) category of Nisnevich neighborhoods $\Omega \to X$ of x of the sets $F(\Omega)$. A Nisnevich neighborhood $f : \Omega \to X$ of x being an étale morphism such that $f^{-1}(x)$ is a set with one point $y \in \Omega$ such that the induced field extension $\kappa(x) \subset \kappa(y)$ on the residue fields respectively of x and y is an isomorphism.

The stalk at x of the affine line $\mathbb{A}^1_k$ is the henselisation $\mathcal{O}^h_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ of X at x. By abuse of notations we also sometimes write $F(\mathcal{O}^h_{X,x})$ for the stalk at x of a sheaf $\mathcal{F}$.

A morphism of spaces over k: $f : \mathcal{X} \to \mathcal{Y}$ is then called a *simplicial weak equivalence*, if for any point $x \in X \in Sm_k$ the induced map of simplicial sets

$$f_x : \mathcal{X}_x \to \mathcal{Y}_x$$

is a weak equivalence, see [41]. We let $\mathcal{H}_s(k)$ be the category obtained by inverting the simplicial weak equivalences, also called the simplicial homotopy category of spaces. This category was known for a long time, and was in fact almost already considered in [19], as the notion of hypercovering, introduced by Verdier, is a particular case of a morphism whose stalks are all trivial Kan fibrations of simplical sets.

The class of $\mathbb{A}^1$-weak equivalences $W_{\mathbb{A}^1}$ can then be otained by a formal process to make each projection $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$, for $X \in Sm_k$, an $\mathbb{A}^1$-weak equivalence. The class of morphisms so obtained is denoted by $W_{\mathbb{A}^1}$, thus somehow $W_{\mathbb{A}^1}$ is "generated" by W_s and the projections $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$. Then the $\mathbb{A}^1$-homotopy category of spaces over k, denoted by $\mathcal{H}_{\mathbb{A}^1}(k)$, is the category obtained by inverting formally any morphism of $W_{\mathbb{A}^1}$ in the category of spaces. As clearly, by definition, any simplicial weak equivalence is an $\mathbb{A}^1$-equivalence, that is $W_s \subset W_{\mathbb{A}^1}$, the category $\mathcal{H}_{\mathbb{A}^1}(k)$ is also obtained from $\mathcal{H}_s(k)$ by inverting (the image of) $W_{\mathbb{A}^1}$. One may show [36] that it is obtained from a simplicial model category structure in the sense of [41][15].

The passage from $\mathcal{H}_s(k)$ to $\mathcal{H}_{\mathbb{A}^1}(k)$ can be explained a bit further:

Proposition 2.1. [36] *The obvious functor $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to \mathcal{H}_{\mathbb{A}^1}(k)$ which inverts the $\mathbb{A}^1$-weak equivalences in $\mathcal{H}_s(k)$ admits a right adjoint*

$$\mathcal{H}_{\mathbb{A}^1}(k) \to \mathcal{H}_s(k)$$

which is a full embedding.

Thus $L_{\mathbb{A}^1} : \mathcal{H}_s(k) \to \mathcal{H}_{\mathbb{A}^1}(k)$ is the left adjoint to the full embedding $\mathcal{H}_{\mathbb{A}^1}(k) \subset \mathcal{H}_s(k)$. For any space $\mathcal{X}$, the unit of the adjunction is a morphism in $\mathcal{H}_s(k)$ of the form:

$$\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$$

A space $\mathcal{X}$ will be called $\mathbb{A}^1$-local if an only if the previous morphism (in $\mathcal{H}_s(k)$): $\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$ is a simplical weak equivalence. The space $L_{\mathbb{A}^1}(\mathcal{X})$ is called the $\mathbb{A}^1$-localization of $\mathcal{X}$. Thus the $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1}(k)$ can also be viewed as the full subcategory of $\mathcal{H}_s(k)$ whose objects are the $\mathbb{A}^1$-local spaces. And in fact, by definition of $W_{\mathbb{A}^1}$, a space $\mathcal{X}$ can be shown to be $\mathbb{A}^1$-local exactly if the morphism (in $\mathcal{H}_s(k)$) : $\mathcal{X} \to R\underline{Hom}(\mathbb{A}^1, \mathcal{X})$ from $\mathcal{X}$ to the (derived) function space of morphisms from $\mathbb{A}^1$ to $\mathcal{X}$ is a simplicial weak equivalence; see [36] where the formal construction of $L_{\mathbb{A}^1}$ is explained.

Spaces which have isomorphic $\mathbb{A}^1$-localization are called (weakly) $\mathbb{A}^1$-equivalent. In the sequel we will also use $\mathbb{A}^1$-equivalence to mean $\mathbb{A}^1$-weak equivalence.

Examples. 1) The projective line over k, $\mathbb{P}^1$ is $\mathbb{A}^1$-equivalent to the simplicial suspension $\Sigma(\mathbb{G}_m)$ of the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ pointed by 1. The simplicial suspension means the one obtained from the model category structure on spaces [41]. To see this use the natural covering of $\mathbb{P}^1$ by its two (contractible) $\mathbb{A}^1 \subset \mathbb{P}^1$ and whose intersection is exactly $\mathbb{G}_m$. The cartesian commutative square:

$$\begin{array}{ccc} \mathbb{G}_m & \subset & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \subset & \mathbb{P}^1 \end{array}$$

is also cocartesian and is a homotopy push-out square as $\mathbb{P}^1$ is the union of the two $\mathbb{A}^1$'s. Mapping it to the homotopy push-out square

$$\begin{array}{ccc} \mathbb{G}_m & \to & * \\ \downarrow & & \downarrow \\ * & \to & \Sigma(\mathbb{G}_m) \end{array}$$

defines an $\mathbb{A}^1$-equivalence in $\mathcal{H}_{\mathbb{A}^1}(k)$:

$$\mathbb{P}^1 \cong \Sigma(\mathbb{G}_m)$$

as the morphisms $\mathbb{A}^1 \to *$ are $\mathbb{A}^1$-equivalences by definition!

2) In the same spirit, $\mathbb{A}^2 - \{0\}$ is $\mathbb{A}^1$-equivalent to the simplicial suspension $\Sigma\big((\mathbb{G}_m)^{\wedge 2} \big)$ of the smash product of $\mathbb{G}_m$ by itself. To see that we observe that $\mathbb{A}^2 - \{0\}$ is covered by $\mathbb{G}_m \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{G}_m$, whose intersection is $\mathbb{G}_m \times \mathbb{G}_m$. Thus the cartesian commutative square:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \subset & \mathbb{G}_m \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{G}_m & \subset & \mathbb{A}^2 - \{0\} \end{array}$$

is also a homotopy push-out square. Collapsing the $\mathbb{A}^1$ we obtain the so-called join of $\mathbb{G}_m$ and $\mathbb{G}_m$, which is well-known to be the simplicial suspension $\Sigma\big((\mathbb{G}_m)^{\wedge 2} \big)$; to see it, embed the following trivial homotopy push-out square:

$$\begin{array}{ccc} \mathbb{G}_m \vee \mathbb{G}_m & \subset & \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \subset & * \end{array}$$

into the above square, using the base point 1 in $\mathbb{G}_m$, and take the quotient termwise in the squares, to obtains an $\mathbb{A}^1$-weak equivalence:

$$\mathbb{A}^2 - \{0\} \cong \Sigma\big((\mathbb{G}_m)^{\wedge 2} \big)$$

More generally, using the same technics, one gets that, for $n \geq 1$, $\mathbb{A}^n - \{0\}$ is $\mathbb{A}^1$-equivalent to the $(n-1)$-suspension $\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$ of the n-th smash power of $(\mathbb{G}_m)$:

$$\mathbb{A}^n - \{0\} \cong \Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$$

2.3. $\mathbb{A}^1$-Homotopy sheaves and Postnikov towers.

From now on, if $\mathcal{X}$ and $\mathcal{Y}$ are spaces, we will simply denote by $[\mathcal{X},\mathcal{Y}]$ the set of morphisms $Hom_{\mathcal{H}_{\mathbb{A}^1}(k)}(\mathcal{X},\mathcal{Y})$ from $\mathcal{X}$ to $\mathcal{Y}$ in the $\mathbb{A}^1$-homotopy category, and if $\mathcal{X}$ and $\mathcal{Y}$ are pointed, $[\mathcal{X},\mathcal{Y}]_\bullet$ will be the pointed set of morphisms in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1,\bullet}$. The latter is the homotopy category obtained from the category of pointed spaces by inverting pointed $\mathbb{A}^1$-weak equivalences between pointed spaces, that is just morphisms of pointed spaces which are $\mathbb{A}^1$-weak equivalence after forgetting the base points.

Sheaf of $\mathbb{A}^1$-connected components

For a space $\mathcal{X}$ we let $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ denote the associated sheaf (of sets) on Sm_k to the presheaf $U \mapsto Hom_{\mathcal{H}_{\mathbb{A}^1}(k)}(U,\mathcal{X}) = [U,\mathcal{X}]$. This sheaf is called the sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$.

A sheaf of sets $\mathcal{F} : (Sm_k)^{op} \to \mathcal{S}et$ is said to be $\mathbb{A}^1$-invariant if for any $X \in Sm_k$, the map $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$ induced by the projection $\mathbb{A}^1 \times X \to X$ is a bijection. A space $\mathcal{X}$ such the canonical morphism:

$$\mathcal{X} \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

is an $\mathbb{A}^1$-equivalence is said to be $\mathbb{A}^1$-discrete. In fact this implies automatically that the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant. A space $\mathcal{X}$ is said to be $\mathbb{A}^1$-rigid if the canonical morphism $\mathcal{X} \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism of sheaves; observe that automatically an $\mathbb{A}^1$-rigid space is a sheaf of sets, with trivial simplicial structure, which is $\mathbb{A}^1$-invariant. For instance $\mathbb{G}_m$, or $\mathbb{A}^1$ minus n points ($n > 0$), or a smooth curve of genus > 0 is $\mathbb{A}^1$-rigid. Abelian varieties, open subsets in those are $\mathbb{A}^1$-rigid.

Remark 2.2. Contrary to an old conjecture of myself, which was disproved recently by Joseph Ayoub [8], the sheaf of sets $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}^1$-invariant in general. However, if X is a smooth k-scheme, in all the known examples, $\pi_0^{\mathbb{A}^1}(X)$ is an $\mathbb{A}^1$-invariant sheaf.

A space $\mathcal{X}$ is said to be $\mathbb{A}^1$-connected if $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is the point (the final object of the category of sheaves of sets). For instance $\pi_0^{\mathbb{A}^1}(\mathbb{P}^1) = \pi_0^{\mathbb{A}^1}(\mathbb{A}^1) = *$; more generally a smooth k-scheme which can be covered by affine spaces is $\mathbb{A}^1$-connected. For instance $\mathbb{P}^n$.

If C is a smooth curve which is $\mathbb{A}^1$-connected, one may show that C is either isomorphic to $\mathbb{A}^1$ or to $\mathbb{P}^1$.

A slightly more subtle example of $\mathbb{A}^1$-connected smooth scheme is $\mathbb{A}^n - \{0\}$ for $n \geq 2$; but this follows from the fact mentionned above, that it is $\mathbb{A}^1$-equivalent to $\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$ and as $n-1 \geq 1$, it is a simplicial suspension, which is always $\mathbb{A}^1$-connected.

Observe that if a smooth scheme X is $\mathbb{A}^1$-connected, it must be irreducible, and it must admit at least a rational k-point.

Higher $\mathbb{A}^1$-Homotopy sheaves

Let $\mathcal{X}$ be a pointed space, and $n \geq 1$. We let $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ denote the sheaf on Sm_k associated to the presheaf of groups

$$U \mapsto [\Sigma^n(U_+), \mathcal{X}]_\bullet$$

Here U_+ mean the space obtained by adding a (disjoint) base point to the smooth k-scheme U, and Σ^n is the n-th suspension functor.

This is a sheaf of groups for $n = 1$ and a sheaf of abelian groups for $n \geq 2$. It is called the n-th $\mathbb{A}^1$-homotopy sheaf of the pointed space $\mathcal{X}$ and is denoted by $\pi_n^{\mathbb{A}^1}(\mathcal{X})$.

Unlike the $\pi_0^{\mathbb{A}^1}$, the higher $\mathbb{A}^1$-homotopy sheaves of a pointed space $\mathcal{X}$ have all the expected properties as we will see below.

Definition 2.3. 1) A sheaf of groups G is said to be *strongly* $\mathbb{A}^1$*-invariant* if for any $X \in Sm_k$ and any $i \in \{0, 1\}$,

$$H^i_{Nis}(X; G) \cong H^i_{Nis}(X \times \mathbb{A}^1; G)$$

2) A sheaf of abelian groups M is said to be *strictly* $\mathbb{A}^1$*-invariant* if for any $X \in Sm_k$ and any $i \in \mathbb{N}$,

$$H^i_{Nis}(X; M) \cong H^i_{Nis}(X \times \mathbb{A}^1; M)$$

Remark 2.4. 1) A non trivial result [32] when the field k is perfect is that a sheaf of abelian groups which is strongly $\mathbb{A}^1$-invariant is automatically strictly $\mathbb{A}^1$-invariant. Thus 1) and 2) coincide when they both make sense at the same time, that is to say for sheaves of abelian groups. The hard part of course is that 1) $\Rightarrow$ 2). The proof of this fact was recently simplified by J. Ayoub [9].

2) A constant sheaf of abelian groups, the sheaf represented by an abelian variety over k, the multiplicative group $\mathbb{G}_m$ are examples of strictly $\mathbb{A}^1$-invariant sheaves.

3) Voevodsky's $\mathbb{A}^1$-invariant sheaves with transfers [53][52] are also strictly $\mathbb{A}^1$-invariant sheaves. In fact these were the basic first examples of strictly $\mathbb{A}^1$-invariant sheaves.

Theorem 2.5. [32] *Assume k is a perfect field. Let $\mathcal{X}$ be a pointed space. Then the sheaf $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant, and the sheaves $\pi_n^{\mathbb{A}^1}$, for $n \geq 2$, are strictly $\mathbb{A}^1$-invariant.*

Remark 2.6. The proof of this fact was also recently simplified by J. Ayoub [9].

From the previous theorem follow formally some fundamental results. Using the same procedure as above in the simplicial homotopy category of pointed spaces, one may define the simplicial homotopy sheaves $\pi_n(\mathcal{X})$ of a pointed space. A space $\mathcal{X}$ is said to be simplicially 0-connected if its simplicial sheaf $\pi_0(\mathcal{X})$ of simplicial connected components is reduced to a point. Then $\mathcal{X}(k)$ is non empty as $\mathcal{X}_0 \to \pi_0(\mathcal{X})$ is always an epimorphism, so there are always k-rational points. One says moreover that $\mathcal{X}$ is $(n-1)$-connected, $n \geq 1$, if for a chosen base point $\pi_i(\mathcal{X})$ is trivial for $i \leq n-1$ (this property is independent of the choice of a base point).

Corollary 2.7. *Assume k is a perfect field. Let $n \geq 1$, and let $\mathcal{X}$ be a simplicially $(n-1)$-connected space. Then its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(\mathcal{X})$ is also $(n-1)$-connected, that is to say*

$$\pi_i^{\mathbb{A}^1}(\mathcal{X}) = 0$$

for $i \leq (n-1)$.

A space with the property of the corollary is said to be $(n-1)$-$\mathbb{A}^1$-connected. For instance the n-th suspension $\Sigma^n(\mathcal{X})$ of a pointed space $\mathcal{X}$ is $(n-1)$-$\mathbb{A}^1$-connected. Thus from what we have seen above, $\mathbb{A}^n - \{0\}$ is always $(n-2)$-$\mathbb{A}^1$-connected as it is $\mathbb{A}^1$-equivalent to $\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$. In the same way $(\mathbb{P}^1)^{\wedge n}$ is $(n-1)$-$\mathbb{A}^1$-connected.

Let $\mathcal{G}r(k)$ be the category of sheaves of groups on Sm_k in the Nisnevich topology, and let $\mathcal{G}r_{\mathbb{A}^1}(k)$ be its full subcategory consisting of strongly $\mathbb{A}^1$-invariant sheaves of groups. The inclusion $\mathcal{G}r_{\mathbb{A}^1}(k) \subset \mathcal{G}r(k)$ has a left adjoint $\mathcal{G}r \to \mathcal{G}r_{\mathbb{A}^1}(k)$, $G \mapsto G_{\mathbb{A}^1}$. For $G \in \mathcal{G}r$, $G_{\mathbb{A}^1}$ is called (for obvious reasons) the free strongly $\mathbb{A}^1$-invariant sheaf of groups on G. This functor is not hard to construct: one starts with the classifying space BG of G, then $G_{\mathbb{A}^1} = \pi_1(\text{Ł}_{\mathbb{A}^1}(BG))$. In other words, $G_{\mathbb{A}^1} = \pi_1^{\mathbb{A}^1}(BG)$.

In the same way, let $\mathcal{A}b(k)$ be the category of sheaves of abelian groups on Sm_k in the Nisnevich topology, and $\mathcal{A}b_{\mathbb{A}^1}(k)$ be its full subcategory consisting of strictly $\mathbb{A}^1$-invariant sheaves of abelian groups. The inclusion $\mathcal{A}b_{\mathbb{A}^1}(k) \subset \mathcal{A}b(k)$ has also a left adjoint $\mathcal{A}b \to \mathcal{A}b_{\mathbb{A}^1}(k)$, $M \mapsto M_{\mathbb{A}^1}$. In fact from what we have seen above, it is the restriction of $\mathcal{G}r \to \mathcal{G}r_{\mathbb{A}^1}(k)$, $G \mapsto G_{\mathbb{A}^1}$ to the category $\mathcal{A}b(k)$. For $M \in \mathcal{A}b(k)$, the sheaf $M_{\mathbb{A}^1}$ is called free strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on M. It is also the free strongly $\mathbb{A}^1$-invariant sheaf of groups on M.

Then it is rather formal to obtain:

Corollary 2.8. *Assume k is a perfect field. Let $n \geq 1$ and let $\mathcal{X}$ be a simplicially $(n-1)$-connected space. Then for $n = 1$, $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is the free strongly $\mathbb{A}^1$-invariant sheaf of groups* *on* $\pi_1(\mathcal{X})$ *and for* $n \geq 2$, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ *is the free strictly* $\mathbb{A}^1$*-invariant sheaf on the sheaf* $\pi_n(\mathcal{X})$.

From the theorem 2.5 one proves the two corollaries by induction on n. To start: if $\mathcal{X}$ is a 0-simplicially connected pointed space, then $\pi_0^{\mathbb{A}^1}(\mathcal{X}) = *$ as well as $\pi_0(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$ is always an epimorphism. This statement is the first corollary for $n = 1$. Then one formally deduce that for G a strongly $\mathbb{A}^1$-invariant sheaf of groups[5],

$$[\mathcal{X}, BG]_\bullet \cong Hom_{\mathcal{H}_{s,\bullet}}(\mathcal{X}, BG) \cong Hom_{\mathcal{G}r(k)}(\pi_1(\mathcal{X}), G) \cong Hom_{\mathcal{G}r(k)}((\pi_1(\mathcal{X}))_{\mathbb{A}^1}, G)$$

and also

$$[\mathcal{X}, BG]_\bullet \cong [L_{\mathbb{A}^1}(\mathcal{X}), BG]_\bullet \cong Hom_{\mathcal{G}r(k)}(\pi_1^{\mathbb{A}^1}(\mathcal{X}), G)$$

Thus we deduce that $(\pi_1(\mathcal{X}))_{\mathbb{A}^1} \to \pi_1^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism. This is the second corollary for $n = 1$. Then we conclude also that if $\mathcal{X}$ is 1-simplicially connected, $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is trivial, which is the first corollary for $n = 2$. And so on, we may go up by an easy induction, using Eilenberg-MacLane spaces[6].□

An important application of the theorem 2.5 and its two corollaries is for $n \geq 2$ for the space $\mathbb{A}^n - \{0\}$, that is the affine space of dimension n minus the origin. We have seen above that $\mathbb{A}^n - \{0\}$ is $\mathbb{A}^1$-equivalent to $\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$ and is thus $(n-2)$-$\mathbb{A}^1$-connected by the first corollary. The second corollary tells us that $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$ is the free strictly $\mathbb{A}^1$-invariant sheaf on $\pi_{n-1}(\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n}))$. We will describe these below in the next section.

Remark 2.9. For $\mathcal{X}$ a space, we may define the simplicial Postnikov tower (see [36] for instance)

$$\mathcal{X} \to \cdots \to P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X}) \to \dots P^1(\mathcal{X}) \to P^0(\mathcal{X}) = \pi_0(\mathcal{X})$$

so that the stalks at each point is the Postnikov tower of the simplicial set stalk of $\mathcal{X}$ at the given point. It is known (*loc. cit.*) that $\mathcal{X}$ is the homotopy inverse limit of its Postnikov tower in $Space(k)$. It follows easily from the theorem 2.5 that for $\mathcal{X}$ pointed connected, the whole Postnikov tower $\mathcal{X} \to \cdots \to P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X}) \to \dots P^1(\mathcal{X}) \to P^0(\mathcal{X}) = \pi_0(\mathcal{X}) = *$ consists of $\mathbb{A}^1$-local spaces.

It would be interesting to know whether more generally, for $\mathcal{X}$ an $\mathbb{A}^1$-local space with $\pi_0(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\mathcal{X})$ an $\mathbb{A}^1$-invariant sheaf of sets, the whole Postnikov tower $\mathcal{X} \to \cdots \to P^n(\mathcal{X}) \to P^{n-1}(\mathcal{X}) \to \dots P^1(\mathcal{X}) \to P^0(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\mathcal{X})$ also consists of $\mathbb{A}^1$-local spaces.

[5] one uses the formal fact that BG is $\mathbb{A}^1$-local if and only if G is strongly $\mathbb{A}^1$-invariant

[6] and the fact that if M is strictly $\mathbb{A}^1$-invariant, $K(M, n)$ is $\mathbb{A}^1$-local for any n

3. $\mathbb{A}^1$-algebraic topology

In this section, the base field k is always assumed to be a perfect field. So that all the results mentionned in the previous part are available.

3.1. The motivic Brouwer degree and Milnor-Witt K-theory.

Let $n \geq 1$ be an integer. The smooth k-scheme $\mathbb{A}^n - \{0\}$ should be thought of as an algebraic "sphere". Through a complex embedding of k it becomes $\mathbb{C}^n - \{0\}$ which is homotopy equivalent to the unit sphere in $\mathbb{R}^{2n}$, that is S^{2n-1}. Through a real embedding it maps to $\mathbb{R}^n - \{0\}$ which homotopy equivalent to S^{n-1}.

For $n = 1$, $\mathbb{A}^1 - \{0\} = \mathbb{G}_m$ is an $\mathbb{A}^1$-rigid space, its $\mathbb{A}^1$-homotopy type is the $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m) = \mathbb{G}_m$ itself, and there are no other higher $\mathbb{A}^1$-homotopy sheaves.

For $n \geq 2$, we have seen above that $\mathbb{A}^n - \{0\}$ is $\mathbb{A}^1$-equivalent to $\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$ and as the latter is $(n-2)$-simplicially connected, both are $(n-2)$-$\mathbb{A}^1$-connected.

Let $SL_2 \subset GL_2$ be the special linear subgroup, kernel of the determinant $GL_2 \to \mathbb{G}_m$. The morphism $SL_2 \to \mathbb{A}^2 - \{0\}$ taking an element to its first column is easily seen to be an $\mathbb{A}^1$-equivalence[7]. It follows that $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\})$ is a sheaf of abelian groups, and as it is strongly invariant, we know it is strictly $\mathbb{A}^1$-invariant by remark 2.4. Thus by corollary 2.8 for $n \geq 2$, $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$ is the free strictly $\mathbb{A}^1$-invariant sheaf associated to $\mathbb{Z}((\mathbb{G}_m)^{\wedge n})$, the free sheaf of abelian groups on the sheaf of pointed sets $(\mathbb{G}_m)^{\wedge n}$ (where the base point of $\mathbb{G}_m$ is 1).

Remark 3.1. Using similar reasoning, and standard results, it is not hard to prove that for $n \geq 2$, $\pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n})$ is canonically isomorphic to $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\})$ as $(\mathbb{P}^1)^{\wedge n}$ is canonically $\mathbb{A}^1$-equivalent to $\Sigma(\mathbb{A}^n - \{0\})$.

We may summarize what precedes as follows:

Theorem 3.2. *For $n \geq 2$, $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n})$ is the free strictly $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathbb{Z}((\mathbb{G}_m)^{\wedge n})_{\mathbb{A}^1}$ generated by the pointed sheaf of sets $(\mathbb{G}_m)^{\wedge n}$.*

This means exactly the following. For any $n \geq 2$ and for any strictly $\mathbb{A}^1$-invariant sheaf of abelian groups M, one has the following property. Any morphism of pointed sheaves of sets:

$$(\mathbb{G}_m)^{\wedge n} \to M$$

[7] by using the covering of $\mathbb{A}^2 - \{0\}$ by two open subsets that we mentionned above

"extends" uniquely to a morphism of strictly $\mathbb{A}^1$-invariant sheaves

$$\mathbb{Z}((\mathbb{G}_m)^{\wedge n})_{\mathbb{A}^1} \to M$$

Surprisingly it is possible to describe these sheaves rather explicitely.

We first need to introduce a purely algebraic object, for any commutative field F, called its Milnor-Witt K-theory. This was introduced by the author, but the following very simple description, which plays a fundamental role in the theory, was found in collaboration with Mike Hopkins:

Definition 3.3. Let F be a commutative field. The *Milnor-Witt K-theory* $K_*^{MW}(F)$ of F is the graded associative ring with units generated by the symbols $[a]$, for each unit $a \in F$, of degree $+1$, and the symbol η of degree -1 subject to the following relations:

(1) (Steinberg relation) For each $a \in F^\times - \{1\}$, one has $[a].[1-a] = 0$;

(2) For each pair $(a, b) \in (F^\times)^2$ one has $[ab] = [a] + [b] + \eta.[a].[b]$;

(3) For each $a \in F^\times$, one has $[a].\eta = \eta.[a]$;

(4) $\eta^2.[-1] + 2.\eta = 0$

The fourth relation sounds very strange *a priori*. For $a \in F^\times$, set $< a >:= \eta.[a] + 1 \in K_0^{MW}(F)$. Set $h := 1+ < -1 >$, this will be later identified as the *hyperbolic plane*. Then $h = \eta.[-1] + 2$ and thus the relation (4) can be rewritten as

$$\eta.h = 0$$

Some simple observations. Given a finite sequence $(a_1, \dots, a_n) \in (F^\times)^n$ of units in F we may form the product $[a_1] \dots [a_n] \in K_n^{MW}(F)$. This induces the symbol map $((\mathbb{G}_m)^{\wedge n})(F) = (F^\times)^{\wedge n} \to K_n^{MW}(F)$ for $n \geq 0$. Here we use the fact that $[1] = 0$ [32].

The quotient $K_*^{MW}(F)/\eta$ by η is clearly the *Milnor K-theory* $K_*^M(F)$ [28].

One can check by hands that the commutative ring $K_0^{MW}(F)$, is generated by the symbols $< a >$ and satisfy exactly the presentation of the Grothendieck-Witt ring $GW(F)$ of symmetric bilinear non degenerate forms described in [30]. This is true over any field, even in characteristic 2; see [32] for more details.

Thus there is an identification of rings $K_0^{MW}(F) = GW(F)$. As $\eta.h = 0$, multiplication by $\eta : K_0^{MW}(F) = GW(F) \to K_{-1}^{MW}(F)$ induces a morphism $GW(F)/h \to K_{-1}^{MW}(F)$. Now $GW(F)/h$ is well known to be the Witt ring $W(F)$ of F, and one may check in fact that multiplication by η^n induces for any $n \geq 1$ and isomorphism:

$$W(F) \cong K^{MW}_{-n}(F)$$

Remark 3.4. Similarly one may introduce the Witt K-theory of F as $K^W_*(F) := K^{MW}_*(F)/h$. Clearly $K^W_*(F) = W(F)$ in degree $* \leq 0$. Using the Milnor conjectures [28] on quadratic forms proved in [37] one may prove that the multiplication by η^n induces for each $n \geq 0$ a monomorphism $K^W_n(F) \subset K^W_0(F) = W(F)$ whose image is the n-th power $I^n(F)$ of the fundamental ideal $I(F)$ of the Witt ring, that is the kernel of the mod 2 rank epimorhism $W(F) \to \mathbb{Z}/2$.

The Milnor conjecture previously mentioned in particular claims that the canonical epimorphism

$$K^M_n(F)/2 \to I^n(F)/I^{n+1}(F)$$

is in fact an isomorphism for each n and one may deduce that the canonical morphism

$$K^{MW}_n(F) \to I^n(F) \times_{I^n(F)/I^{n+1}(F)} K^M_n(F)$$

is an isomorphism for all n. Observe that this also makes sense for $n < 0$ as $K^M_n(F) = 0$, as well as $I^n(F)/I^{n+1}(F)$ and the right hand side is just $W(F)$

Quickly said, the Milnor-Witt K-theory of F is the fiber product of the Milnor K-theory $K^M_*(F)$ and the Witt K-theory $K^W_*(F)$ over $K^M_*(F)/2 = K^W_*(F)/\eta$.

These results won't be used in the sequel.

One technical step, explained in details in [32] is that one may construct for any $n \in \mathbb{Z}$ a canonical strictly $\mathbb{A}^1$-invariant sheaf $\mathbf{K}^{\mathrm{MW}}_n$ of "unramified" Milnor-Witt K-theory in weight n whose stalk a the generic point $\xi \in X \in Sm_k$, X irreducible with function field F, is the group $K^{MW}_n(F)$. Moreover the restriction morphism $\mathbf{K}^{\mathrm{MW}}_n(X) \to \mathbf{K}^{\mathrm{MW}}_n(F) = K^{MW}_n(F)$ is injective, with image explicitely described inside $K^{MW}_n(F)$. Of course for $n \geq 1$, the canonical symbol map in Milnor-Witt K-theory of fields mentioned above extends to a morphism of pointed sheaves of sets:

$$(\mathbb{G}_m)^{\wedge n} \to \mathbf{K}^{\mathrm{MW}}_n$$

and one of the main computations in $\mathbb{A}^1$-algebraic topology is the fact that:

Theorem 3.5. [32] *For each $n \geq 1$, the morphism $(\mathbb{G}_m)^{\wedge n} \to \mathbf{K}^{\mathrm{MW}}_n$ is the universal one to a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups.*

At once we get the following consequence:

Corollary 3.6. *For any $n \geq 2$, we have canonical isomorphisms*

$$\pi_{n-1}^{\mathbb{A}^1}(\Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})) \cong \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \cong \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong \mathbf{K}_n^{\mathrm{MW}}$$

An important and simple construction which naturally occurs when manipulating strictly $\mathbb{A}^1$-invariant sheaves M is the contraction $M \mapsto M_{-1}$, first used a lot by V. Voevodsky [52][53]. Roughly speaking M_{-1} is the sheaf of pointed $\mathbb{G}_m$-loops in M. It has the property that for any $X \in Sm_k$, there is a functorial splitting $M(\mathbb{G}_m \times X) = M(X) \oplus M_{-1}(X)$.

It is not hard to check (see [32]) that for any n, there is an canonical identification $(\mathbf{K}_n^{\mathrm{MW}})_{-1} = \mathbf{K}_{n-1}^{\mathrm{MW}}$. Moreover, playing with the definitions and the previous results, it is not hard to check that for any $M \in \mathcal{A}b_{\mathbb{A}^1}(k)$, the group of morphisms $Hom_{\mathcal{A}b_{\mathbb{A}^1}(k)}(\mathbf{K}_n^{\mathrm{MW}}, M)$ is canonically isomorphic to $M_{-n}(k)$, for $n \geq 1$, where $M \mapsto M_{-n}$ is the iteration of n times the contraction on M. In particular there is a canonical identification $Hom_{\mathcal{A}b_{\mathbb{A}^1}(k)}(\mathbf{K}_n^{\mathrm{MW}}, \mathbf{K}_n^{\mathrm{M}}) = K_0^{MW}(k) = GW(k)$. From the computation above 3.6 we get the so called motivic version of the theory of the Brouwer degree:

Corollary 3.7. (Motivic Brouwer degree). *For any $n \geq 2$, there exists a "degree" isomorphism*

$$[(\mathbb{A}^n - \{0\}), (\mathbb{A}^n - \{0\})] \cong [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}] \cong GW(k)$$

For $n = 1$ it is an epimorphism

$$[\mathbb{P}^1, \mathbb{P}^1] \twoheadrightarrow GW(k)$$

with kernel isomorphic to $k^{\times 2}$.

This computation reflects the two possible topological intuitions that arise in $\mathbb{A}^1$-homotopy theory. Given a smooth k-scheme X and a real embedding $k \subset \mathbb{R}$, one may consider the topological space $X(\mathbb{R})$ of real points with it classical topology; it is a differentiable manifold. One may also consider the topological space $X(\mathbb{C})$ of complex points with it classical topology; it is also a differentiable manifold, in fact a complex analytic manifold as well.

We already mentionned above, that in the case of the algebraic sphere $\mathbb{A}^n - \{0\}$, the real and complex realizations are spheres of different dimensions. So on one hand we get a complex Brouwer degree, an integer, independent in fact of the choice of the real embedding, and an real Brouwer degree, also an integer which now depends on the real embedding. The above formula is taking care of all these phenomena at the same time. This is reflected by the cartesian square mentioned above in degree 0 for Milnor-Witt K-theory:

$$\begin{array}{ccc} GW(k) & \to & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(k) & \to & \mathbb{Z}/2 \end{array}$$

The top horizontal morphism corresponds to the complex Brouwer degree, independ of any embedding, and the left vertival one takes care of the real Brouwer degrees: each real embedding $k \subset \mathbb{R}$ defines a signature homomorphism $W(k) \to \mathbb{Z}$ which composed with the morphism $GW(k) \twoheadrightarrow W(k)$ gives the corresponding real Brouwer degree.

The motivic Brouwer degree has been used to enrich a lot of geometric computations, leading to quadratic enumerative geometry where instead of usual integers one gets a quadratic refinement in $GW(k)$; see for instance [23][38]. See also [12] for an other way of using $GW(k)$ as an enrichement of $\mathbb{Z}$.

3.2. $\mathbb{A}^1$-coverings and $\pi_1^{\mathbb{A}^1}$.

An $\mathbb{A}^1$-covering $\mathcal{Y} \to \mathcal{X}$ is a morphism which has the unique right lifting property for $\mathbb{A}^1$-trivial cofibrations. This means the following. For any commutative square of spaces of the form

$$\begin{array}{ccc} \mathcal{A} & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{B} & \to & \mathcal{X} \end{array}$$

where $\mathcal{A} \to \mathcal{B}$ is a monomorphism and an $\mathbb{A}^1$-equivalence, also called an $\mathbb{A}^1$-trivial cofibration, there exists a *unique* morphism $\mathcal{B} \to \mathcal{Y}$ which let the whole diagram commutative.

This definition is analogous to the classical definition of coverings, for nice spaces, involving the so called unique lifting property of homotopies. The $\pi_1^{\mathbb{A}^1}$ and the theory $\mathbb{A}^1$-coverings of a given $\mathbb{A}^1$-connected space are connected in a similar way as in the classical theory, as we will see below.

A finite étale covering $Y \to X$ between smooth k-varieties in characteristic 0 is an $\mathbb{A}^1$-covering. A Galois étale covering $Y \to X$ with Galois group of order prime to the characteristic of k is an $\mathbb{A}^1$-covering.

A $\mathbb{G}_m$-torsor $\mathcal{Y} \to \mathcal{X}$ is an $\mathbb{A}^1$-covering ! Remember that through a real embedding of k, $\mathbb{R}^\times$ is up to homotopy $\{\pm 1\}$ and a $\mathbb{G}_m$-torsor becomes up to homotopy a $\mathbb{Z}/2$-covering.

More generally if G is a strongly $\mathbb{A}^1$-invariant sheaf of groups, any G-torsor (in the Nisnevich topology) is an $\mathbb{A}^1$-covering.

The following can be proved along the same lines as the classical result:

Theorem 3.8. [32] *Any pointed $\mathbb{A}^1$-connected space $\mathcal{X}$ admits a unique pointed $\mathbb{A}^1$-covering $\widetilde{\mathcal{X}} \to \mathcal{X}$ with $\widetilde{\mathcal{X}}$ simply $\mathbb{A}^1$-connected, up to canonical isomorphism. Forgetting the base points, the morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a torsor under the strongly $\mathbb{A}^1$-invariant sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{X})$. If $Cov_{\mathbb{A}^1}(\mathcal{X})$ denotes the category of $\mathbb{A}^1$-coverings of $\mathcal{X}$, then for any $\mathbb{A}^1$-coverings $\pi : \mathcal{Y} \to \mathcal{X}$, if $\Gamma_\pi \subset \mathcal{Y}$ denotes the inverse image of the base point x_0 of $\mathcal{X}$, then the map*

$$Hom_{Cov_{\mathbb{A}^1}(\mathcal{X})}(\widetilde{\mathcal{X}}, \mathcal{Y}) \to \Gamma_\pi(k) \ , \ \big(\phi : \widetilde{\mathcal{X}} \to \mathcal{Y}\big) \mapsto \phi(\widetilde{x}_0)$$

(where $\widetilde{x}_0$ is the chosen base point of $\widetilde{\mathcal{X}}$) is a bijection. More generally, there is a canonical right action of $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ on the fiber $\Gamma_\pi \subset \mathcal{Y}$ of π at x_0 and this action induce an equivalence of categories:

$$Cov_{\mathbb{A}^1}(\mathcal{X}) \;\cong\; \pi_1^{\mathbb{A}^1}(\mathcal{X}) - Shv_{\mathbb{A}^1}(k)$$

where the right hand side is the category of $\mathbb{A}^1$-invariant sheaves of sets endowed with a right action of $\pi_1^{\mathbb{A}^1}(\mathcal{X})$.

Here are a some examples:

Theorem 3.9. *Let $n \geq 2$. The canonical $\mathbb{G}_m$-torsor*

$$(\mathbb{A}^{n+1} - \{0\}) \to \mathbb{P}^n$$

is the universal covering of $\mathbb{P}^n$ and defines an isomorphism

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbb{G}_m$$

Indeed the morphism is a $\mathbb{G}_m$-torsor (in fact already in the Zariski topology) and we have seen that if $n \geq 2$, $(\mathbb{A}^{n+1} - \{0\})$ is simply $\mathbb{A}^1$-connected.

Observe however that the $\mathbb{G}_m$-torsor $(\mathbb{A}^2 - \{0\}) \to \mathbb{P}^1$ is not the universal covering of $\mathbb{P}^1$ because $\mathbb{A}^2 - \{0\}$ is not simply $\mathbb{A}^1$-connected, as its $\pi_1^{\mathbb{A}^1}$ was seen to be isomorphic to $\mathbf{K}_2^{\mathrm{MW}}$. We will describe the $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ below.

We already observed that the projection $SL_2 \to (\mathbb{A}^2 - \{0\})$ is an $\mathbb{A}^1$-equivalence, so $\pi_1^{\mathbb{A}^1}(SL_2) \cong \mathbf{K}_2^{\mathrm{MW}}$. We have the following more precise result:

Theorem 3.10. *The universal $\mathbb{A}^1$-covering of SL_2 admits a group structure and is given by a central extension of the form*

$$0 \to \mathbf{K}_2^{\mathrm{MW}} \to \widetilde{SL}_2 \to SL_2 \to 1$$

We already know the last affirmation of the $\mathbb{A}^1$-fundamental sheaf of groups of $SL_2 \cong (\mathbb{A}^2 - \{0\})$. One has just to observe, like in classical topology, that the universal $\mathbb{A}^1$-covering of an $\mathbb{A}^1$-connected sheaf of groups admits a canonical and unique group structure which turns the projection into a group homomorphism. The fact that the extension is central can also be deduced along the same lines: quickly said the action of K_2^{MW} on $\widetilde{SL}_2$ by conjugations defines automorphisms $\widetilde{SL}_2 \cong \widetilde{SL}_2$ of covering of SL_2. As they all take the base point (which is the 0 of K_2^{MW}) to itself, they all are the identity.

Remark 3.11. This statement has been generalized in [34] to any $\mathbb{A}^1$-connected split semi-simple algebraic group. In fact we observed first that a split semi-simple algebraic group G is $\mathbb{A}^1$-connected if and only if it is simply connected in the sense of algebraic group theory

[45]. We then proved in *loc. cit.* that for G a split, semi-simple, almost simple, algebraic group, $\pi_1^{\mathbb{A}^1}(G) \cong \mathbf{K}_2^{\mathrm{MW}}$ for G of symplectic type, and $\pi_1^{\mathbb{A}^1}(G) \cong \mathbf{K}_2^{\mathrm{M}}$ for G not of symplectic type.

To conclude this section, we want to describe $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. To do this we use the $\mathbb{A}^1$-fibration sequence

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \to \mathbb{P}^\infty \cong B(\mathbb{G}_m)$$

which, using the long exact sequence of $\mathbb{A}^1$-homotopy sheaves, gives an exact sequence of the form:

$$1 \to \mathbf{K}_2^{\mathrm{MW}} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m \to 1$$

This extension is central. Indeed, as we already observed above, the projection to the first column $SL_2 \overset{\simeq}{\to} \mathbb{A}^2 - \{0\}$ is an $\mathbb{A}^1$-equivalence. If one embedds $\mathbb{G}_m \subset SL_2$ as diagonal matrices, one gets a fibration sequence:

$$SL_2/(\mathbb{G}_m) \to B(\mathbb{G}_m) \to BSL_2$$

Then the induced morphism $SL_2/(\mathbb{G}_m) \to \mathbb{P}^1$ is an $\mathbb{A}^1$-weak equivalence; now BSL_2 is simply $\mathbb{A}^1$-connected and its $\pi_2^{\mathbb{A}^1}$ is $\mathbf{K}_2^{\mathrm{MW}}$. We thus get a morphism in $\mathcal{H}_{\mathbb{A}^1}(k) : B(\mathbb{G}_m) \to K(\mathbf{K}_2^{\mathrm{MW}}, 2)$, whose homotopy fiber is $B(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1))$. And the fibration sequence

$$B(\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)) \to B(\mathbb{G}_m) \to K(\mathbf{K}_2^{\mathrm{MW}}, 2)$$

induces with the exact sequence of $\mathbb{A}^1$-homotopy sheaves, a central extension, which is exactly the above one.

The epimorhism of groups $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m$ admits a canonical section of pointed sheaves of sets $\sigma : \mathbb{G}_m \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$: it is the universal morphism expressing $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ as the free strongly $\mathbb{A}^1$-invariant sheaf of groups on $\mathbb{G}_m$ (see 2.8). This explain why the long exact sequences considered above are in fact short.

Then one may check [32] that the morphism $\mathbb{G}_m \wedge \mathbb{G}_m \to \mathbf{K}_2^{\mathrm{MW}}$ induced by the formula $\sigma(U).\sigma(V).\sigma(U.V)^{-1}$, which exactly measures the defect of σ to be a morphism of sheaves of groups, is the universal symbol !

One may deduce for instance that the commutator $[\sigma(U), \sigma(V)] \in \mathbf{K}_2^{\mathrm{MW}}$ is $h.[U].[V]$ and in particular one sees that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is never a sheaf of abelian groups.

There will be more examples of computations of $\mathbb{A}^1$-fundamental groups in the last section below.

3.3. $\mathbb{A}^1$-homotopy classification of algebraic vector bundles.

It is classical that the set $\Phi_n(X)$ of isomorphism classes of rank n algebraic vector bundles over $X \in Sm_k$ is canonically in bijection with the set $H^1(X;GL_n)$ of GL_n-torsors (either in the Zariski, Nisnevich or étale topology) over X.

Let $\mathbb{G}r_n = \cup_i \mathbb{G}r_{n,i}$ be the infinite Grassmanian, that is the union of the finite Grasmannians $\mathbb{G}r_{n,i}$ of n-planes in the affine space $\mathbb{A}^{n+i}$; it is in fact the same as the homogeneous scheme $GL_{n+i}/B_{n,i}$, with the $B_{n,i} \subset GL_{n+i}$ the subgroup of matrices of the form:

$$\begin{pmatrix} M & P \\ 0 & N \end{pmatrix}$$

with $M \in GL_n$, $N \in GL_i$, and P an arbitrary $n \times i$ matrix.

It is well known that the quotient $\mathbb{V}_{n,i} := GL_{n+i}/B'_{n,i}$, where $B'_{n,i} \subset B_{n,i}$ is the subgroup such that $M = Id_n$ (as above), is a highly $\mathbb{A}^1$-connected space, as it is, up to $\mathbb{A}^1$-homotopy, a successive fibration of spheres of the form $GL_{n'}/GL_{n'-1} \cong_{\mathbb{A}^1} \left(\mathbb{A}^{n'} - \{0\}\right)$, for $n' \in \{i+1, \dots i+n\}$. Thus the union of the $\mathbb{V}_{n,i}$ over i is $\mathbb{A}^1$-equivalent to the point. Now there is an obvious free action of GL_n on $\mathbb{V}_{n,i}$ which makes the diagram:

$$GL_n \to \mathbb{V}_{n,i} \to \mathbb{G}r_{n,i}$$

a GL_n-torsor on $\mathbb{G}r_{n,i}$. Thus setting $\mathbb{V}_n := \cup_i \mathbb{V}_{n,i}$ we see that

$$GL_n \to \mathbb{V}_n \to \mathbb{G}r_n$$

is a GL_n-torsor over $\mathbb{G}r_n$, with $\mathbb{V}_n$ weakly $\mathbb{A}^1$-contractible. It follows that $\mathbb{G}r_n$ is $\mathbb{A}^1$-equivalent to the classifying space BGL_n, generalizing the fact that $\mathbb{P}^\infty = \cup_i \mathbb{P}^i$ is $\mathbb{A}^1$-equivalent to $B\mathbb{G}_m$.

Let $T \to X$ be a GL_n-torsor on $X \in Sm_k$. The morphism:

$$T \times_{GL_n} \mathbb{V}_n \to T/GL_n = X$$

is an $\mathbb{A}^1$-equivalence, as it is an $\mathbb{A}^1$-fibration with $\mathbb{A}^1$-contractible fibers ($\mathbb{V}_n$ indeed). Thus by inverting that $\mathbb{A}^1$-equivalence the "classifying" diagram:

$$\begin{array}{ccc} T \times_{GL_n} \mathbb{V}_n & \to & \mathbb{V}_n/GL_n = \mathbb{G}r_n \\ \downarrow & & \\ X & & \end{array}$$

defines a classifying morphism in $\mathcal{H}_{\mathbb{A}^1}(k)$:

$$X \to \mathbb{G}r_n$$

an element of $[X, \mathbb{G}r_n]$.

Theorem 3.12. *([32][5]) For any integer $n \geq 1$ and any affine smooth k-scheme X the previous map*

$$\Phi_n(X) = H^1(X; GL_n) \to [X, \mathbb{G}r_n]$$

is a bijection.

Remark 3.13. It is well-known that this map can't be a bijection in general because the functor $X \mapsto \Phi_n(X)$ is not $\mathbb{A}^1$-invariant in general. Even for $X = \mathbb{P}^1$, the map $\Phi_2(\mathbb{P}^1) \to \Phi_2(\mathbb{A}^1 \times \mathbb{P}^1)$ is not a bijection. Now the right hand side $X \mapsto [X, \mathbb{G}r_n]$ is tautologically $\mathbb{A}^1$-invariant. Over projectives spaces (or schemes) the classification of vector bundles is a very difficult subject, much more complicated than over affine smooth schemes.

The proof of theorem 3.12 uses amongst others the fact that for any n and for any smooth affine k-scheme X the projection $X \times \mathbb{A}^1 \to X$ induces indeed a bijection

$$\Phi_n(X) \cong \Phi_n(X \times \mathbb{A}^1)$$

This is due to Lindel [25] in this generality, after the fundamental cases obtained by Quillen [41] and Suslin [46] on the Serre problem, when X is some affine space $\mathbb{A}^d$.

The theorem 3.12 was first proven in [32] for $n \geq 3$ (the case $n = 1$ was already known in [36]) and was proven in a much greater generality and the proof drastically simplified in [5] [6].

One may deduce from theorem 3.12 some nice applications, which have had a lot of further developments. For instance, one observes that there is an $\mathbb{A}^1$-fibration sequence of pointed spaces:

$$\mathbb{A}^n - \{0\} \to \mathbb{G}r_{n-1} \to \mathbb{G}r_n$$

or equivalently

$$\mathbb{A}^n - \{0\} \to BGL_{n-1} \to BGL_n$$

where the morphism $BGL_{n-1} \to BGL_n$ is obtained by the inclusion of groups $GL_{n-1} \subset GL_n$, $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$.

For a given smooth affine k-scheme X, it follows that the map induced by the inclusion above $GL_{n-1} \subset GL_n$

$$\Phi_{n-1}(X) \to \Phi_n(X)$$

is the map taking a vector bundle ξ of rank $n-1$ on X to the direct sum $\xi \oplus \epsilon^1$, with ϵ^1 the trivial line bundle $\mathbb{A}^1 \times X$ over X.

Thus, for affine smooth k-schemes, we may study the map "adding the trivial line bundle" by obstruction theory using the $\mathbb{A}^1$-fibration sequence :

$$\mathbb{A}^n - \{0\} \to BGL_{n-1} \to BGL_n$$

and theorem 3.12,

The problem here is that $\pi_0^{\mathbb{A}^1}(GL_n) = \mathbb{G}_m$ (through the determinant), so the $\mathbb{A}^1$-connected space BGL_n is not simply $\mathbb{A}^1$-connected. In general the obstruction involves twisted cohomology groups. If ξ is a rank n algebraic vector bundle over X, and $f_\xi : X \to BGL_n$ classifies ξ, the composition $f_\xi : X \to BGL_n \to B\mathbb{G}_m$ is an element in $Pic(X) = [X, B\mathbb{G}_m]$ which is its first Chern class $c_1(\xi)$, or determinant line bundle $\Lambda^n(\xi)$. In case ξ is oriented in the sense that $c_1(\xi) = 0$, it follows that ξ is actually classified by a morphism

$$f_\xi : X \to BSL_n$$

Now the fibration sequence

$$\mathbb{A}^n - \{0\} \to BSL_{n-1} \to BSL_n$$

is a bit simpler to use because the base space BSL_n is now simply $\mathbb{A}^1$-connected, and the obstruction theory easier to explain and to use. A standard use of the Postnikov tower of the morphism $BSL_{n-1} \to BSL_n$ leads to the following.

The fact that $\mathbb{A}^n - \{0\}$ is $(n-2)$-$\mathbb{A}^1$-connected and $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \mathbf{K}_n^{\mathrm{MW}}$ show that the first non trivial stage of the Postnikov tower of the morphism $BSL_{n-1} \to BSL_n$ is a morphism

$$E_n : BSL_n \to K(\mathbf{K}_n^{\mathrm{MW}}, n)$$

where $K(\mathbf{K}_n^{\mathrm{MW}}, n)$ is the Eilenberg-MacLane space which is $(n-1)$-$\mathbb{A}^1$-connected and with $\pi_n^{\mathbb{A}^1}$ equal to $\mathbf{K}_n^{\mathrm{MW}}$. The homotopy fiber of that morphism is the second non-trivial stage of the Postnikov tower.

The Euler class $e(\xi) \in H^n(X; K_n^{MW})$ of ξ is the composition

$$X \to BSL_n \to K(\mathbf{K}_n^{\mathrm{MW}}, n)$$

of E_n and f_ξ. Now standard obstruction theory gives:

Theorem 3.14. *(***Theory of the Euler class***)* [32][31] *Assume*[8] $n \geq 2$. *Let X be smooth affine over k, and let ξ be an oriented algebraic vector bundle of rank n ($\Lambda^n(\xi)$ is trivialized). If dimension $X \leq n$:*

$$\big(\ \xi \text{ split off a trivial line bundle}\ \big) \Leftrightarrow \big(\ e(\xi) = 0 \in H^n(X; K_n^{MW})\ \big)$$

The groups $H^n(X; K_n^{MW})$ are now called Chow-Witt groups and were defined first in [10], as well as a very concrete version of $e(\xi)$. These groups have been extensively studied since then, see for instance the survey [16]. There the twisted Euler class for non orientable

[8] In the references [32][31] n was supposed to be ≥ 4 as theorem 3.12 was not yet known for $n = 2$

vector bundle ξ of rank n with determinant $\lambda = \Lambda^n(\xi)$ was defined in details in the twisted Chow-Witt group $H^n(X; K_n^{MW}(\lambda))$ and the generalisation of the previous theorem in that generality follows along the same lines by obstruction theory. See also [3] and [4] for more complete surveys of the actual state of the art and recent developpments.

Observe that the theorem applied in case $dim(X) < n$ implies that any ξ splits off a trivial line bundle, a result which was long known over noetherian affine schemes as Bass-Serre Theorem [44].

The theory of the Euler class is thus the first obstruction that appears to split off a trivial line bundle.

Amazingly, Asok and Fasel [3] proved that one may actually use this approach to attack the problem one step further: when can one split a trivial line bundle for a rank $(n-1)$ vector bundle over a dimension n smooth affine scheme? In the obstruction theory approach, one has to deal now with the first two non trivial $\mathbb{A}^1$-homotopy sheaves of $\mathbb{A}^n - \{0\}$. The first one is well understood in the Corollary 3.6 to our main computation, and was just used for the proof of the previous theorem. The second $\mathbb{A}^1$-homotopy sheaf, becomes stably the first stable $\mathbb{A}^1$-homotopy group of the sphere spectrum; it was computed recently in [43]. More recently Asok, Bachmann and Hopkins proved some important new result concerning an analogue of the Freundenthal suspension theorem involving the smash product with $\mathbb{G}_m$ [2], which enabled them, using the results of Asok-Fasel just mentioned, to get enough informations on the second non trivial $\mathbb{A}^1$-homotopy sheaf of $\mathbb{A}^n - \{0\}$ to prove a conjecture of Murthy. See the talks by Asok and Hopkins at PCMI 2024 [1] [21].

Another very interesting area related to the previous one is the study of the geometric classifying space of reductive groups. The situation is more complicated here, as the geometric classifying space of a reductive group is in general not $\mathbb{A}^1$-connected. This is related to the theory of cohomological invariants [17]. The Motivic Cohomology of these and their Motives is still mysterious, althought there has been a lot of results, see for instance [49] [50].

3.4. Geometric $\mathbb{A}^1$-Topology.

We want to describe shortly some works, mostly in progress, which show the potentiel use of $\mathbb{A}^1$-homotopy theory to the study of smooth projectives schemes over a field, in analogy to what we mentioned in the classical case how geometric topology can study compact differentiable manifolds. These smooth projectives schemes over a field are really the analogues of compact differentiable manifolds.

$\mathbb{A}^1$-connectedness.

First we deal with the notion of $\mathbb{A}^1$-connectedness. In classical algebraic topology, a compact differentiable manifold is the disjoint union of finitely many connected compact differentiable manifolds. The situation in algebraic geometry is more complicated. But we will describe the smooth projective k-schemes which are $\mathbb{A}^1$-connected. Recall that a space is said to be $\mathbb{A}^1$-connected if its $\pi_0^{\mathbb{A}^1}$ is trivial. As for any space $\mathcal{X}$ the morphism $\mathcal{X} \to \pi_0^{\mathbb{A}^1}$ is always an epimorphism of sheaves and as the point $* = Spec(k)$ is the spectrum of a field, it follows[9] that $\mathcal{X}(k) \neq \emptyset$. We already saw above examples of $\mathbb{A}^1$-connected spaces.

A more naive notion is the notion of being $\mathbb{A}^1$-chain connected. A smooth k-scheme X is said to be $\mathbb{A}^1$-chain connected if for any finite type field extension K of k any two elements of $X(K)$, the set of K-rational points of X, or k-morphisms $Spec(K) \to X$, can be connected by a finite chain of k-morphisms $\mathbb{A}^1_K \to X$, in the obvious way. As we are assuming that k is a perfect field, a finite type field extension K of k is always the function field of an integral smooth k-scheme. It is known [33] that an $\mathbb{A}^1$-chain connected smooth k-scheme X (or space) is $\mathbb{A}^1$-connected. In [7] we proved the converse when X is smooth proper over k.

The only $\mathbb{A}^1$-connected proper smooth curve is $\mathbb{P}^1$. The only $\mathbb{A}^1$-connected smooth curves are $\mathbb{A}^1$ and $\mathbb{P}^1$.

For higher dimensional smooth schemes it is more complicated and interesting. In [7] we proved that over an algebraically closed field k, a smooth projective surface is $\mathbb{A}^1$-connected if and only if it is rational. It means basically that such a surface can be reached from $\mathbb{P}^2$ by a finite sequence of blowing up a point or blowing down a rational curve.

However, it is known that in general, over perfect fields, there are examples of $\mathbb{A}^1$-connected smooth projective surfaces which are not rational, for instance[10] some conic bundle over $\mathbb{P}^1$ are sometimes $\mathbb{A}^1$-connected but not rational [11]. In fact if a smooth projective surface over k is $\mathbb{A}^1$-connected, it is either rational, or a conic bundle over $\mathbb{P}^1$ [35]; this fact uses all the known results on smooth projective surfaces over a perfect field (see [35] for a survey of what we need).

In [7] we proved that if one blows up a closed smooth k-scheme in a smooth projective scheme X and if Y is the blow-up, then (X is $\mathbb{A}^1$-connected) $\Leftrightarrow$ (Y is $\mathbb{A}^1$-connected). This is not so hard using the criterium mentioned above that for smooth projective, $\mathbb{A}^1$-connected is equivalent to being $\mathbb{A}^1$-chain connected. Using this we could prove that in characteristic 0, any rational smooth projective scheme is $\mathbb{A}^1$-chain connected, so $\mathbb{A}^1$-connected.

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[9]Because we are using the Nisnevich topology. It would be wrong in the étale toplogy

[10]This was mentioned to us by Aravind Asok

Here is an easy observation which explains a huge difference between algebraic geometry and algebraic topology; a smooth projective k-scheme of dimension > 0 which is $\mathbb{A}^1$-connected is never simply $\mathbb{A}^1$-connected:

Lemma 3.15. *Let X be a smooth projective scheme over k. If X is $\mathbb{A}^1$-connected, and of dimension > 0, the sheaf $\pi_1^{\mathbb{A}^1}(X)$ at any (rational) point is non trivial.*

Proof. It is well known that if $dim(X) > 0$, then $Pic(X) \neq 0$. To see that, embedd X into some $\mathbb{P}^N$. Then the pull-back $\mathcal{L}$ of $\mathcal{O}(1)$ through the closed embedding $X \subset \mathbb{P}^N$ is non trivial. Indeed $\mathcal{L}$ is generated by $N+1$-sections, so there is an epimorphism

$$(\mathcal{O}_X)^{N+1} \twoheadrightarrow \mathcal{L}$$

If $\mathcal{L} \cong \mathcal{O}_X$ is trivial, then any global section of $\mathcal{L}$ on X is constant, and there is no epimorphism of $\mathcal{O}_X$-modules $(\mathcal{O}_X)^{N+1} \twoheadrightarrow \mathcal{L}$ except if $dim(X) = 0$.

Now we proved in [7] that if X is $\mathbb{A}^1$-connected, for any base (rational) point of X, there is an isomorphism bewteen $Pic(X)$ and $Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(X), \mathbb{G}_m)$. So clearly, if $Pic(X) \neq 0$, the $\pi_1^{\mathbb{A}^1}(X)$ has to be non trivial. □

$\pi_1^{\mathbb{A}^1}$ of smooth $\mathbb{A}^1$-connected k-schemes

The previous result shows why the study of smooth projective $\mathbb{A}^1$-connected k-scheme from the $\mathbb{A}^1$-homotopical point of view is much less intuitive than the study of compact differential manifolds, because the latter was understood first in case of highly connected compact differential manifolds.

Let us give now some examples, some computations, and some facts concerning the $\pi_1^{\mathbb{A}^1}(X)$ of $\mathbb{A}^1$-connected smooth projective schemes X.

The $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ was already described above, it is a very explicit central extension of $\mathbb{G}_m$ by $\mathbf{K}_2^{\mathrm{MW}}$, and is a non commutative sheaf of groups. For $n \geq 2$ we also have seen that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) = \mathbb{G}_m$.

As we seen in the proof above, given an $\mathbb{A}^1$-connected smooth projective scheme X, for any choice of base (rational) point there is an isomorphism $Pic(X) \cong Hom_{\mathcal{G}r}(\pi_1^{\mathbb{A}^1}(X), \mathbb{G}_m$. It follows that if we blow up a closed smooth k-scheme in X and if Y is the blow-up, then it is easy to check that (for any convenient choice of base point) $\pi_1^{\mathbb{A}^1}(Y) \to \pi_1^{\mathbb{A}^1}(X)$ is an epimorphism. And it follows that $\pi_1^{\mathbb{A}^1}(Y)$ is strictly bigger than $\pi_1^{\mathbb{A}^1}(X)$ as the Picard group of Y is $Pic(X) \oplus \mathbb{Z}$. In the case we blow up a rational point $x \in X(k)$ the diagram:

$$\begin{array}{ccc} E & \subset & Y \\ \downarrow & & \downarrow \\ \{x\} & \subset & X \end{array}$$

in which E is the exceptional divisor of the blow up, thus isomorphic to $\mathbb{P}^{n-1}$ with $n = dim(X)$, is very often homotopy cocartesian. So the Van-Kampen theorem tells us that $\pi_1^{\mathbb{A}^1}(X)$ is the cokernel in the category of strongly $\mathbb{A}^1$-invariant sheaves of groups of the non trivial morphism

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^{n-1}) = \pi_1^{\mathbb{A}^1}(E) \to \pi_1^{\mathbb{A}^1}(Y)$$

In general there is no simple formula.

Let us try to understand the simplest possible cases, the smooth projective $\mathbb{A}^1$-connected surfaces; remember that over any field a smooth projective $\mathbb{A}^1$-connected curve is isomorphic to $\mathbb{P}^1$ so everything is known. In general, if k is not assumed to be algebraically closed, we have mentioned that there are examples of smooth projective $\mathbb{A}^1$-connected surfaces which are conic bundles over $\mathbb{P}^1$, and which are not rational. In [35] one may find some further discussion of rational smooth projective k-surfaces for k perfect. We will here assume for simplicity that the field k is now algebraically closed so that all the results and ideas in [7] can be used.

In that case we know that the smooth projective $\mathbb{A}^1$-connected surfaces are exactly the rational ones. There is a list of so called minimal models, consisting of $\mathbb{P}^2$ and the Hirzebruch surfaces Σ_n, $n \in \mathbb{N}$, so that any smooth projective $\mathbb{A}^1$-connected surface X is obtained by inductively blowing up a closed point finitely many times from one of the minimal model from that list.

Recall that $\mathcal{O}(n)$ denotes the n-th tensor product of the canonical line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$. Then Σ_n is equal to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ the projective bundle of the rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(n)$ on $\mathbb{P}^1$, for $n \in \mathbb{N}$.

The $\pi_1^{\mathbb{A}^1}$ of Hirzeburch surfaces was describe in [7], using the $\mathbb{A}^1$-fibration sequence

$$\mathbb{P}^1 \to \Sigma_n \to \mathbb{P}^1$$

which induces a long exact sequence in $\mathbb{A}^1$-homotopy sheaves. In that case, as the morphism $\Sigma_n \to \mathbb{P}^1$ has a section, we see that the induced diagram

$$1 \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \pi_1^{\mathbb{A}^1}(\Sigma_n) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to 1$$

is a split short exact sequence, so the sheaves $\pi_1^{\mathbb{A}^1}(\Sigma_n)$ are semi-direct products. We proved that for n even the semi-direct product is actually a product. For instance Σ_0 is $\mathbb{P}^1 \times \mathbb{P}^1$. For n odd the sheaves $\pi_1^{\mathbb{A}^1}(\Sigma_n)$ are all isomorphic to an explicit semi-direct product through a non trivial tautological operation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ on itself. In fact we proved:

Theorem 3.16. [7] *For $n \geq 0$, the Σ_n's are distinguished up to $\mathbb{A}^1$-equivalence by their $\pi_1^{\mathbb{A}^1}$; in other words Σ_n and Σ_m are $\mathbb{A}^1$-homotopy equivalent if and only $\pi_1^{\mathbb{A}^1}(\Sigma_n)$ and $\pi_1^{\mathbb{A}^1}(\Sigma_m)$ are isomorphic (and the latter occurs if and only if n and m have the same parity).*

In fact we proved slightly more: Σ_n and Σ_m are $\mathbb{A}^1$-homotopy equivalent if and only if Σ_n and Σ_m are $\mathbb{A}^1$-h-cobordant [7]. We conjectured then that this holds in general: two smooth projective $\mathbb{A}^1$-connected surfaces are $\mathbb{A}^1$-homotopy equivalent if and only if are $\mathbb{A}^1$-h-cobordant. A proof of that conjecture was only sketched there, but it is known in a big proportion of cases (depending on the configurations of points one blows up).

Recall the following classical result of differential topology. Let S be a compact connected differentiable surface. Let $x \in S$ be a point. Then $S - s$ has the homotopy type of a wedge of circles.

Conjecture 3.17. *Let X be smooth projective $\mathbb{A}^1$-connected k-surface (still assuming k is algebraically closed). Let $x \in X$ be a closed point. Then there is an $\mathbb{A}^1$-equivalence*

$$X - \{x\} \cong \vee_{i=1}^{r} \mathbb{P}^1$$

with r the rank of the Picard group of X (a finite type free abelian group).

This is known in many cases, for instance the minimal surfaces, and some work in progress following [7] might prove the general case.

Remark 3.18. If one removes more than one rational point, the conjecture predicts that X to which we remove finitely many points is $\mathbb{A}^1$-equivalent to a wedge of $\mathbb{P}^1$'s (same number of them) to which we add a wedge of an $\mathbb{A}^2 - \{0\}$ for each extra point removed.

To illustrate this kind of ideas, if $\mathcal{F}$ is a finite set of rational k-points of $\mathbb{A}^2$, it is known for instance that $\mathbb{A}^2 - \mathcal{F}$ is $\mathbb{A}^1$-connected, and that there is an $\mathbb{A}^1$-weak equivalence:

$$\mathbb{A}^2 - \mathcal{F} \cong \vee_{\mathcal{F}}(\mathbb{A}^2 - \{0\})$$

Remark 3.19. If k is only assumed to be perfect, one may generalise the previous conjectures as follows. If L is a finite separable extension of k, we denote by $\vee_L \mathbb{P}^1$ the quotient of $\mathbb{P}^1_L$ by any L-rational point in the category of sheaves of sets on $(Sm_k)_{Nis}$. We call this pointed space a wedge of $\mathbb{P}^1$'s parametrized by L. Observe that it is $\mathbb{A}^1$-connected and that if $[L : k] = n$, then the extension of $\vee_L \mathbb{P}^1$ to an algebraic closure $\overline{k}$ of k becomes a wedge of n copies of $\mathbb{P}^1$. We conjecture that if X is a projective smooth $\mathbb{A}^1$-connected surface over a perfect field, and if $x \in X(k)$ is a k-rational point, then $X - \{x\}$ is $\mathbb{A}^1$-equivalent to a finite wedge of $\vee_{L_i} \mathbb{P}^1$, for finitely many L_i's, finite separable extensions of k. An abelian version of that is proven in [35] using the cellular $\mathbb{A}^1$-homology we are going to discuss now.

Cellular $\mathbb{A}^1$-homology, Poincaré duality and $\mathbb{A}^1$-geometric topology

To conclude, we will introduce the cellular $\mathbb{A}^1$-homology (see [34]) and show some computations . To give a geometric intuition, one may introduce the cellular $\mathbb{A}^1$-homology objects of a smooth k-scheme X (or in fact of any space) using the Verdier formula [19], or more precisely using the dual Verdier formula. The Verdier formula gives a computation of the étale cohomology of X with coefficients in a given étale sheaf of abelian groups M as:

$$H^*_{\acute{e}t}(X;M) \cong colim_{\mathcal{X}_\bullet \tilde{\to} X} H^*(M(\mathcal{X}_\bullet))$$

where the $\mathcal{X}_\bullet \xrightarrow{\sim} X$'s run over the category of hypercoverings $\mathcal{X}_\bullet$ of X in the étale topology, up to simplicial homotopy of hypercoverings. These form a left filtering essentially small category so the colimit is well defined. Moreover, each $\mathcal{X}_\bullet$ is a simplicial object in the category of étale morphisms to X and $M(\mathcal{X}_\bullet)$ is the associated cosimplicial abelian group obtained by taking sections of M; $H^*(M(\mathcal{X}_\bullet))$ means then its obvious cohomology.

If ℓ is a prime number, different from $char(k)$, and one would try to define the mod ℓ étale homology directly, not by a dualizing process, inspired by the previous formula, one would take the formula

$$H_*^{\acute{e}t}(X;\mathbb{Z}/\ell) \cong lim_{\mathcal{X}_\bullet \tilde{\to} X} H_*(\mathbb{Z}/\ell(\mathcal{X}_\bullet))$$

In the previous formula, by $\mathbb{Z}/\ell(\mathcal{X}_\bullet)$ we mean the simplicial $\mathbb{Z}/\ell$-vector space obtained by taking, in each simplicial degree n, the sections $\mathbb{Z}/\ell(\mathcal{X}_n)$ on the smooth k-scheme $\mathcal{X}_n$ of the constant étale sheaf $\mathbb{Z}/\ell$. In case k is separably closed, the objects $H_*^{\acute{e}t}(X;\mathbb{Z}/\ell)$ of the previous definition, which look like profinite $\mathbb{Z}/\ell$-vector spaces, can be shown to be constant $\mathbb{Z}/\ell$-vector spaces, finite dimentional, and are the dual of the étale cohomology groups

$$H^*_{\acute{e}t}(X;\mathbb{Z}/\ell)$$

In some sense, the mod ℓ étale homology groups of a smooth k-scheme X can be seen as the left derived functors of the functor $X \mapsto H_0^{\acute{e}t}(X;\mathbb{Z}/\ell) = \mathbb{Z}/\ell(X)$. For the definition of the cellular $\mathbb{A}^1$-homology, we will proceed in an analogous way, using the $H_0^{\mathbb{A}^1}$ instead of the $H_0^{\acute{e}t}(-;\mathbb{Z}/\ell)$.

Now k is again assumed to be any perfect field. For X a smooth k-scheme, we denote by $H_0^{\mathbb{A}^1}(X)$ the free strictly $\mathbb{A}^1$-invariant sheaf on X. With the notations we introduced in 2.3 it is just $\mathbb{Z}(X)_{\mathbb{A}^1}$. It has the property that, for any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathcal{A}b_{\mathbb{A}^1}(k)$ there is an identification of the form:

$$M(X) \cong Hom_{\mathcal{A}b_{\mathbb{A}^1}(k)}(H_0^{\mathbb{A}^1}(X), M)$$

In other words any morphism of sheaves of sets $X \to M$ extends uniquely to a morphism of sheaves of abelian groups $H_0^{\mathbb{A}^1}(X) \to M$.

We define the cellular $\mathbb{A}^1$-homology objects of X with integral coefficients as the graded pro-object $H_*^{cell}(X)$ of the category $\mathcal{A}b_{\mathbb{A}^1}(k)$ of strictly $\mathbb{A}^1$-invariant sheaves defined by the formula:

$$H_*^{cell}(X) := lim_{\mathcal{X}_\bullet \tilde{\to} X} H_*(H_0^{\mathbb{A}^1}(\mathcal{X}))$$

where the $\mathcal{X}_\bullet \tilde{\to} X$ run this time over the category of hypercoverings $\mathcal{X}_\bullet \to X$ of X in the Nisnevich topology, up to simplicial homotopy of hypercoverings. These form a left filtering essentially small category so the limit is well defined. Moreover, each $\mathcal{X}_\bullet$ is a simplicial sheaf of sets (which can always be chosen to be a simplicial smooth k-scheme) and $H_*(H_0^{\mathbb{A}^1}(\mathcal{X}_\bullet))$ means then the homology in the abelian category $\mathcal{A}b_{\mathbb{A}^1}(k)$ of the simplicial object $H_0^{\mathbb{A}^1}(\mathcal{X}_\bullet) \in \Delta^{op}\mathcal{A}b_{\mathbb{A}^1}(k)$.

These $H_*^{cell}(X)$ are obviously pro-object in $\mathcal{A}b_{\mathbb{A}^1}(k)$ and one may show that they agree with the one defined in [34]. We conjecture in [34] and [35] that these are in fact always constant. We can prove this in many cases, but the adaptation of the analogous result in the étale topology (using for instance Artin's notion of good neighborhoods) is not completely clear.

However we may formally prove that if X is a smooth k-scheme of Krull dimension $\leq d$, the objects $H_n^{cell}(X)$ vanish for $n > d$. These cellular homology of X are in general easier to compute than the $\mathbb{A}^1$-homology [32], and are to some extent the left derived functors of $X \mapsto H_0^{\mathbb{A}^1}(X)$.

One may obtain a universal coefficient cohomological spectral sequence, for any $M \in \mathcal{A}b_{\mathbb{A}^1}(k)$, of the form $E_r^{p,q}(X;M) \Rightarrow H_{Nis}^*(X;M)$ with E_2-term

$$E_2^{p,q}(X;M) = Ext^p_{\mathcal{A}b_{\mathbb{A}^1}(k)}(H_q^{cell}(X), M)$$

For smooth schemes X with a reasonable "cellular" structure [34], like $\mathbb{P}^n$, or any split reductive k-group, or reasonable homogeneous varieties over those, it is know that the cellular homology are constant. For $\mathbb{P}^n$ we may compute the whole cellular homology.

In general, this cellular homology is much more computable. In [35] we are able to entirely compute $H_*^{cell}(X)$ for X a smooth projective k-rational surface, which is what we ment above by the generalisation to perfect field of the abelian version of what we have done with Asok over an algebraically closed field for the π_1.

Poincaré duality should holds in an explicit form. One of the most concrete consequence of that would be the following:

Conjecture 3.20. *Let X be a smooth projective $\mathbb{A}^1$-connected k-scheme of dimension n. Then:*

$$\big(X \text{ is orientable}\big) \Leftrightarrow \big(H_n^{cell}(X) \cong \mathbf{K}_n^{\mathrm{MW}}\big)$$

Here we mean orientable in the sense that the canonical line bundle $\Lambda^n(T_X)$, the n-exterior power of the tangent bundle T_X, is a square. Of course an orientation is a choice of the square root. Once the orientation is choosen, the isomorphism $H_n^{cell}(X) \cong \mathbf{K}_n^{\mathrm{MW}}$ would be canonical.

This statement is an obvious analogue of the fact that a compact connected differentiable manifold M of dimension n is orientable if and only if $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

This conjecture is true for the $\mathbb{P}^n$'s, as for n odd we compute that $H_n^{cell}(\mathbb{P}^n) = \mathbf{K}_n^{\mathrm{MW}}$ and for n even, we know that $H_n^{cell}(\mathbb{P}^n) = Ker(\eta : \mathbf{K}_n^{\mathrm{MW}} \to \mathbf{K}_{n-1}^{\mathrm{MW}})$. We can also prove this conjecture in several cases, and for instance in [35] we prove it for the case of rational projective smooth surfaces over a perfect field.

One on the nice applications of such a result, is the following. Let $f : X \to Y$ be a morphism in $\mathcal{H}_{\mathbb{A}^1}(k)$ between two oriented smooth projective $\mathbb{A}^1$-connected k-schemes of dimension n; then one may define $deg(f) \in GW(k) = \mathbf{K}_0^{\mathrm{MW}}(k)$ as the induced morphism

$$H_n^{cell}(f) : \mathbf{K}_n^{\mathrm{MW}} = H_n^{cell}(X) \to H_n^{cell}(Y) = \mathbf{K}_n^{\mathrm{MW}}$$

and using the canonical identification $GW(k) = Hom_{\mathcal{A}b_{\mathbb{A}^1}(k)}(\mathbf{K}_n^{\mathrm{MW}}, \mathbf{K}_n^{\mathrm{MW}})$ [32].

An other direction of applications, is the fact that in many interesting cases we know the existence for a nice X of an explicit cellular chain complex $C_*^{cell}(X)$ in the abelian category $\mathcal{A}b_{\mathbb{A}^1}(k)$. It could be used to understand the nature of the signature of the smooth projective $\mathbb{A}^1$-connected k-scheme X, as this problem was addressed and described in a talk of the author at the conference [39].

Another exciting potential development[11] of these new technics is the possibility to define, for nice[12] smooth projective $\mathbb{A}^1$-connected k-schemes X and Y, the Whitehead torsion $\tau(f)$ of an $\mathbb{A}^1$-weak equivalence $f : X \to Y$ using the associated morphism of cellular chain complexes

$$C^{cell}(f) : C_*^{cell}(X) \to C_*^{cell}(Y)$$

by mimicking the standard definition. We think that in this way we may distinguished Hirzebruch surfaces Σ_n and Σ_m with same parity of n and m, which we know from [7] are $\mathbb{A}^1$-h-cobordant, thus $\mathbb{A}^1$-weak equivalent. This means that given an $\mathbb{A}^1$-weak equivalence $f : \Sigma_n \cong \Sigma_m$, one may define the torsion $\tau(f)$ in some motivic Whitehead group, so that if n and m are different, then $\tau(f) \neq 0$. Of course there should be a generalisation of this to handle all the cases of $\mathbb{A}^1$-weak equivalences between nice smooth projective $\mathbb{A}^1$-connected k-schemes. These idea where also described in [7].

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